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Discrete version of the Pythagorean theorem

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Abstract

The areas of the squares in the Pythagorean theorem are replaced by polygonal numbers and some new Pythagorean-type propositions are proved. The hatching length of regular m-gons as a new parameter quantifying the area of polygons is defined and the related propositions are found.
keywords: Pythagorean theorem, polygonal numbers hatching length

1 Introduction

The following observations are motivated by the facts that the area of a planar figure displayed on a screen can be expressed by a certain number of pixels; and if the figure is drawn by a plotter, then its area can be characterized by total length of a line which fills it in.

The generalizations of the Pythagorean theorem are of the three kinds. Firstly, the squares on the sides of the right triangle are substituted by other geometrically similar planar figures (Euclids Elements Book VI, Proposition 31 [5], see also J. Edgren [3]). Secondly, the assumption of the right angle is omitted (the law of cosines), or both of these generalizations occur simultaneously (Pappus' area theorem [7], see also H. W. Eves [4]). Thirdly, the other mathematical spaces than the plane are considered (de Gua - Faulhaber theorem about trirectangular tetrahedrons [4], further generalized by Tinseau [8], Euclidean n-spaces, Banach spaces [6], see also [1]).

We will describe several pythagorean-type results close to the first kind mentioned above.

2 Results

If positive integers a, b, c denote the lengths of the sides of a right triangle and $a < b < c$, then by the Pythagorean theorem $c^2 = a^2 + b^2$. Let us denote the surface area of a regular m-gon with s as the length of its side by $A_m(s)$ and let us rewrite the Pythagorean equality as $A_4(c) = A_4(a) + A_4(b)$. It is known that $A_m(s) = s^2 \cdot \frac{m}{4} \cdot \cot \frac{\pi}{m}$. It follows that if the areas of squares are substituted by the areas of regular m-gons corresponding to the sides of the given right triangle, then $A_m(c) = (a^2 + b^2) \cdot \frac{m}{4} \cdot \cot \frac{\pi}{m} = A_m(a) + A_m(b)$, the equality remains valid. We will show a similar relation holding for polygonal numbers.

Let us recall that a polygonal (triangular, square, pentagonal, m-gonal) number is a positive integer which can be represented by regular and discrete geometric pattern of equally spaced points (points in triangle, square, pentagon, m-gon). In the next, the n-th m-gonal number $S_m(n)$ is defined for positive integers m, n where $m \geq 3$, as the sum of the first n elements of the arithmetic progression starting from 1, with $d = m - 2$ as its difference, i.e. $S_m(n) = 1 + (1 + d) + \ldots + (1 + (n-1) \cdot d) = \frac{(m-2)\cdot(n^2-n)}{2} + n$ (see Deza and Deza [2]).

If we substitute the areas of squares on the sides of a right triangle with integer lengths of sides by square numbers, the equality $S_4(c) = S_4(a) + S_4(b)$ will hold. However, if we substitute the square numbers by the pentagonal (or by the triangular) numbers, then the equality will not longer be true. We will show that the difference between the polygonal number on the hypotenuse and the sum of the polygonal numbers of the same type on the other two sides is a multiple of the incirle of the given triangle (namely multiple of $r = \frac{a+b-c}{2}$).

Proposition 2.1. *Let the positive integers a, b, c such that $a < b < c$ denote the lengths of the sides of a right triangle and let r be the inradius of this triangle. Then $S_m(c) = S_m(a) + S_m(b) + (m-4) \cdot r$.*

Proof.

$$S_m(a) + S_m(b) = \frac{(m-2)\cdot(a^2-a)}{2} + a + \frac{(m-2)\cdot(b^2-b)}{2} + b =$$

$$= \frac{(m-2)\cdot(a^2+b^2-a-b+c-c)}{2} + a + b - c + c =$$

$$= \frac{(m-2)\cdot(c^2-2r-c)}{2} + 2r + c = \frac{(m-2)\cdot(c^2-c)}{2} + c - \frac{(m-2)\cdot 2r}{2} + 2r =$$

$$= S_m(c) - (m-4)\cdot r.$$

□

The Figure 1 shows a right triangle where $a = 3, b = 4, c = 5, r = 1$ and $S_5(3) = 12, S_5(4) = 22, S_5(5) = 35$. It is easy to see that the correspondent identity holds: $S_5(5) = 35 = 12 + 22 + 1 = S_5(3) + S_5(4) + (5-4) \cdot 1$.

