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House-swapping with divorcing and engaged pairs*

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Abstract. We study a real-life modification of the classical house-swapping problem that is motivated by the existence of engaged pairs and divorcing couples. We show that the problem of finding a new house for the maximum number of agents is inapproximable, but fixed-parameter tractable.

Keywords: House-swapping, Inapproximability, Fixed-parameter tractability

AMS classification: 68Q25

1 Introduction

A popular Czech movie from the 1970's, called Kulový blesk (Ball lightning), describes the 'greatest event in the history of migration'. The story starts by a meeting. The main organizer, Doctor Radosta, hangs a schematics of Kulový blesk (see Figure 1) on the wall and explains to the participants of the meeting:

"Gentlemen, on the day D, the circle will rotate. Mr and Mrs Kadlec are divorcing, therefore they will separate into two apartments. Fortunately, this

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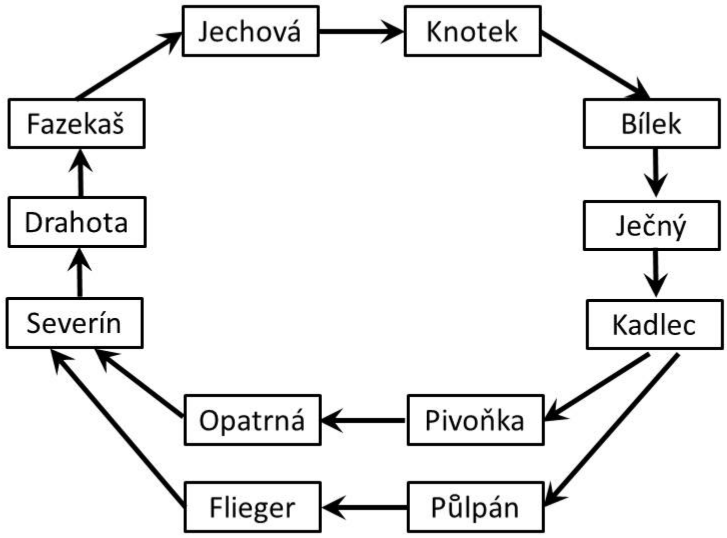

Figure 1: Picture from the film

> *divorce is compensated for by wedding Opatrná - Flieger, so starting with Mr Severin, the chain joins and further it continues normally."*

Behind the scene one can feel the difficult housing situation in the Czechoslovak capital Prague back then. Mr Knotek explains:

> *"We read 286 advertisements and answered 165 of them. We visited 32 apartments, 29 people came to visit us. We have two children, one kitchen and one bedroom. We need more space. And so we found ourselves in this giant carussel of Dr. Radosta."*[1]

The film directly deals with a variant of a model, known in the Economics and Computer science literature under the name the *house-swapping game* or the *housing market* [15]. The main idea is an exchange of a certain indivisible good, traditionally called *house*, although the same model applies to the exchange of other goods, for example undergraduate housing at various universities [16], assigning professors to offices, kidney donors to patients in need [13, 10, 8, 6, 11]. A class of models assumes property rights, i.e. each

[1] The original Czech text of both quotations is as follows:
"Přátelé, ten kruh se nám ve stanovený den pootočí. Kadlecovi se rozvádějí, takže se nám rozdělí do dvou bytů, naštěstí ten rozvod máme kompenzován svatbou Opatrná - Flieger, takže počínaje inženýrem Severínem se nám řetěz spojí a dál už pokračuje normálně."
"Přečetli jsme 286 inzerátů, na 165 jsme odpověděli. 32 bytů jsme si prohlédli my, 29 zájemců se zase dostavilo k nám. Máme dvě děti, pokoj a kuchyň. Chceme do většího. A tak jsme se dostali až na tenhle mamutí kolotoč doktora Radosty."

good has its owner and this owner will participate in the exchange only if she can thus improve her situation [1].

The above quotations from the movie illustrate two aspects of the problem. First, it might be extremely difficult to collect the necessary data. This aspect is also touched in the scentific literature. Yuan [17] describes the large residence exchange fairs for subsidized public housing in China (80 000 person-attendances in Beijing, May 1991). In other setting, finding suitable kidney donors might involve extensive and expensive testing of possible patient-donor pairs [13].

