On invariants of hereditary graph properties

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Abstract

The product $\mathcal{P}\circ\mathcal{Q}$ of graph properties $\mathcal{P}, \mathcal{Q}$ is a class of all graphs having a vertex-partition into two parts inducing subgraphs with properties $\mathcal{P}$ and $\mathcal{Q}$, respectively. For a graph invariant φ and a graph property $\mathcal{P}$ we define $\varphi(\mathcal{P})$ as the minimum of $\varphi(F)$ taken over all minimal forbidden subgraphs F of $\mathcal{P}$. An invariant of graph properties φ is said to be *additive with respect to reducible hereditary properties* if there is a constant c such that $\varphi(\mathcal{P}\circ\mathcal{Q}) = \varphi(\mathcal{P})+\varphi(\mathcal{Q})+c$ for every pair of hereditary properties $\mathcal{P}, \mathcal{Q}$. In this paper we provide a necessary and sufficient condition for invariants that are additive with respect to reducible hereditary graph properties. We prove that the order of the largest tree, the chromatic number, the colouring number, the tree-width and some other invariants of hereditary graph properties are of such type.

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1 Introduction

Investigating the structure of a graph G we deal with its properties, numerical characteristics, usually called invariants, and other attributes. Graph are very often branded with different attributes like outerplanar, claw-free, perfect, non-planar, 3-colourable, 3-regular, 3-connected, dense, hamiltonian, vertex-transitive, etc.; we determine their order, size, minimum and maximum degree, chromatic number, colouring number, independence number, domination number, crossing number, etc.; they receive different names like Petersen graph, Mycielski graphs; they are denoted by various symbols e.g. P_n, $K_{m,n}$; they are drawn as pictures $\ldots$. All such attributes of a graph G are inter-related each other in a mysterious way and they all together describe the object of the investigation of graph theory: GRAPH G.

In this paper we study invariants of graph properties related to graph invariants. More precisely, let $\mathcal{I}$ be the class of all finite simple graphs. A *graph property* is any non-empty isomorphism-closed proper subset of $\mathcal{I}$. A graph property $\mathcal{P} \subset \mathcal{I}$ is called *hereditary*, if from the fact that a graph G has the property $\mathcal{P}$, it follows that all subgraphs of G also belong to $\mathcal{P}$. A property is called *additive* if it is closed under taking disjoint union of graphs. The *completeness* of a hereditary property $\mathcal{P}$, denoted by $c(\mathcal{P})$, is defined as $c(\mathcal{P}) = \max\{p : K_{p+1} \in \mathcal{P}\}$.

It is well-known (cf. [2, 5]) that a hereditary property $\mathcal{P}$ can be uniquely characterized in terms of maximal graphs belonging to $\mathcal{P}$ (i.e. maximal graphs, with respect to subgraph partial order, possessing given property) or by the set of graphs not contained in $\mathcal{P}$. To be more accurate, the set $\mathbf{M}(n, \mathcal{P})$ of $\mathcal{P}$-*maximal graphs* of order n is defined as follows:

$$\mathbf{M}(n, \mathcal{P}) = \{H \in \mathcal{P} : |V(H)| = n \text{ and for each } e \in E(\overline{H})\ H + e \notin \mathcal{P}\}.$$

The set $\mathbf{F}(\mathcal{P})$ of *minimal forbidden subgraphs* of $\mathcal{P}$ is defined by:

$$\mathbf{F}(\mathcal{P}) = \{H \notin \mathcal{P} : \text{ each proper subgraph of } H \text{ belongs to } \mathcal{P}\}.$$

For other terminology related to hereditary graph properties we follow [2].

