Eigenvectors of interval matrices over max-plus algebra

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Abstract

The behaviour of a discrete-event dynamic system is often conveniently described using a matrix algebra with operations max and plus. Such a system moves forward in regular steps of length equal to the eigenvalue of the system matrix, if it is set to operation at time instants corresponding to one of its eigenvectors. However, due to imprecise measurements it is often unappropriate to use exact matrices. One possibility to model imprecision is to use interval matrices. We show that the problem to decide whether a given vector is an eigenvector of one of the matrices in the given matrix interval is polynomial, while the complexity of the existence problem of a universal eigenvector remains open. As an aside, we propose an alternative combinatorial method for solving two-sided systems of linear equations over the max-plus algebra.

Keywords: Discrete-event dynamic systems, max-plus algebra, interval matrices, eigenproblem, two-sided systems of linear equations.

AMS classification: 15A06, 90C48

1 Introduction

The behaviour of a **discrete-event dynamic system** is often conveniently described using a matrix algebra with operations max and plus. The basic idea is as follows. Suppose that the operation of the system is performed in cycles and consists of n interrelated jobs. The number a_{ij} denotes the duration of the operation of job j needed for job i. (If job i does not need job j then $a_{ij} = -\infty$.)

*The work was supported by the Slovak Agency for Science, contract #1/0425/03 "Combinatorial Structures and Complexity of Algorithms"

If vector $\mathbf{x}(k) = (x_1(k), x_2(k), \ldots, x_n(k))$ denotes the time instants in which all jobs started for the k^{th} time, then if all the jobs wait for all the preceding jobs to finish their operation, the earliest possible starting times of $k+1^{st}$ cycle are expressed by vector $\mathbf{x}(k+1)$, with

$$x_i(k+1) = max\{a_{i1} + x_1(k), a_{i2} + x_2(k), \ldots, a_{in} + x_n(k)\}.$$

Cuninghame-Green [5] introduced a convenient formalism for the description of such situations. In the **max-plus algebra** $(\mathcal{R}, \oplus, \otimes)$, the set $\mathcal{R}$ is equal to the set of all real numbers appended with $-\infty$, $\oplus = max$ and $\otimes$ =normal addition of real numbers. The max-plus algebra is a semiring ([1], [15]) with the additive zero equal to $-\infty$ and multiplicative unit equal to the real number 0.

Matrices can be multiplied formally in the same manner as in the classical algebra (over the field of reals), just using the $\oplus$ and $\otimes$ operations instead of addition and multiplication. Then the development of a system described above can be expressed by a vector equation over max-plus algebra of the form:

$$\mathbf{x}(k+1) = A \otimes \mathbf{x}(k).$$

In what follows, the set of all $m \times n$ matrices over $\mathcal{R}$ will be denoted by $\mathcal{R}(m,n)$ and the set of all column n-vectors over $\mathcal{R}$ by $\mathcal{R}(n)$. Vectors are usually denoted by boldface letters, matrices by capitals, the i^{th} column of matrix A by A_i.

A finite number λ and a vector $\mathbf{x} \in \mathcal{R}_n$ with at least one finite entry are called an **eigenvalue** and an **eigenvector** of a matrix $A \in \mathcal{R}(n,n)$ if

$$A \otimes \mathbf{x} = \lambda \otimes \mathbf{x}.$$

We suppose that the reader is familiar with the basic eigenvalue-eigenvector theory for max-plus algebra, in particular with the role played by the **associated digraph** $G(A)$ of the matrix A. Nevertheless, we summarize here the most important facts.

Cuninghame-Green [6] was the first one to derive the celebrated result, stating that each irreducible matrix (i.e. one with strongly connected associated digraph) has one and only one eigenvalue $\lambda(A)$, which is equal to the maximum cycle mean of the associated digraph of A, i.e.

$$\lambda(A) = \max\left\{\frac{w(\rho)}{\ell(\rho)};\ \rho \text{ is a cycle in } G(A)\right\}, \tag{1}$$

where for a cycle $\rho = (i_1, i_2, \ldots, i_k)$ its weight $w(\rho) = a_{i_1 i_2} \otimes a_{i_2 i_3} \otimes \ldots \otimes a_{i_k i_1}$ and its length $\ell(\rho) = k$.