Second, when trying to find possible exchange for as many subjects as possible, it might be necessary to include long cycles. These are undesired, as when just one exchange on the cycle fails for any reasons, the whole exchange cycle falls apart. Therefore, for example in the organization of kidney exchanges in UK only two-exchanges and three-exchanges with a back-up are admitted [11].

2 Related work

Models of exchange of indivisible goods have commonly been studied under the name *housing market* or the *house allocation problem*, as the term *house* has been adopted as a synonym for the indivisible good in question.

In these models, there is usually a set of n agents, each one owns one unit of an indivisible good and each one wants to end up again with one unit of this good, perhaps a better one than his original. An obtained exchange can be evaluated according to several optimization criteria: Pareto efficiency [3], popularity [4], rank maximality [9], core or strict core [15, 12, 8, 6].

In our model, we suppose that each agent only indicates the set of acceptable houses (i.e. she is indifferent between her acceptable houses) and our aim is to find a new house for as many agents as possible. Notice that in this simple housing model, a maximum size exchange corresponds to a maximum cardinality vertex disjoint cycle packing in a directed graph $\mathcal{G}$ with vertices representing agents and arcs acceptable houses. This can be solved efficiently by a method that is now a folklore and that can be described as follows. Create the following bipartite graph $\mathcal{H}$: replace each vertex v of $\mathcal{G}$ by two copies v' and v", connect vertex v' with v" by an edge of weight 0 and vertex v' with vertex u" by an edge with weight 1 if (v, u) is an arc in $\mathcal{G}$. Each maximum weight perfect matching in $\mathcal{H}$ corresponds to a maximum cardinality cycle packing in $\mathcal{G}$ and vice versa. As finding a maximum weight perfect matching in a bipartite graph with N vertices and M edges can be found in time $O(N(M + N\ log\ N))$ (see Theorem 17.3. in [14]), when applied to the simple housing market this gives time complexity $O(n(\ell + n\ log\ n))$ where ℓ is the total number of houses that all agents list as acceptable in their preference lists.

However when a restriction on the maximum length k of involved cycles is included, this problem becomes intractable for any $k \geq 3$ (NP-hard, or even APX-complete, see

[2, 7]).

This paper deals with another modification of the house allocation problem, brought about by the existence of divorcing couples and engaged pairs. We shall show that this modification is also hard. This is because the combinatorial structure of the problem is much more complicated, the exchange chain may split several times, there might even not be one single "master" chain. Two examples of such more elaborate exchanges can be seen in Figure 2.

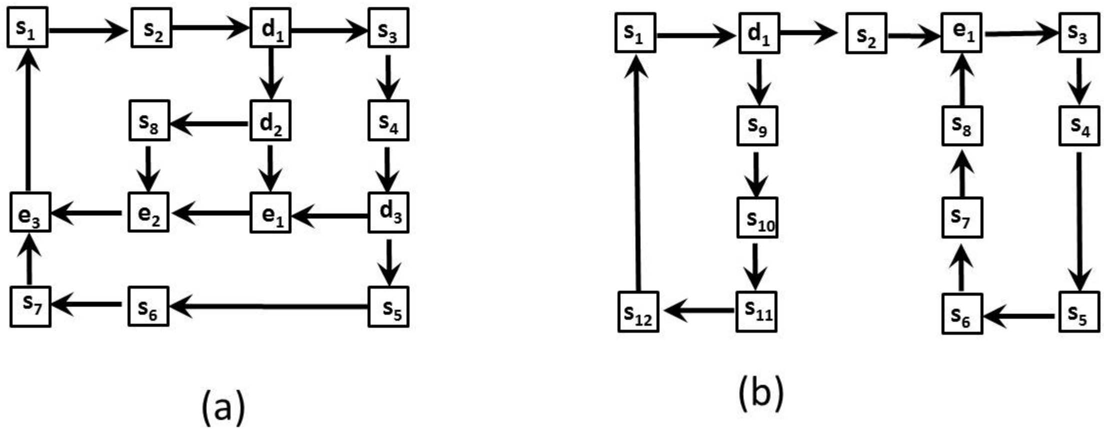

Figure 2: Examples of more complicated exchanges.