By a graph invariant φ we mean any integer-valued (real-valued) function defined on $\mathcal{I}$ such that $\varphi(G) = \varphi(H)$ for each pair G, H of isomorphic graphs. In accordance with S. Zhou [14], we say that the invariant φ interpolates over the class $\mathcal{P}$ of graphs if for any $G, H \in \mathcal{P}$ and each integer k between $\varphi(G)$ and $\varphi(H)$ there exists a graph $F \in \mathcal{P}$ such that $\varphi(F) = k$. A graph invariant φ is called *monotone* if $G \subseteq H$ implies $\varphi(G) \leq \varphi(H)$. According to this definition, the maximum degree Δ, the chromatic number χ, the choice number ch, the tree-width tw and the clique number ω are examples of monotone invariants, whereas the independence number α, the minimum degree δ, the vertex-connectivity number κ, the edge-connectivity λ are not. Given a graph invariant φ, we define the associated *invariant of a property* $\mathcal{P}$ in the following manner:

$$\varphi(\mathcal{P}) = \min\{\varphi(F) : F \in \mathbf{F}(\mathcal{P})\}.$$

The motivation for the investigation of invariants related to hereditary graph properties comes from extremal and chromatic graph theory. The classical Erdős-Stone-Simonovits formula provides a relationship between the maximum number of edges in an $\mathcal{P}$-maximal graph of order n and the invariant $\chi(\mathcal{P})$ - the chromatic number of $\mathcal{P}$ (see e.g. [13]).

The generalized colouring can be introduced as follows: Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be any properties of graphs. A *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partition* of a graph G is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$ the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$. A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is defined as the set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. If a property $\mathcal{R}$ can be expressed as the product of at least two properties, then it is said to be *reducible*; otherwise it is called *irreducible*.

We say that a graph invariant φ is *additive with respect to reducible hereditary properties* (abbreviated by ARHP) if there exists a constant c such that for any reducible property $\mathcal{P} \circ \mathcal{Q}$ the equality $\varphi(\mathcal{P} \circ \mathcal{Q}) = \varphi(\mathcal{P}) + \varphi(\mathcal{Q}) + c$ is valid. In [11] we proved that the chromatic number χ is ARHP.

In this paper we present a necessary and sufficient condition for monotone graph invariants that are ARHP and we show that among others the order of a graph, colouring number and tree-width are ARHP. In Section 2 we investigate fundamental properties of graph invariants. The main results are presented in Section 3.

2 Preliminaries

A. Berger [1] proved that any reducible additive hereditary property of graphs has infinitely many minimal forbidden graphs. But only very little is known about the structure of $\mathbf{F}(\mathcal{P} \circ \mathcal{Q})$ even in the case, when the structure of $\mathbf{F}(\mathcal{P})$ and $\mathbf{F}(\mathcal{Q})$ is known. Moreover, A. Farrugia proved in [6], that the recognition whether a graph belongs to a property $\mathcal{P} \circ \mathcal{Q}$ (it means whether it contains a graph from $\mathbf{F}(\mathcal{P} \circ \mathcal{Q})$ as a subgraph) is polynomial only in the simplest case : if the property $\mathcal{P} \circ \mathcal{Q}$ is the property "to be bipartite". Useful information on the structure of $\mathbf{F}(\mathcal{P} \circ \mathcal{Q})$ can be obtained by investigation of graph invariants associated with a property $\mathcal{P} \circ \mathcal{Q}$.

For a graph invariant φ we can define an associated graph invariant $\hat{\varphi}$ in the following way:

$$\hat{\varphi}(G) = \max_{H \subseteq G} \varphi(H).$$

Proposition 2.1 *Let φ be a graph invariant. Then φ is monotone if and only if for every graph G it holds $\varphi(G) = \hat{\varphi}(G)$.*

Proof. It follows immediately from the definition, that if φ is monotone, then $\varphi(G) = \hat{\varphi}(G)$. Thus, it is sufficient to prove that $\hat{\varphi}$ is monotone. Let G be a graph and F be its subgraph. Then evidently

$$\hat{\varphi}(F) = \max_{H \subseteq F} \varphi(H) \leq \max_{H \subseteq G} \varphi(H) = \hat{\varphi}(G). \blacksquare$$

An interesting and important example of such an invariant is the *degeneracy number* (called also *Wilf-Szekeres number*) $\hat{\delta} = \max_{H\subseteq G}\delta(H)$. This invariant is related to invariant *colouring number*, denoted col (see e.g. [7]), which is defined as $col(G) = \hat{\delta}(G) + 1$. Another examples are given by Matula in [9].

The following basic statements follow immediately from the definitions.