This result has later been rediscovered by many other authors, for more references see e.g. [1], [4] or [15]. Cuninghame-Green [7] showed that $\lambda(A)$ is the optimal solution of the linear program

$$\lambda \rightarrow min \tag{2}$$

$$\lambda + x_i - x_j \geq a_{ij} \text{ for all pairs } i, j \text{ with } a_{ij} \text{ finite} \tag{3}$$

but a more practical algorithm of complexity $O(n^3)$ is due to Karp [10]. Moreover, numerical experiments show that $\lambda(A)$ can be computed in almost linear time by Howard's algorithm [4].

Cuninghame-Green [7] also gave a complete description of the set of all eigenvectors of a given matrix. For a summary of this result, we need some more concepts. A circuit of the associated digraph $G(A)$ is called **critical**, if its mean is equal to $\lambda(A)$. The **critical digraph** $G^C(A)$ consists of those vertices (called **critical vertices**) and arcs of $G(A)$ that belong to some critical circuit. Two critical vertices that belong to the same strongly connected component of $G^C(A)$ are said to be **equivalent**.

Now, if we subtract $\lambda(A)$ from each entry of $A \in \mathcal{R}(n, n)$, the obtained matrix B has $\lambda(B) = 0$ but the eigenspaces of A and B are identical. In the metric matrix $\Gamma(B) = B \oplus B^2 \oplus \ldots \oplus B^n$ of B necessarily some columns have their main-diagonal entries equal to 0; they are said to be critical, as they correspond to critical vertices of $G(A)$ (and as well as of $G(B)$). These columns are eigenvectors of B and of A and are called **fundamental eigenvectors**. Then each eigenvector of A can be expressed as a max-plus linear combination of a set of m fundamental eigenvectors corresponding to mutually nonequivalent critical vertices of $G(A)$; here m denotes the number of strongly connected components of $G^C(A)$.

There are many important practical interpretations of the eigenvalue and eigenvector of a matrix. We shall mainly use the following two of them:

1. If the system with matrix A has been started according to some eigenvector $\mathbf{x}$ of matrix A, then it will move forward in regular steps with the same length, i.e. the time elapsed between the consecutive starts of all jobs will be equal to $\lambda(A)$.

2. Suppose that there is a given schedule for the system, requiring that the time interval between two consecutive jobs should not exceed a certain value μ. Is it possible to start the system in such a way that the schedule will be kept? In other words, does there exist a vector $\mathbf{x}$ such that

$$A \otimes \mathbf{x} \leq \mu \otimes \mathbf{x}? \tag{4}$$

The fact that $\lambda(A)$ is the optimal solution of the linear program (2–3) implies

Lemma 1 *For any square matrix A, inequality (4) is soluble if and only if $\mu \geq \lambda(A)$.*

2 The eigenproblem for interval matrices

In some cases, the use of a precise matrix is inappropriate for the modelled situation, as the measurement imprecisions imply that the computations performed with the exact matrix do not correspond to the real behaviour of the system. The importance of dealing with interval data in the classical algebra has been recognized for a very long time, however, we do not know any other publications in the area of interval algebraic problems for max-plus algebra except for [2], [3].

We shall study some extensions of the eigenvector-eigenvalue problem over max-plus algebra for interval square matrices. An **interval matrix** is the set of matrices in the matrix interval of the form $\mathbf{A} = \langle \underline{A}, \overline{A} \rangle$ with given $\underline{A}, \overline{A} \in \mathcal{R}(n,n)$, $\underline{A} \leq \overline{A}$. For simplicity, let us suppose that all the entries in $\underline{A}$ as well as in $\overline{A}$ are finite. This implies that the associated digraphs of all matrices $A \in \mathbf{A}$ are strongly connected and hence each $A \in \mathbf{A}$ has a unique eigenvalue $\lambda(A)$ given by (1).