3 Definitions and preliminary observations

The set of agents is $A = S \cup B \cup G \cup M \cup W$, where $S = \{s_1, s_2, \dots, s_{n_1}\}$, $B = \{b_1, b_2, \dots, b_{n_2}\}$, $G = \{g_1, g_2, \dots, g_{n_2}\}$, $M = \{m_1, m_2, \dots, m_{n_3}\}$ and $W = \{w_1, w_2, \dots, w_{n_3}\}$ are the sets of single agents, boys, girls, men and women, respectively. For $i = 1, 2, \dots, n_2$, boy b_i and girl g_i form an *engaged pair* $e_i = (b_i, g_i)$, the set of these pairs is denoted by E. For $i = 1, 2, \dots, n_3$, man m_i and woman w_i form a *divorcing couple* $d_i = (m_i, w_i)$ and their set is denoted by D.

Each agent $a \in S \cup B \cup G$ and each divorcing couple $d \in D$ own a house $h(a)$ and $h(d)$, respectively. Hence the set of houses is $H = \{h(a); a \in S \cup B \cup G \cup D\}$. There are no other houses in the market.

Each agent $a \in S \cup M \cup W$ and each engaged pair $e \in E$ provide a set of *acceptable* houses $P(a)$ and $P(e)$ respectively. This means that the boy and the girl of an engaged pair want to move together into a common house. On the other hand, each member m, w

of a divorcing couple $d = (m, w)$ has his and her own set of acceptable houses $P(m)$ and $P(w)$, respectively. Clearly, m and w want to split into two different houses.

We suppose that an agent (a divorcing couple) can stay in their own house and that when an engaged pair move together to a new house then one of their original houses may stay empty. However, when participating in an exchange, it is not allowed to keep one's own house (both houses of an engaged pair). This corresponds to the situation when there is no money in the market (or its use is illegal), so people who move must "pay" for the good they obtain (the house they receive) by the good they originally own. In case of divorcing couples this means that it is not allowed that the man or the woman stay in their original house and the other spouse moves away, since in this case they have nothing to give for the new good.

Further, nobody is allowed to become homeless, at least in this paper. This implies that if a divorcing pair lets their house to a new tenant then both the wife and the husband have to get a new house.

The system of acceptable sets $(P(a), a \in S \cup E \cup M \cup W)$ is called the *preference profile* and we shall denote it by $\mathcal{P}$.

An instance of the House-swapping problem with engaged and divorcing pair (HSED in brief) is a triple $(A, H, \mathcal{P})$.

Definition 1 *An* exchange *is a 1-1 mapping $\mathcal{M} : A' \to H$, where $A' \subseteq S \cup E \cup M \cup W$, such that*

(i) *$\mathcal{M}(a) \in P(a)$ for each $a \in A'$;*

(ii) *if $s \in A'$ then there exists an agent $a' \in A'$ such that $\mathcal{M}(a') = h(s)$;*

(iii) *if $e = (b, g) \in A'$ then there exists $a' \in A'$ such that $\mathcal{M}(a') \in \{h(b), h(g)\}$;*

(iv) *for each $d = (m, w) \in D$, either both m, w belong to A' or none. In the former case there exists an agent $a' \in A'$ such that $\mathcal{M}(a') = h(d)$.*

Conditions of Definition 1 express the requirements that each single agent moves into an acceptable house, both members of an engaged pair move together into a house they consider acceptable and the man and woman of a divorcing couple move into two different houses such that each person finds this new house acceptable. The people in A' will be said to *move* in the exchange $\mathcal{M}$ (notice that when $e \in A'$ then *two* people move, namely, the corresponding boy and girl). Those who are outside A' are said to *stay at home*. Furher, the conditions express the requirements that if somebody moves then he/she must give up his/her house, or, at least one of the two houses of the engaged pair. In the case of the engaged pairs, if the boy and the girl decide to move together into one of their original houses and to keep the second one, then they do not need to participate in the exchange and so we do not consider such pairs.