Proposition 2.2 *If $\mathcal{P}_1 \subseteq \mathcal{Q}_1$, $\mathcal{P}_2 \subseteq \mathcal{Q}_2$ are hereditary properties of graphs then $\mathcal{P}_1\circ\mathcal{P}_2 \subseteq \mathcal{Q}_1\circ\mathcal{Q}_2$.*

Proposition 2.3 *If φ is a monotone graph invariant and $\mathcal{P} \subseteq \mathcal{Q}$ are hereditary graph properties, then $\varphi(\mathcal{P}) \leq \varphi(\mathcal{Q})$.*

A graph invariant $\varphi(\mathcal{P})$ strongly depends on the features of the minimal forbidden subgraphs. The following lemma provides a lower bound of φ for $\mathcal{P}$-maximal graphs. It generalizes the results from [8, 11].

Lemma 2.4 *Let $\varphi(G)$ be a monotone graph invariant satisfying $\varphi(G+e) \leq \varphi(G)+1$ for any edge e from the complement of G. Then for any graph $G \in \boldsymbol{M}(n,\mathcal{P})$ with $n \geq c(\mathcal{P}) + 2$ the following holds: $\varphi(G) \geq \varphi(\mathcal{P}) - 1$.*

Proof. Since $G \in \boldsymbol{M}(n,\mathcal{P})$, according to the definition of $\mathcal{P}$-maximal graphs, we immediately have that $G + e \notin \mathcal{P}$ for any e belonging to $E(\overline{G})$. Hence, there exists $F \in \boldsymbol{F}(\mathcal{P})$ such that $F \subseteq G + e$. And therefore

$$\varphi(\mathcal{P}) \leq \varphi(F) \leq \varphi(G+e) \leq \varphi(G) + 1.$$

■

The next corollary summarizes some important graph invariants satisfying assumptions of Lemma 2.4.

Corollary 2.5 *Let $G \in \boldsymbol{M}(n,\mathcal{P})$, $n \geq c(\mathcal{P}) + 2$. Then*

1. $\chi(G) \geq \chi(\mathcal{P}) - 1$ *(the chromatic number);*
2. $\Delta(G) \geq \Delta(\mathcal{P}) - 1$ *(the maximum degree);*
3. $\omega(G) \geq \omega(\mathcal{P}) - 1$ *(the clique number);*
4. $\hat{\delta}(G) \geq \hat{\delta}(\mathcal{P}) - 1$;
5. $\hat{\kappa}(G) \geq \hat{\kappa}(\mathcal{P}) - 1$ *(see [9, 14]);*
6. $\hat{\lambda}(G) \geq \hat{\lambda}(\mathcal{P}) - 1$ *(see [9, 14]);*

Now we are going to investigate ARHP invariants of graph properties. One can rather easily see that the invariant $p(G)$ - the order of a graph G - is related to the completeness of a property $\mathcal{P}$ in the following way: $c(\mathcal{P}) = p(P) - 2$. (In the case of additive hereditary properties we can consider the invariant $\pi(G)$ - the largest tree contained in a graph G, i.e. the order of the largest component of G, and we obtain the same relationship.) In [3] it is proved that $c(\mathcal{P}\circ\mathcal{Q}) = c(\mathcal{P}) + c(\mathcal{Q}) + 1$ and it immediately implies that $\pi^*(\mathcal{P}\circ\mathcal{Q}) = \pi^*(\mathcal{P}) + \pi^*(\mathcal{Q})$, where $\pi^*(G) = \pi(G) - 1$.

The following problem is stated in [11]:

Problem 1 *For which graph invariant φ is it true that $\varphi(\mathcal{P}\circ\mathcal{Q}) = \varphi(\mathcal{P}) + \varphi(\mathcal{Q})$ for all hereditary properties $\mathcal{P}$ and $\mathcal{Q}$?*

The next simple proposition shows, that invariants which are ARHP provides, after a small modifi cation, a solution of the problem:

Proposition 2.6 *If for some invariant φ there is an constant c, such that the equality $\varphi(\mathcal{P}\circ\mathcal{Q}) = \varphi(\mathcal{P}) + \varphi(\mathcal{Q}) + c$ holds for all hereditary properties $\mathcal{P}$ and $\mathcal{Q}$, then the invariant $\varphi^*(G) = \varphi(G) + c$ satisfies $\varphi^*(\mathcal{P}\circ\mathcal{Q}) = \varphi^*(\mathcal{P}) + \varphi^*(\mathcal{Q}) + c$.*