Definition 1 *We say that a real number λ is a* **possible eigenvalue** *of an interval matrix $\mathbf{A}$ if it is an eigenvalue of at least one $A \in \mathbf{A}$. A real number λ is a* **universal eigenvalue** *of an interval matrix $\mathbf{A}$ if it is an eigenvalue of each $A \in \mathbf{A}$.*

The expression of the eigenvalue $\lambda(A)$ of a matrix A in the form (1) by a continuous and isotone function of matrix entries immediately leads to the following full description of the set of all eigenvalues of an interval matrix:

Theorem 1 *A number λ is a possible eigenvalue of an interval matrix $\mathbf{A}$ if and only if $\lambda \in \langle \lambda(\underline{A}), \lambda(\overline{A}) \rangle$. An interval matrix $\mathbf{A}$ has a universal eigenvalue if and only if $\lambda(\underline{A}) = \lambda(\overline{A})$.*

Let us mention here, that this result is in a marked contrast with the situation in the classical algebra. The problem to decide whether a given rational number is a possible eigenvalue of an interval matrix is NP-hard, which follows from the results in [13] and [12] and is formulated as Theorem 21.17. in [11]. As far as we know, an analogy of the universal eigenvalue has not been studied in classical algebra.

Now we can turn our attention to eigenvectors.

Definition 2 *We say that a vector $\mathbf{x} \in \mathcal{R}(n)$ is a* **possible eigenvector** *of an interval matrix $\mathbf{A}$ if there exists $A \in \mathbf{A}$ such that $A \otimes \mathbf{x} = \lambda(A) \otimes \mathbf{x}$. A vector $\mathbf{x} \in \mathcal{R}(n)$ is a* **universal eigenvector** *of an interval matrix $\mathbf{A}$ if $A \otimes \mathbf{x} = \lambda(A) \otimes \mathbf{x}$ for each $A \in \mathbf{A}$.*

In the following sections we shall deal with the following problems: Given an interval matrix $\mathbf{A}$ of order n decide whether:

Problem 1. a given vector $\mathbf{x} \in \mathcal{R}(n)$ is a possible eigenvector of $\mathbf{A}$;

Problem 2. $\mathbf{A}$ has a universal eigenvector.

An interpretation of a possible eigenvector $\mathbf{x}$ might be as follows: if a system is started according to $\mathbf{x}$, then there exists a concrete realization of the operation of the system, materialized by a concrete matrix A from the given interval $\mathbf{A}$, for which the following starting times of all jobs will be delayed by the same value, equal to the eigenvalue $\lambda(A)$. If in the next cycles the system again happens to be controlled by A, then it may continue this regular behaviour forever. It would be interesting to get some bounds for a kind of a 'neighbourhood' of matrix A ensuring this regular behaviour with vector $\mathbf{x}$, but this question is left open in the present paper.

On the other hand, there is a plausible interpretation of a universal eigenvector if the given interval matrix $\mathbf{A}$ has a universal eigenvalue λ. Namely, if $\mathbf{x}$ is a universal eigenvector and the system is started according to $\mathbf{x}$, then no matter what the concrete realisations of the system in the following cycles will be, the system will always move forward regularly in steps of length equal to λ.

In the classical algebra it can be tested in polynomial time whether a given rational vector is a possible eigenvector of an interval matrix [14]. Again, the notion of a universal eigenvector seems not to have been studied.

3 Possible eigenvector

If we want to test whether a given vector $\mathbf{x}$ is a possible eigenvector of $\mathbf{A}$, then the difficulty is that we know neither the concrete matrix from $\mathbf{A}$ nor the corresponding eigenvalue. Let us look at the example.

Example Consider an interval matrix $\mathbf{A} = \begin{pmatrix} (3,8) & (2,4) & (6,9) \\ (2,6) & (3,5) & (1,2) \\ (4,5) & (2,8) & (3,7) \end{pmatrix}$. Then $\underline{A} = \begin{pmatrix} 3 & 2 & 6 \\ 2 & 3 & 1 \\ 4 & 2 & 3 \end{pmatrix}$ with $\lambda(\underline{A}) = 5$ and $\overline{A} = \begin{pmatrix} 8 & 4 & 9 \\ 6 & 5 & 2 \\ 5 & 8 & 7 \end{pmatrix}$ with $\lambda(\overline{A}) = 8$.

Now, if we take $\mathbf{x} = (1,2,3)^T$, then $\underline{A} \otimes \mathbf{x} = (9,5,6)^T$, and as $A \geq \underline{A}$ for each $A \in \mathbf{A}$, we see that if $\mathbf{x}$ is to be a possible eigenvector, its corresponding eigenvalue must be at least 8 and $A \otimes \mathbf{x}$ must be at least $(9,8,7)^T$. However, $\overline{A} \otimes \mathbf{x} = (12,7,10)^T$, hence for no matrix $A \in \mathbf{A}$ the vector $A \otimes \mathbf{x}$ can have in its second and third coordinates the desired values. Hence $\mathbf{x} = (1,2,3)^T$ is not a possible eigenvector of this interval matrix.