MAX-HSED denotes the problem of finding an exchange that moves a maximum possible number of agents, given an instance J of HSED.

First we identify an efficiently solvable special case.

Theorem 1 *In an instance J of* HSED*, a necessary condition for the existence of an exchange such that everybody moves, is $|D| \leq |E|$. If this is the case, then the existence of such an exchange can be decided and found in polynomial time.*

Proof. The necessity of the condition is clear: the single agents demand the same number of houses as they supply; and as each divorcing couple demands one more house than it supplies, these houses must be provided by engaged pairs.

Now create a bipartite graph $\mathcal{G}(J) = (V, U, E)$ for J as follows. The color class V of vertices contains one vertex for each single agent, each engaged pair and each member of a divorcing couple, i.e. $V = S \cup E \cup M \cup W$. The color class $U = H$, i.e. it corresponds to houses. There is an edge between $a \in V$ and $h \in U$ if agent a finds the house h acceptable. Further, for each $e = (b, g) \in E$ introduce one virtual homeless player $v(e)$ who is interested only in the two houses $h(b), h(g)$. Now there is an exchange where everyone moves if and only if there is a matching in $\mathcal{G}(J)$ that covers all the houses and all the nonvirtual agents. By the Mendelsohn-Dulmage theorem (see Theorem 16.8. and Corollary16.8b. in [14]), to decide the existence of such a matching, we only have to check whether there exist two matchings in $\mathcal{G}(J)$: one that covers all the houses and another one that covers all the nonvirtual agents. These matching problems can be solved efficiently, as finding a matching that covers n vertices needs n augmentations, while each augmentation is a BFS that takes $O(m)$ time in a graph with m edges. Finally, combining the two matchings a'la Mendelsohn-Dulmage is linear in the size of the matchings, since the union of the two matchings consists of cycles and paths and we have to pick one of the matchings in each of the components. ■

However, below we show that MAX-HSED is an NP-complete problem, moreover, it does not admit a polynomial approximation algorithm if $P \neq NP$.

4 Inapproximability

We shall reduce from the NP-complete problem (2,2)-E3-SAT. This is the problem of deciding, given a Boolean formula $\mathcal{B}$ in CNF in which each clause contains exactly 3 literals and, for each variable v_i, each of literals v_i and $\bar{v}_i$ appears exactly twice in $\mathcal{B}$, whether $\mathcal{B}$ is satisfiable. Berman et al. [5] showed that (2,2)-E3-SAT is NP-complete. [5].

The intuition of the proof can be explained as follows. We start from a boolean formula $\mathcal{B}$. There will be a divorcing couple for each clause c_i and another divorcing couple for each variable. These divorcing couples (thanks to the preferences of women) together with a big set of *dummy* single agents, create one long cycle. The preferences are such that either all the people on this cycle move or none of them does. There will be other agents for each variable (providing houses for the men of the 'clause' divorcing pairs). The trick is to ensure that for one variable, only the two houses corresponding to the occurence of nonnegated variable or only the two houses corresponding to the occurences of negated variable in the formula are available for these men, thus making the connection to the satisfiability of the formula.

Theorem 2 MAX-HSED *is not approximable within* $N^{1-\varepsilon}$ *where* N *is the number of agents, for any* $\varepsilon > 0$, *unless P=NP.*

Proof. Let $\mathcal{B}$ be an instance of (2,2)-E3-SAT, i.e., a Boolean formula in CNF such that each clause contains exactly three literals and each variable occurs exactly twice in B negated and exactly twice nonnegated. Let the set of variables be $X = \{x_1, x_2, \ldots, x_n\}$ and the set of clauses $C = \{c_1, c_2, \ldots, c_m\}$.

We shall construct an instance of the HSED as follows.

We take an arbitrary $\varepsilon > 0$ and denote $t = \lceil \frac{1}{\varepsilon} \rceil$. Notice that $t \geq 1$ and $t\varepsilon \geq 1$.