Proof. One can easily see that $\varphi^*(\mathcal{P}\circ\mathcal{Q}) = \varphi(\mathcal{P}\circ\mathcal{Q}) + c = \varphi(\mathcal{P}) + \varphi(\mathcal{Q}) + 2c = \varphi^*(\mathcal{P}) - c + \varphi^*(\mathcal{Q}) - c + 2c = \varphi^*(\mathcal{P}) + \varphi^*(\mathcal{Q})$. ■

3 Main results

We are going to establish a necessary and suffi cient condition for invariants that are ARHP. First we need some notations and important lemmas. Let φ be a monotone graph invariant which interpolates over $\mathcal{I}$, k_0 be the minimum over $\varphi(G), G \in \mathcal{I}$ and $\mathcal{P}_{(\varphi,k)} = \{G \in \mathcal{I} : \varphi(G) \leq k + k_0\}$.

Then the chain $\mathcal{P}_{(\varphi,0)} \subset \mathcal{P}_{(\varphi,1)} \subset \cdots \subset \mathcal{P}_{(\varphi,n)} \subset \ldots$ of hereditary properties is called *the chain associated to φ*.

Lemma 3.1 *Let φ be a monotone graph invariant interpolating over $\mathcal{I}$ and $\mathcal{P}_{(\varphi,0)} \subset \mathcal{P}_{(\varphi,1)} \subset \ldots$ be the chain associated to φ. Then the following two statements are equivalent:*

(i) $\varphi(\mathcal{P}\circ\mathcal{Q}) \geq \varphi(\mathcal{P}) + \varphi(\mathcal{Q})$ *for each hereditary properties $\mathcal{P}$ and $\mathcal{Q}$;*

(ii) $\mathcal{P}_{(\varphi,k+l+1)} \subset \mathcal{P}_{(\varphi,k)}\circ\mathcal{P}_{(\varphi,l)}$ *for each non-negative integers k, l.*

Proof.
(i)⇒(ii) Let us suppose that the inequality $\varphi(\mathcal{P}\circ\mathcal{Q}) \geq \varphi(\mathcal{P}) + \varphi(\mathcal{Q})$ is valid for arbitrary hereditary properties $\mathcal{P}$ and $\mathcal{Q}$. Let k, l be two nonnegative integers. Then,

according to the definition of an invariant of a property, we have $\varphi(\mathcal{P}_{(\varphi,k)}) = k+1$ and $\varphi(\mathcal{P}_{(\varphi,l)}) = l+1$. Therefore

$$\varphi(\mathcal{P}_{(\varphi,k)}\circ\mathcal{P}_{(\varphi,l)}) \geq \varphi(\mathcal{P}_{(\varphi,k)}) + \varphi(\mathcal{P}_{(\varphi,l)}) = k+1+l+1 = k+l+2.$$

It implies that any graph G with $\varphi(G) \leq k+l+1$ belongs to $\mathcal{P}_{(\varphi,k)}\circ\mathcal{P}_{(\varphi,l)}$ and we obtain the desired inclusion $\mathcal{P}_{(\varphi,k+l+1)} \subset \mathcal{P}_{(\varphi,k)}\circ\mathcal{P}_{(\varphi,l)}$.
(ii)$\Rightarrow$ (i) Assume now that the inclusion $\mathcal{P}_{(\varphi,k+l+1)} \subset \mathcal{P}_{(\varphi,k)}\circ\mathcal{P}_{(\varphi,l)}$ holds for all non-negative integers k,l. Let $\mathcal{P},\mathcal{Q}$ be arbitrary two hereditary properties of graphs. Let us denote by $a+1$ and $b+1$ the values $\varphi(\mathcal{P})$ and $\varphi(\mathcal{Q})$ respectively. Then evidently $\mathcal{P}_{(\varphi,a)} \subset \mathcal{P}$ and $\mathcal{P}_{(\varphi,b)} \subset \mathcal{Q}$. Therefore, according to our assumption, $\mathcal{P}_{(\varphi,a+b+1)} \subset \mathcal{P}_{(\varphi,a)}\circ\mathcal{P}_{(\varphi,b)} \subset \mathcal{P}\circ\mathcal{Q}$. But the previous inclusions mean that

$$\varphi(\mathcal{P}\circ\mathcal{Q}) \geq \varphi(\mathcal{P}_{(\varphi,a+b+1)}) = a+b+2 = \varphi(\mathcal{P}) + \varphi(\mathcal{Q}).$$