On the other hand, for $\mathbf{x} = (3,2,1)^T$ we have $\underline{A} \otimes \mathbf{x} = (7,5,7)^T$, so the eigenvalue of $\mathbf{x}$ should be at least 6. Denote $\lambda(\mathbf{x}) = 6$. Now we try to increase the matrix $\underline{A}$ in each coordinate in such a way that we do not exceed $\overline{A}$ but for

Input: an interval matrix $\mathbf{A}$, a vector $\mathbf{x}$.

1. **begin** $\mathbf{y} := \underline{A} \otimes \mathbf{x}$;
2.   **if** $\mathbf{y} - \mathbf{x}$ is a constant vector
3.   **then** $\mathbf{x}$ is a possible eigenvector, STOP
4.   **else begin** $\lambda(\mathbf{x}) := \max_i\{y_i - x_i\}$;
5.   **for all** $i, j : a^*_{ij} = min\{\overline{a}_{ij}, \lambda(\mathbf{x}) + x_i - x_j\}$;
6.   **if** $(A^* \otimes \mathbf{x}) - \mathbf{x}$ is a constant vector
7.   **then** $\mathbf{x}$ is a possible eigenvector of $\mathbf{A}$; STOP
8.   **else** $\mathbf{x}$ is a not possible eigenvector of $\mathbf{A}$; STOP
9.   **end**
10. **end**

Figure 1: Algorithm PossibleEigenvector

each coordinate i of $A \otimes \mathbf{x}$ we ensure the value at least $\lambda(\mathbf{x}) \otimes x_i$. It is easy to see that we should set $a^*_{ij} = min\{\overline{a}_{ij}, \lambda(\mathbf{x}) + x_i - x_j\}$ so that in this example

$$A^* = \begin{pmatrix} 6 & 4 & 8 \\ 5 & 5 & 2 \\ 4 & 5 & 6 \end{pmatrix}.$$

Since $\underline{A} \leq A^* \leq \overline{A}$ and $A^* \otimes \mathbf{x} = 6 \otimes \mathbf{x}$, we see that $\mathbf{x}$ is a possible eigenvector of $\mathbf{A}$ with the corresponding eigenvalue $\lambda = 6$. ■

Algorithm PossibleEigenvector for testing whether a given vector $\mathbf{x}$ is a possible eigenvector of an interval matrix $\mathbf{A}$ is given in Figure 1.

Theorem 2 *Algorithm in Figure 1 is correct; its computational complexity is* $O(n^2)$.

Proof. Algorithm PossibleEigenvector ends with the positive answer in rows 3 and 7, where it has detected that $\mathbf{x}$ is an eigenvector of matrices $\underline{A}$ and A^*, respectively.

If $\mathbf{x}$ is not an eigenvector of matrix $\underline{A}$ (this will correspond to lines 4 and later in the algorithm) then, due to Lemma 1 necessarily $\lambda(\mathbf{x}) \geq \lambda(\underline{A})$. Further, as matrix $\underline{A}$ is the smallest one in $\mathbf{A}$, vector $A \otimes \mathbf{x}$ will always be componentwise greater or equal to $\mathbf{y}$, so the smallest possible increase in the values of its entries will be $\lambda(\mathbf{x})$. So if we want $\mathbf{x}$ to be translated by the same amount in each coordinate, the only possibility is to increase some entries of $\underline{A}$, as it is done in row 5 of the algorithm. Condition in row 6 is not fulfilled only if some entries of the newly defined matrix A^* have exceeded the upper bound given by $\overline{A}$, so in this case $\mathbf{x}$ cannot be an eigenvector of any matrix in $\mathbf{A}$.

The verification of the computational complexity of the algorithm is trivial.
■

4 Universal eigenvectors

As said before, universal eigenvectors have a plausible interpretation if $\lambda(\underline{A}) = \lambda(\overline{A})$. Therefore we shall restrict ourselves to this case only. For simplicity, from now on we shall assume that the given interval matrix has a universal eigenvalue equal to 0.