For each variable x_i there will be a set of three divorcing couples $d_i^\ell = (m_i^\ell, w_i^\ell)$, $\ell = 1, 2, 3$, four engaged pairs $e_i^k = (b_i^k, g_i^k)$ and 4 single agents s_i^k, $k = 1, 2, 3, 4$. The boys b_i^1 and b_i^2 correspond to the first and the second occurence of literal x_i and boys b_i^3 and b_i^4 correspond to the first and the second occurence of literal $\bar{x}_i$. Let us call the subset of the market corresponding to variable x_i *town* T_i.

For each clause c_j there is one divorcing couple $\delta_j = (\mu_j, \nu_j)$. Denote by v_j^1, v_j^2 and v_j^3 the boys corresponding to the first, second and third literal in clause c_j. (For easier reference, agents corresponding to variables are denoted by Latin alphabet, those that correspond to clause by Greek letters.)

The agents introduced so far will be called *proper* agents; their number is thus $r = 2m + n(6 + 8 + 4) = 2m + 18n$.

Now introduce $R = (2r)^t$ single *dummy* agents σ_i, $i = 1, 2, \ldots, R$. So, the total number of agents in the market is $N = r + R$.

The acceptable houses of the agents are given in Figure 3 and the structure of a town is illustrated in Figure 4. The dashed arrows indicate that these houses are acceptable for people outside the town.

The long cycle consists of divorcing couples $\delta_j, j = 1, 2, \ldots, m$, followed by the divorcing couples $d_i, i = 1, 2, \ldots, n$ and the dummy single agents $\sigma_i, i = 1, 2, \ldots, R$.

Let us first suppose that there exists a boolean valuation f of variables that makes $\mathcal{B}$ true. Notice that except for agents μ_j and agents m_i^1 everybody in the constructed instance has only one acceptable house. Let us send the agents $\nu_j, \sigma_i, e_i^k, s_i^k, w_i^k, m_i^2, m_i^3$ to these houses. Further define

$$\mathcal{M}(\mu_j) = h(v_j^k) \text{ where } v_j^k \text{ is the first true literal for } f \text{ in clause } c_j.$$

If x_i is true then we add to the exchange

$$\mathcal{M}(m_i^1) = h(d_i^3); \mathcal{M}(m_i^3) = h(b_i^3); \mathcal{M}(w_i^3) = h(b_i^4)$$

$P(\nu_j):$	$h(\delta_{j+1})$	for $j = 1, \dots, m-1$
$P(\nu_m):$	$h(d_1^1)$	
$P(\mu_j):$	$h(v_j^1), h(v_j^2), h(v_j^3)$	for $j = 1, 2, \dots, m$
$P(e_i^k):$	$h(s_i^k)$	for $i = 1, 2, \dots, n, k = 1, 2, 3, 4$
$P(s_i^k):$	$h(g_i^k)$	for $i = 1, 2, \dots, n, k = 1, 2, 3, 4$
$P(w_i^1):$	$h(d_{i+1}^1)$	for $i = 1, \dots, n-1$
$P(w_n^1):$	$h(\sigma_1)$	
$P(m_i^1):$	$h(d_i^2), h(d_i^3)$	for $i = 1, 2, \dots, n$
$P(m_i^2):$	$h(b_i^1)$	for $i = 1, 2, \dots, n$
$P(w_i^2):$	$h(b_i^2)$	for $i = 1, 2, \dots, n$
$P(m_i^3):$	$h(b_i^3)$	for $i = 1, 2, \dots, n$
$P(w_i^3):$	$h(b_i^4)$	for $i = 1, 2, \dots, n$
$P(\sigma_i):$	$h(\sigma_{i+1})$	for $i = 1, \dots, R-1$
$P(\sigma_R):$	$h(\delta_1)$	

Figure 3: Acceptable houses for agents.

This means that divorcing couple d_i^2 does not participate in the exchange and the houses owned by boys b_i^1, b_i^2 are free for the men from the 'clause' divorcing pairs. If x_i is false then we move these agents as follows:

$$\mathcal{M}(m_i^1) = h(d_i^2); \mathcal{M}(m_i^2) = h(b_i^1); \mathcal{M}(w_i^2) = h(b_i^2)$$

Now divorcing couple d_i^3 does not participate in the exchange and the houses owned by boys b_i^3, b_i^4 are free for the men from the 'clause' divorcing pairs.