■

The generalized Ramsey arrow relation for graph properties have been used in the paper [10] to prove minimal reducible bounds for the class of k-degenerate graphs. Let us recall the definition of generalized *Ramsey arrow relation*: Let G, F_1, F_2 be graphs and $\mathcal{P},\mathcal{Q}_1,\mathcal{Q}_2$ be graph properties. We write $G \to (F_1,F_2)$ if for any vertex partition $\{V_1,V_2\}$ of $V(G)$ either $F_1 \subset G[V_1]$ or $F_2 \subset G[V_2]$; and $\mathcal{P} \to (\mathcal{Q}_1,\mathcal{Q}_2)$ if for every pair of graphs $F_1 \in \mathcal{Q}_1$ and $F_2 \in \mathcal{Q}_2$ there exists a graph $G \in \mathcal{P}$ such that $G \to (F_1,F_2)$.

Lemma 3.2 *Let φ be a monotone graph invariant which interpolates over $\mathcal{I}$ and $\mathcal{P}_{(\varphi,0)} \subset \mathcal{P}_{(\varphi,1)} \subset \ldots$ be the chain associated to φ. Then the following two statements are equivalent:*

(i) $\varphi(\mathcal{P}\circ\mathcal{Q}) \leq \varphi(\mathcal{P}) + \varphi(\mathcal{Q})$ *for each hereditary properties $\mathcal{P}$ and $\mathcal{Q}$;*

(ii) $\mathcal{P}_{(\varphi,k+l)} \to (\mathcal{P}_{(\varphi,k)},\mathcal{P}_{(\varphi,l)})$ *for each non-negative integers k,l.*

Proof.
(i)$\Rightarrow$(ii) Let k,l be any non-negative integers. Let F and G be arbitrary graphs from $\mathcal{P}_{(\varphi,k)}$ and $\mathcal{P}_{(\varphi,l)}$ respectively. Consider the properties $\mathcal{P} = -F$ and $\mathcal{Q} = -G$ (i.e. the properties with exactly one minimal forbidden graph).

According to our assumption, $\varphi(\mathcal{P}\circ\mathcal{Q}) \leq \varphi(\mathcal{P}) + \varphi(\mathcal{Q})$ for any pair of hereditary properties $\mathcal{P},\mathcal{Q}$ and therefore we obtain the inequalities:

$$\varphi\left((-F)\circ(-G)\right) \leq \varphi(-F) + \varphi(-G) = \varphi(F) + \varphi(G) \leq k+l.$$

It implies that $\mathcal{P}_{(\varphi,k+l)} \not\subset (-F)\circ(-G)$ and there exists a graph $H \in \mathcal{P}_{(\varphi,k+l)}$ such that $H \to (F,G)$. Since integers k,l and the graphs F,H are chosen arbitrarily, we have the desired relation $\mathcal{P}_{(\varphi,k+l)} \to (\mathcal{P}_{(\varphi,k)},\mathcal{P}_{(\varphi,l)})$.