We already know that each eigenvector of a given matrix is a max-plus linear combination of a generating set of mutually nonequivalent fundamental eigenvectors [7]. Hence, if a vector $\mathbf{x}$ is to be a universal eigenvector, it must be a max-plus linear combination of a generating set of mutually nonequivalent fundamental eigenvectors of matrix $\underline{A}$ (say $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$) and simultaneously an eigenvector od $\overline{A}$. Hence we look for a vector $\mathbf{y}$ of the form $\sum_{i+1}^{m}{}^{\oplus}(y_i \otimes \mathbf{u}_i)$ such that

$$\overline{A} \otimes \sum_{i=1}^{m}{}^{\oplus}(y_i \otimes \mathbf{u}_i) = \sum_{i=1}^{m}{}^{\oplus} y_i \otimes \mathbf{u}_i$$

This can be rewritten as

$$\sum_{i=1}^{m}{}^{\oplus}(\overline{A} \otimes \mathbf{u}_i) \otimes y_i = \sum_{i=1}^{m}{}^{\oplus} \mathbf{u}_i \otimes y_i.$$

Since vectors $\mathbf{u}_i$ are eigenvectors of $\underline{A}$ with eigenvalue 0 and $\overline{A} \geq \underline{A}$, we have that $\overline{A} \otimes \mathbf{u}_i \geq \underline{A} \otimes \mathbf{u}_i = \mathbf{u}_i$ for all i. Hence looking for universal eigenvectors reduces to solving special systems of two-sided max-plus equations with the same variables in each side and one matrix elementwise greater or equal to the second one. The previous discussion can be summarized by

Theorem 3 *Let an interval matrix* $\mathbf{A}$ *with* $\lambda(\underline{A}) = \lambda(\overline{A}) = 0$ *be given and let* $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ *be any generating system of fundamental eigenvectors of* $\underline{A}$. *Then there exists a universal eigenvector of* $\mathbf{A}$ *if and only if the two-sided system*

$$C \otimes \mathbf{y} = D \otimes \mathbf{y}$$

with $C_i = \overline{A} \otimes \mathbf{u}_i$ *and* $D_i = \mathbf{u}_i$ *for all* $i = 1, 2, \ldots, m$ *has a solution.*

Although several method for solving general systems of two-sided max-plus equations have been derived, e.g. in [8], [9], all of them are of iterative nature. Moreover, even if [8] proves pseudopolynomiality of the proposed algorithm, the existence of a polynomial algorithm is still an open question. We tried a more combinatorial approach, which is for the special systems involved in the universal eigenvector problem partially successful.

5 Solving special two-sided systems of equations

In this section we deal with two-sided systems of linear equations over max-plus algebra of the form

$$C \otimes \mathbf{y} = D \otimes \mathbf{y} \text{ with } C \geq D. \tag{5}$$

N denotes the set of row indices $\{1, 2, \ldots, n\}$ and M is the set of column indices $\{1, 2, \ldots, m\}$ of matrices C and D.

Lemma 2 *If there exists a row i such that $c_{ij} > d_{ij}$ for each j then system (5) is insoluble.*

Proof. To get a contradiction, suppose that a vector $\mathbf{y}$ is a solution of (5) and that the maxima in the left-hand side and right-hand side of row i have been achieved in terms j and l respectively, i.e.

$$c_{ij} + y_j = \max_k\{c_{ik} + y_k\} \text{ and } d_{il} + y_l = \max_k\{d_{ik} + y_k\}.$$

Then

$$c_{ij} + y_j \geq c_{il} + y_l > d_{il} + y_l,$$

a contradiction. ■

Corollary 1 *For each soluble system of the form (5) there exists for each row i an index j such that $c_{ij} = d_{ij}$.*

We shall call an index j from the previous Corollary **critical** for row i. In what follows, let us denote the set of critical indices for row i by K_i. If a vector $\mathbf{y}$ is given, then an index j is **maximizing** for row i in $C \otimes \mathbf{y}$, if $c_{ij} + y_j = \max_k\{c_{ik} + y_k\}$.