It is easy to see that $\mathcal{M}$ is really an exchange and the number of moving agents is $2m$ for the clauses, 16 for each variable plus all the dummy agents. This is together $2m + 16n + R$.

Now suppose that $\mathcal{B}$ is not satisfiable. We shall show that it is impossible that the agents on the long cycle will move. Since, if this cycle moves, then the man of each divorcing 'clause' pair moves to a house of a boy corresponding to a literal contained in this clause. Further, in each town j, the man of the central divorcing couple d_j^1 will move either to the house of divorcing pair d_j^2 or to the house of divorcing pair d_j^3, forcing their members either to the two houses $h(b_j^1)$ and $h(b_j^2)$ or to the two houses $h(b_j^3)$ and $h(b_j^4)$. In the former case, variable v_j can be defined to be *true*, in the latter it can be defined to be *false*. This means that for each clause c_j , the man μ_j of the 'clause' divorcing pair δ_j moves to a house that corresponds to a true literal. Thus $\mathcal{B}$ is satisfiable, after all.

So, if formula $\mathcal{B}$ is not safisfiable, all the agents on the long cycle stay at home, and

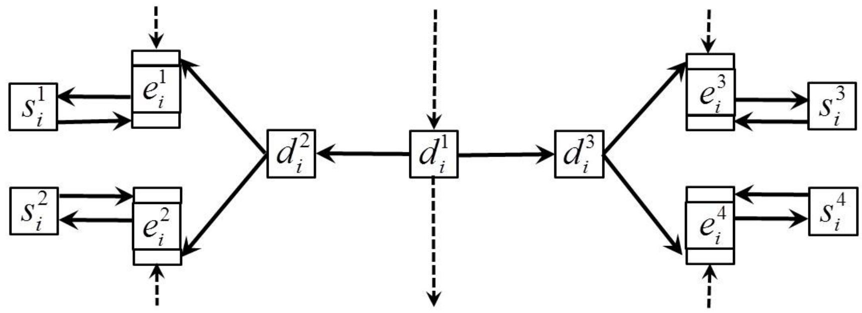

Figure 4: Structure of a town.

hence the number of moving agents is at most $12n$ (i.e., the engaged pairs and single agents in the town for each variable).

Now, if there were a polynomial approximation algorithm $\mathcal{A}$ with approximation guarantee $N^{1-\varepsilon}$, then if this algorithm got as input an instance with at least $2m + 16n + R$ moving agents, then it must output an exchange that moves at least the following number of agents:

$$\begin{aligned}
\frac{2m+16n+R}{N^{1-\varepsilon}} &= \frac{2m+16n+R}{(2m+18n+R)^{1-\varepsilon}} \\
&> \frac{R}{(r+R)^{1-\varepsilon}} = \frac{(r+R)^{\varepsilon}}{\frac{r+R}{R}} = \frac{(r+R)^{\varepsilon}}{\frac{r}{(2r)^t}+1} \\
&\geq \frac{(r+R)^{\varepsilon}}{\frac{1}{2}+1} \text{ (since } (2r)^t \geq 2r \text{ as } t \geq 1) \\
&= \frac{2}{3}(r+R)^{\varepsilon} > \frac{2}{3}\left((2r)^t\right)^{\varepsilon} \geq \frac{4r}{3} \text{ (since } t\varepsilon \geq 1) \\
&= \frac{4}{3}(2m+18n) > \frac{4}{3}18n > 12n.
\end{aligned}$$

Hence $\mathcal{A}$ would be able to distinguish between instances where the long cycle moves and those it does not, consequently, this polynomial algorithm would be able to distinguish between Boolean formulas that are satisfiable and those that are not.