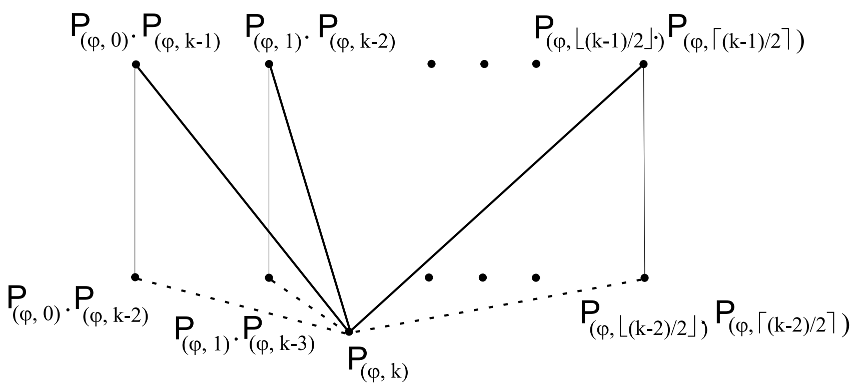

Figure 1: Depiction of the dependencies described by Theorem 3.3

(ii)⇒ (i) Let $\mathcal{P}, \mathcal{Q}$ be arbitrary hereditary properties of graphs. Let us denote by k and l the values $\varphi(\mathcal{P})$ and $\varphi(\mathcal{Q})$ respectively. Then there exist graphs F_1, F_2 such that $F_1 \in \mathbf{F}(\mathcal{P}), \varphi(F_1) = k$ and $F_2 \in \mathbf{F}(\mathcal{Q}), \varphi(F_2) = l$.

Since we assume that $\mathcal{P}_{(\varphi,k+l)} \to (\mathcal{P}_{(\varphi,k)}, \mathcal{P}_{(\varphi,l)})$, there is a graph $F \in \mathcal{P}_{(\varphi,k+l)}$ such that $F \to (F_1, F_2)$. We point out that either $l \leq k$ or $k \leq l$ and therefore either $\mathcal{P}_{(\varphi,k)} \subset \mathcal{P}_{(\varphi,l)}$ or $\mathcal{P}_{(\varphi,l)} \subset \mathcal{P}_{(\varphi,k)}$. Thus we obtain $F \notin \mathcal{P}\circ\mathcal{Q}$. But it means that $\mathcal{P}_{(\varphi,k+l)} \not\subset \mathcal{P}\circ\mathcal{Q}$ and $\varphi(\mathcal{P}\circ\mathcal{Q}) \leq k + l = \varphi(\mathcal{P}) + \varphi(\mathcal{Q})$. ■

Combining the previous two lemmas we obtain a necessary and suffi cient condition for graph invariants that are additive with respect to reducible hereditary properties. The condition provides a relationship between monotone invariants and the properties of their associated chains in $\mathbb{L}_{\subset}$. Figure 1 illustrates the inclusions (thick lines) and the Ramsey arrow relations (dashed lines) described by statements (i) and (ii) of Theorem 3.3 respectively.

Theorem 3.3 *Let φ be a monotone graph invariant and $\mathcal{P}_{(\varphi,0)} \subset \mathcal{P}_{(\varphi,1)} \subset \dots$ be the chain associated to φ. Then φ is additive with respect to hereditary properties if and only if for every non-negative integers k, l the following two conditions hold:*

(i) $\mathcal{P}_{(\varphi,k+l+1)} \subset \mathcal{P}_{(\varphi,k)}\circ\mathcal{P}_{(\varphi,l)}$;

(ii) $\mathcal{P}_{(\varphi,k+l)} \to (\mathcal{P}_{(\varphi,k)}, \mathcal{P}_{(\varphi,l)})$.

Using the characterisation provided by the previous theorem, we can show that some important graph invariants are ARHP.

Corollary 3.4 *The order of the graph $p(G)$ is ARHP.*

Proof. Let us consider the properties $\mathcal{Q}_k = \{G \in \mathcal{I} : p(G) \leq k + 1\}$, $k = 0, 1, \dots$. Then it is rather easy to see that $\mathcal{Q}_{k+l+1} \subset \mathcal{Q}_k\circ\mathcal{Q}_l$ for each non-negative integers k, l. Moreover, if $G \in \mathcal{Q}_k$ and $H \in \mathcal{Q}_l$ then obviously the complete graph $K_{|V(G)|+|V(H)|-1}$

satisfies $K_{|V(G)|+|V(H)|-1} \to (G, H)$. Therefore $\mathcal{Q}_{k+l} \to (\mathcal{Q}_k, \mathcal{Q}_l)$ and Theorem 3.3 yields that $p(G)$ is ARHP. ■

Corollary 3.5 *The order of the largest tree contained in a graph G, denoted by $\pi(G)$, is ARHP.*