Lemma 3 *For each solution $\mathbf{y}$ of (5) there exists for each row i a critical maximizing index.*

Proof. Let us suppose that $\mathbf{y}$ is a solution of (5) and in row i of $C \otimes \mathbf{y}$ and $D \otimes \mathbf{y}$ maximizing terms are $c_{ij} + y_j$ and $d_{ik} + y_k$, respectively. This means $c_{ij} + y_j = d_{ik} + y_k$, hence $y_k = y_j + c_{ij} - d_{ik}$. Then for the k^{th} term in row i of $C \otimes \mathbf{y}$ we have

$$c_{ik} + y_k = c_{ik} + y_j + c_{ij} - d_{ik} \geq c_{ij} + y_j,$$

since $c_{ik} \geq d_{ik}$. Should $c_{ik} - d_{ik} > 0$ then index j is not maximizing in row i of $C \otimes \mathbf{y}$, a contradiction. Therefore index k is maximizing and critical for row i in $C \otimes \mathbf{y}$. ■

Hence looking for a solution of a two-sided system of equations of the form (5) leads to choosing a suitable maximizing term from the critical indices for each row of $C \otimes \mathbf{y}$. The choice of these terms will be formalized in the following theorem by the function j. This leads to the following necessary and sufficient solvability condition.

Theorem 4 *A system of the form (5) is soluble if and only if each K_i is nonempty and there exists a function $j : N \to M$ such that $j(i) \in K_i$ for all $i \in N$ and for each subset $\{i_1, i_2, \ldots, i_k\}$ of N such that the indices $j(i_1), j(i_2), \ldots, j(i_k)$ are mutually different, we have*

$$c_{i_1 j(i_1)} + c_{i_2 j(i_2)} + \ldots + c_{i_k j(i_k)} \geq c_{i_1 j(i_2)} + c_{i_2 j(i_3)} + \ldots + c_{i_k j(i_1)} \tag{6}$$

Proof. For the 'only if' direction suppose that $\mathbf{y}$ is a solution of (5). Let us take any subset $\{i_1, i_2, \ldots, i_k\} \subseteq N$. By Lemma 3, in each row there exists a maximizing critical index of $C \otimes \mathbf{y}$, let us denote one such index for each row by $j(i_l), l = 1, 2, \ldots, k$ and suppose that the chosen column indices are mutually different. Then we have

$$c_{i_l j(i_l)} + y_{j(i_l)} \geq c_{i_l j(i_{l+1})} + y_{j(i_{l+1})}$$

for all $l = 1, 2, \ldots, k$, while $k + 1 = 1$. When we add the above inequalities, we get (6).

Conversely, let us suppose that it is possible to choose for each row i a critical column index $j(i)$ in such a way that inequality (6) holds. Let us now denote by $J(N) = \{j(i); i \in N\}$ the set of column indices of chosen columns and let C' stand for the submatrix of C created from columns of $J(N)$. Denote by $p(j)$ the number of rows with the chosen index in column j and consider the following transportation problem (TP) with matrix C':

$$\sum_{i \in N} \sum_{j \in J(N)} c'_{ij} x_{ij} \quad \to \quad \max \tag{7}$$

$$\sum_{j \in J(N)} x_{ij} \quad = \quad 1 \text{ for all } i \in N \tag{8}$$

$$\sum_{i \in N} x_{ij} \quad = \quad p(j) \text{ for all } j \in J(N) \tag{9}$$

$$x_{ij} \quad \geq \quad 0 \text{ for all } i \in N, j \in J(N) \tag{10}$$

The TP (7–10) is balanced, hence soluble, and we show that one possible optimal solution is given by $x_{ij(i)} = 1$ for all $i \in N$ and $x_{ij} = 0$ otherwise. For, let us suppose that $\mathbf{x}$ is an integer optimal solution and $x_{i_1 j} = 1$ for some $i_1 \in N$ and $j \neq j(i_1)$. As capacity of row i_1 is 1, necessarily $x_{i_1 j(i_1)} = 0$. Further, as $j \in J(N)$ and the capacity of column j is $p(j)$, there exists a row $i_2 \in N$ such that $j = j(i_2)$ and $x_{i_2 j(i_2)} = 0$. Then, capacity of row i_2 is 1, so there exists a column index $j = j(i_3)$ for some row i_3 such that $x_{i_2 j(i_3)} = 1$ and $x_{i_3 j(i_3)} = 0$ etc. In this way we get a cycle of row indices (without loss of generality let this cycle be $(i_1, i_2, \ldots, i_k)$) such that $x_{i_l j(i_l)} = 0$ and $x_{i_l j(i_{l+1})} = 1$ for all $l = 1, 2, \ldots, k$, while $k + 1 = 1$. Let us define a new vector $\mathbf{x}'$ by setting $x'_{i_l j(i_l)} = 1$ and $x'_{i_l j(i_{l+1})} = 0$ for all $l = 1, 2, \ldots, k$ and $x'_{ij} = 0$ for all other pairs of indices. Then, since C' fulfills (6), the cost of vector of $\mathbf{x}'$ is not smaller than the cost of vector $\mathbf{x}$ and after several steps of this kind we get an optimal solution of the TP (7–10) with the property $x_{ij} = 1$ if and only if $j = j(i)$ for all $i \in N$.