Hence, if P≠NP, this is a contradiction. ■

5 Fixed parameter tractability

Given the strong inapproximability result of the previous section, we want to explore the problem from the parameterized complexity prespective. A natural candidate for a parameter is the number of engaged pairs $|E| = n_2$ or the number of divorcing pairs $|D| = n_3$. First we prove a structural result that gives some intuition on how an exchange could look like.

Given an exchange $\mathcal{M}$, let us introduce the *exchange digraph* $\mathcal{G}(\mathcal{M})$ (examples of such digraphs are given in Figure 2). Its vertex set is $V = S \cup E \cup D$ and its set of arcs encodes the exchange in the following way. The tail of an arc is either a single agent, or an engaged pair or a divorcing couple and its head is the vertex corresponding to the house into which the agent (a single person, an engaged pair or a member of a divorcing couple) at the tail of the arc moves. This means that there are three types of nonisolated vertices in $\mathcal{G}(\mathcal{M})$:

- if $x \in S$, then the indegree of x as well as its outdegree is 1;
- if $x \in E$, then the indegree of x is 1 or 2, but the outdegree is 1;
- if $x \in D$, then the indegree of x is 1 and the outdegree is 2.

Lemma 1 *Let $\mathcal{M}$ be an exchange for an instance J of* HSED *and let $D' \subseteq D$ be the set of moving divorcing couples and $E' \subseteq E$ the set of moving engaged pairs who give up both their houses. Then there is a perfect matching φ between D' and E' and a collection of arc-disjoint paths $(p(e), e \in E')$ such $p(e)$ starts in $e \in E'$ and ends in $\varphi(e)$.*

Proof. The total indegree in $\mathcal{G}(\mathcal{M})$ is the same as the total outdegree, so the number of indegree 2 vertices (endgaged pairs in E') is the same as the number of outdegree 2 vertices (couples in D'). Assume that this number is m. Add m new arcs to $\mathcal{G}(\mathcal{M})$ so that we add a *new* arc from each $e \in E'$ to a different $d \in D'$. The obtained digraph is Eulerian. It is well-known that the arc set of an Eulerian digraph can be decomposed into arc disjoint cycles. Take the set of cycles C in this decomposition that contain at least one new arcs. As each new arc goes from $e \in E'$ to a $d \in D'$, taking these arcs out of cycles of C leaves the paths as required. Hence the assertion is proved. ■

Examples of these arc-disjoint paths can again be seen in Figure 2 (a):

$$d_1 \to d_2 \to s_8 \to e_2;\ d_2 \to e_1;\ d_3 \to s_5 \to s_6 \to s_7 \to e_3.$$

Lemma 1 implies that when we want to move m chosen divorcing couples, we need (in a certain sense) m different engaged pairs; other engaged pairs in the instance (if any) could keep one of their original houses. However, the task of finding a combination of such paths and cycles to include the maximum possible number of agents is not obvious. Therefore we shall use for an instance J of HSED an approach that we just describe.

Let us choose in J a set Σ of m divorcing couples and m engaged pairs (notice that this construction applies also if $m = 0$, i.e., no divorcing couple moves) and create a digraph $\mathcal{G}_\Sigma(J)$ as follows. We start with the set of vertices $V = S \cup D \cup E$, from which we delete all the non-chosen divorcing couples. Each chosen divorcing pair $d \in D \cap \Sigma$ is replaced by four vertices d_w, d_m, d_h and d_x. They represent the woman, the man, the house of the couple and a dummy vertex. For each chosen engaged pair $e \in E \cap \Sigma$ we also have four vertices e_b, e_g, e_t and e_y (the boy's house, the girl's house, the pair together and a dummy vertex); the non-chosen engaged pairs are represented by just one vertex.

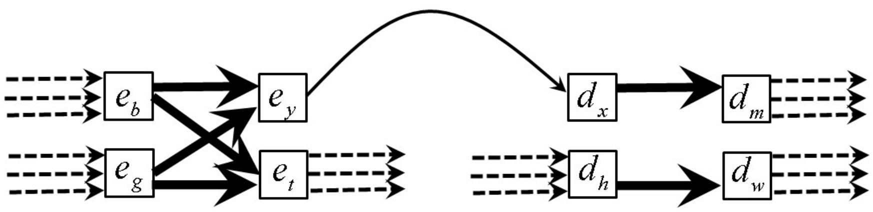

Figure 5: Splitting the vertices for engaged pairs and divorcing couples.