Proof. The invariant π is associated with the chain of properties $\mathcal{O}_k = \{G \in \mathcal{I} : \pi(G) \leq k+1\}$, $k = 0, 1, \ldots$. The inclusions $\mathcal{O}_{k+l+1} \subset \mathcal{O}_k \circ \mathcal{O}_l$ and the relations $\mathcal{O}_{k+l} \to (\mathcal{O}_k, \mathcal{O}_l)$ were proved in [12] (see also [2]). Therefore Theorem 3.3 immediately implies that $\pi(\mathcal{P} \circ \mathcal{Q}) = \pi(\mathcal{P}) + \pi(\mathcal{Q})$ for all hereditary properties $\mathcal{P}, \mathcal{Q}$. ■

The next result was already proved in [11] using well-known Erdős-Stone-Simonovits theorem and another arguments. We recall, that *subchromatic number* ψ is defined as $\chi - 1$.

Corollary 3.6 [11] *The subchromatic number ψ is ARHP.*

Proof. Let us consider the properties $\hat{\mathcal{O}}_k = \{G \in \mathcal{I} : \psi(G) \leq k+1\}$, $k = 0, 1, \ldots$. Since for any graph G the value of $\psi(G)$ is equal to $\chi(G) - 1$, one can easily see that $\hat{\mathcal{O}}_k = \mathcal{O}^{k+1}$. Therefore we have the equations

$$\hat{\mathcal{O}}_{k+l+1} = \mathcal{O}^{k+l+2} = \mathcal{O}^{k+1} \circ \mathcal{O}^{l+1} = \hat{\mathcal{O}}_k \circ \hat{\mathcal{O}}_l.$$

It implies that the chain $\hat{\mathcal{O}}_0 \subset \hat{\mathcal{O}}_1 \subset \ldots$ satisfies the condition (i) of Theorem 3.3.

To prove the relation $\hat{\mathcal{O}}_{k+l} \to (\hat{\mathcal{O}}_k, \hat{\mathcal{O}}_l)$, where k, l are arbitrary non-negative integers, let us consider two graphs $G \in \hat{\mathcal{O}}_k = \mathcal{O}^{k+1}$ and $H \in \hat{\mathcal{O}}_l = \mathcal{O}^{l+1}$. If we denote by $K_s^{(r)}$ the complete r-partite graph with the order of each partition equal to s, then there exist positive integers n_1, n_2 such that $G \subset K_{n_1}^{(k+1)}$ and $H \subset K_{n_2}^{(q+1)}$. Consider now the graph $K_{n_1+n_2-1}^{(k+l+1)}$. By an application of Pigeonhole principle we have, that for any two-colouring of the vertices of $K_{n_1+n_2-1}^{(k+l+1)}$, either there are at least $k+1$ partitions with at least n_1 vertices of the first colour or there are at least $l+1$ partitions with at least n_2 vertices of the second colour. It implies that $K_{n_1+n_2-1}^{(k+l+1)} \to (K_{n_1}^{(k+1)}, K_{n_2}^{(l+1)})$. But then obviously $K_{n_1+n_2-1}^{(k+l+1)} \to (G, H)$. Since the graphs G, H were chosen arbitrarily, we have $\hat{\mathcal{O}}_{k+l} \to (\hat{\mathcal{O}}_k, \hat{\mathcal{O}}_l)$. It means that the chain $\hat{\mathcal{O}}_0 \subset \hat{\mathcal{O}}_1 \subset \ldots$ satisfies also the condition (ii) of Theorem 3.3.

Hence, by an application of Theorem 3.3 we obtain that the subchromatic number is ARHP. ■

Corollary 3.7 [11] *The chromatic number χ is ARHP.*

Proof. Since for any graph G we have the equality $\chi(G) = \psi(G) + 1$, Proposition 2.6 and Corollary 3.6 immediately imply, that χ is ARHP. ■

Corollary 3.8 *The colouring number col is ARHP.*

Proof. It was already mentioned that the colouring number is related to the invariant $\hat{\delta}$. And the invariant $\hat{\delta}$ is associated with the chain of properties $\mathcal{D}_k = \{G \in \mathcal{I} : G$ is k$-$degenerate$\}$, $k = 0, 1, \ldots$. The inclusions $\mathcal{D}_{k+l+1} \subset \mathcal{D}_k \circ \mathcal{D}_l$ and the relations $\mathcal{D}_{k+l} \to (\mathcal{D}_k, \mathcal{D}_l)$ were proved in [10] (see also [12]). Therefore Theorem 3.3 yields that $\hat{\delta}(\mathcal{P} \circ \mathcal{Q}) = \hat{\delta}(\mathcal{P}) + \delta(\mathcal{Q})$ for all hereditary properties $\mathcal{P}, \mathcal{Q}$. By an application of Proposition 2.6 we obtain that the colouring number is ARHP. ■