Now let us take the dual of the TP (7–10):

$$\begin{aligned} \sum_{i\in N} u_i + \sum_{j\in J(N)} p(j)v_j &\to \min \\ u_i + v_j &\geq c'_{ij} \text{ for all } i \in N, j \in J(N) \end{aligned}$$

We assert that vector $\mathbf{y}$ defined by

$$y_j = \begin{cases} -\infty & \text{for } j \notin J(N) \\ -v_j & \text{for } j \in J(N) \end{cases}$$

is an optimal solution of (5). First of all, since in each row i we have chosen a critical index to be $j(i)$, we have

$$c_{ij(i)} + y_{j(i)} = d_{ij(i)} + y_{j(i)}$$

and we need further to show that for all other columns l we have:

$$c_{il} + y_l \leq c_{ij(i)} + y_{j(i)}.$$

For $l \notin J(N)$ this inequality is trivial, for the columns of $J(N)$ let us recall that $c_{ij} = c'_{ij}$. Since the chosen entries $(i, j(i))$ correspond to nonzero components of the optimal solution of TP (7–10), the complementarity theorem of linear programming ensures that $u_i = c_{ij(i)} - v_{j(i)}$ for them and for all other entries $v_l \geq c_{il} - u_i$. Then we have

$$c_{il} + y_l = c_{il} - v_l \leq c_{il} - c_{il} + u_i = c_{ij(i)} - v_{j(i)} = c_{ij(i)} + y_{j(i)}$$

and we are done. ∎

Example. The proposed procedure for solving (5) will be illustrated for matrices

$$C = \begin{pmatrix} 1 & 2 & 4 & 8 \\ 6 & 3 & 9 & 4 \\ 7 & 3 & 8 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & 1 & 4 & 5 \\ 2 & 3 & 9 & 2 \\ 7 & 3 & 7 & 0 \end{pmatrix}.$$

Matrix C with critical entries encircled is

$$C = \begin{pmatrix} 1 & 2 & \textcircled{4} & 8 \\ 6 & \textcircled{3} & \textcircled{9} & 4 \\ \textcircled{7} & \textcircled{3} & 8 & \textcircled{0} \end{pmatrix}.$$

By inspection, we can choose the entries c_{13}, c_{23}, c_{31} and it is easy to see that they fulfill (6). So we have a solution of the transportation problem with matrix

$$C' = \begin{pmatrix} 1 & 4 \\ 6 & 9 \\ 7 & 8 \end{pmatrix}$$

and row capacities and capacity of column 1 equal 1, capacity of column 2 equal 2 given by $x_{12} = x_{22} = x_{31} = 1$ and all other $x_{ij} = 0$. Now, using the complementarity theorem we get one possible solution of the dual equal to $u_1 = 3, u_2 = 8, u_3 = 7$ and $v_1 = 0, v_2 = 1$. Hence, a solution of our system is $\mathbf{y} = (0, -\infty, -1, -\infty)$ which is easy to verify.

However, although for some special cases function j is easy to find (e.g. if all entries in one column are critical), in general we do not know whether it is possible to decide the existence of the function j polynomially.

6 Conclusion and open questions

As the influence of imprecisions in real systems can be quite high, interval computations for max-plus algebraic problem shoud be given greater attention. For the interval eigenvalue and eigenvector problem we propose as further research topics for example the following questions:

- Find an efficient procedure to decide whether a given interval matrix has a universal eigenvector or prove that this problem is NP-hard.
- For a given possible eigenvector $\mathbf{x}$ of an interval matrix $\mathbf{A}$ find the greatest set of matrices in $\mathbf{A}$ for which $\mathbf{x}$ is an eigenvector.
- Describe some kind of generators for the set of all possible eigenvectors of a given interval matrix $\mathbf{A}$.

7 Acknowledgement

The author would like to thank Ray Cuninghame-Green, Rainer Burkard and Karel Zimmermann for stimulating discussions concerning the problems described in this paper.

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