Unless otherwise specified, arcs have weight 1 and they correspond to acceptable houses: everbody points to the agent who owns an acceptable house of his/her. For single agents this creates no confusion, for other agents we make the following specifications. If $e \in E \setminus \Sigma$ then the incoming arcs correspond to the house of the girl as well as to the house of the boy of e. The outgoing arcs will have weight 2, as they will represent two people moving (the boy and the girl). For vertices corresponding to the set Σ we further introduce the arcs with weights according to Figure 6, where the y-vertex of each engaged pair is connected by an arc to the x-vertex of each divorcing couple. K denotes a sufficiently big number, for example we can set $K = 2(n_1 + n_2 + n_3)^2$. The arcs with weight K are called *thick* arcs, for brevity. The construction is also illustrated in Figure 5, where the dashed arcs correspond to original preferences of agents.

Lemma 2 *There exists an exchange in J moving the agents in set Σ and further ℓ engaged pairs and k single agents if and only if the digraph $\mathcal{G}_\Sigma(J)$ admits a cycle packing of weight $4K + 2\ell + k$.*

Proof. If the maximum weight of a cycle packing $\mathcal{C}$ in $\mathcal{G}_\Sigma(J)$ is less than $4K$, then $\mathcal{C}$ leaves out at least one of the thick edges. If one of $(d_x, d_m), (d_h, d_w)$ is missing then either the man or the woman of d does not move, so it is not an exchange according to Definition 1. If the four vertices e_b, e_g, e_d and e_t are not matched by a matching of size 2 then the engaged pair e did not give away their second house that was needed for a divorcing pair in Σ.

arcs	*weight*
$(d_x, d_m), (d_h, d_w)$	K
(a, d_h)	1 (if a finds the house of couple d acceptable)
(d_m, a)	1 (if m finds the house of a acceptable)
(d_w, a)	1 (if w finds the house of a acceptable)
$(e_b, e_t), (e_g, e_t), (e_b, e_y), (e_g, e_y)$	K
(a, e_b)	1 (if a finds the house of boy b acceptable)
(a, e_g)	1 (if a finds the house of girl g acceptable)
(e_t, a)	2 (if the pair e finds the house of a acceptable)
(e_y, d_x)	0 the dummy arc

Figure 6: Arcs for the engaged pair and the divorcing couple.

On the other hand, if the weight of a maximum cycle packing $\mathcal{C}$ is more than $4K$ then for each $d \in \Sigma$, $\mathcal{C}$ must include the dummy arc (e_y, d_x) for some $e \in \Sigma$. The cycle in $\mathcal{C}$ that contains this arc corresponds to a path from the divorcing couple d to the engaged pair e. The arcs of weight 1 count the number of moving single agents, men and women and the arcs with weight 2 count for the two moving members of engaged pairs in $E \setminus \Sigma$. ■

Now we can prove the main theorem in this section.

Theorem 3 MAX-HSED *is FPT when parameterized by* $\kappa = \max\{|E|, |D|\}$.

Proof. The number of ways m divorcing couples and m engaged pairs in an instance I of MAX-HSED can be chosen is $\binom{|D|}{m}\binom{|E|}{m}$. Summarizing, the number of the maximum weight cycle packing problems we have to solve is

$$\sum_{m=0}^{min\{|D|,|E|\}} \binom{|D|}{m}\binom{|E|}{m},$$

bounded by $f(\kappa) = (\kappa + 1)\kappa^\kappa$. Hence MAX-HSED can be solved by an algortihm whose time complexity is $f(\kappa)|I|^{O(1)}$, where f is a computable function of the parameter κ only, so MAX-HSED is fixed-parameter tractable. ■

6 Conclusion

We showed that the house-swapping with the engaged pairs and divorcing couples is inapproximable, but it is fixed-parameter tractable when parameterized by the maximum of the number of divorcing couples and engaged pairs.

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