Corollary 3.9 *The tree-width tw is ARHP.*

Proof. The invariant tree-width tw is associated with the chain of properties $\mathcal{PT}_k = \{G \in \mathcal{I} : G$ is a subgraph of a k$-$tree$\}$, $k = 0, 1, \ldots$. The inclusions $\mathcal{PT}_{k+l+1} \subset \mathcal{PT}_k \circ \mathcal{PT}_l$ and the relations $\mathcal{PT}_{k+l} \to (\mathcal{PT}_k, \mathcal{PT}_l)$ were, in fact, proved in [4] (see also [12]). Therefore Theorem 3.3 yields that $tw(\mathcal{P} \circ \mathcal{Q}) = tw(\mathcal{P}) + tw(\mathcal{Q})$ for all hereditary properties $\mathcal{P}, \mathcal{Q}$. ■

Another importance of Theorem 3.3 is that it provides a suffi cient condition for the existence of minimal reducible bounds of degenerate hereditary additive properties of graphs (a property is *degenerate* if it has at least one bipartite graph forbidden and it is *additive* if it is closed under taking disjoin union). Let $(\mathbb{L}^a, \subset)$ be the lattice of additive hereditary properties of graphs. A property $\in \mathbb{L}^a$ is called a *minimal reducible bound* for a property $\in \mathbb{L}^a$ if in the interval $(\mathcal{P}, \mathcal{R})$ of the lattice $\mathbb{L}^a$ there are only irreducible properties. The determination of minimal reducible bounds is, in general, very diffi cult problem, but the following theorem, proved in [12], provides one useful method.

Theorem 3.10 *Let $\mathcal{O} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots$ be a chain of additive hereditary degenerate properties of graphs. If for arbitrary non-negative integers r, s, t, u, $r + s + 1 = k$, $t + u = k$ the properties $\mathcal{P}_r, \mathcal{P}_s, \mathcal{P}_t, \mathcal{P}_u$ satisfy the following two conditions*

(i) $\mathcal{P}_k \subset \mathcal{P}_r \circ \mathcal{P}_s$;

(ii) $\mathcal{P}_k \to (\mathcal{P}_t, \mathcal{P}_u)$,

then the set of minimal reducible bounds for $\mathcal{P}_k$ in the lattice $\mathbb{L}^a$ is of the form $\boldsymbol{B_L}(\mathcal{P}_k) = \{\mathcal{P}_p \circ \mathcal{P}_q : p + q + 1 = k\}$.

Combining Theorem 3.3 and Theorem 3.10 we obtain:

Theorem 3.11 *Let φ be an additive monotone graph invariant that is additive with respect to hereditary properties and $\mathcal{P}_k = \{G : \varphi(G) \leq k\}$. If for any positive integer k there exist a bipartite graph $B \in \mathcal{P}_k$ such that $\varphi(B) > k$, then the set of minimal reducible bounds for $\mathcal{P}_k$ in the lattice $\mathbb{L}^a$ is of the form $\boldsymbol{B_L}(\mathcal{P}_k) = \{\mathcal{P}_p \circ \mathcal{P}_q : p + q + 1 = k\}$.*

Let us remark that the maximum degree Δ is not ARHP because it does not satisfies the condition (ii) of Lemma 3.2. The set of minimal reducible bounds for the class S_k, of graphs of maximum degree less or equal to k, is not determined even for $k = 3$. It is well known, that for any graph G we have

$$\chi(G) \leq ch(G) \leq col(G) \leq \Delta(G) + 1.$$

We proved that the chromatic number χ and the colouring number col are ARHP. We conjecture that the choice number is ARHP too.

Conjecture 1 *The choice number $ch(\mathcal{P})$ is an additive invariant with respect to reducible hereditary graph properties.*

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