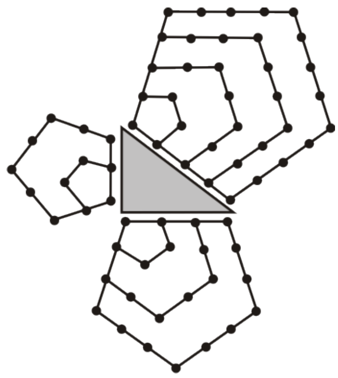

Figure 1: $a = 3, b = 4, c = 5, r = 1$ and $S_5(3) = 12, S_5(4) = 22, S_5(5) = 35$

Another class of the figural numbers corresponding to the other polygonal arrangement is the class of the centered polygonal numbers (called also polygonal numbers of the second order). They arise by surrounding a central point by the polygonal layers with the subsequently increasing length of the sides. Precisely, the n-th centered m-gonal number $CS_m(n)$ is defined for positive integers m, n where $m \geq 3$, as the sum of the first n elements of the sequence starting with 1 and continuing with the arithmetic progression $m, 2m, 3m, \ldots$, i.e. $CS_m(n) = 1 + m + 2m + \ldots + (n-1) \cdot m = \frac{m \cdot (n^2 - n)}{2} + 1$ (see Deza and Deza [2]). It is interesting that for this class, we obtain a similar result as above.

Proposition 2.2. *Let the positive integers a, b, c such that $a < b < c$ denote the lengths of the sides of a right triangle and let r be the inradius of this triangle. Then $CS_m(c) = CS_m(a) + CS_m(b) + (mr - 1)$.*

Proof.

$$CS_m(a) + CS_m(b) = \frac{m \cdot (a^2 - a)}{2} + 1 + \frac{m \cdot (b^2 - b)}{2} + 1 =$$

$$= \frac{m \cdot (a^2 + b^2 - a - b + c - c)}{2} + 2 = \frac{m \cdot (c^2 - 2r - c)}{2} + 2 =$$

$$= \frac{m \cdot (c^2 - c)}{2} + 1 - mr + 1 = CS_m(c) - (mr - 1).$$

□

The Figure 2 shows a triangle where $a = 3, b = 4, c = 5, r = 1$ and $CS_4(3) = 13, CS_4(4) = 25, CS_4(5) = 41$. By the Proposition 2.2, $CS_4(5) = CS_4(3) + CS_4(4) + (4 - 1)$.

Remark 2.3. *Let a, b, c denote the positive integers such that $a^2 + b^2 = c^2$. Then the identity $S_m(c) = S_m(a) + S_m(b)$ holds if and only if $m = 4$ (the squares). If $m = 3$, then $S_3(c) < S_3(a) + S_3(b)$, and if $m > 4$, then $S_m(c) > S_m(a) + S_m(b)$. The inequality $CS_m(c) > CS_m(a) + CS_m(b)$ holds for every m.*

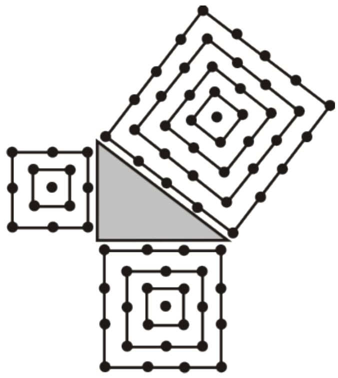

Figure 2: $a = 3, b = 4, c = 5, r = 1$ and $CS_4(3) = 13, CS_4(4) = 25, CS_4(5) = 41$

In the following part, we will consider regular m-gons with their every side divided by points into n line segments of the length 1. These dividing points together with the vertices of the polygon will be denoted by $A_0, A_1, \ldots, A_{mn-1}$ as the Figure 3 shows. Let us define **the hatching length** of the regular m-gon for an odd number m as the sum of lengths of some line segments A_iA_j, and denote it by $H_m(n)$. Precisely, let $H_m(n) = \sum_{i=1}^{kn-1} |A_iA_{mn-i}|$, where k is a positive integer such that $m = 2k+1$. The Figure 3 shows the specific line segments A_iA_j in the case $m = 5, n = 4$.

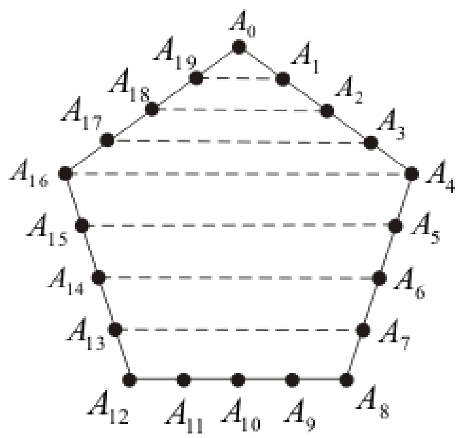

Figure 3: The hatching length in the regular 5-gon

The next proposition presents a relation between the hatching lengths of $(2k+1)-$gons on the sides of a pythagorean triangle (a right triangle with integer side lengths).

Proposition 2.4. *Let the positive integers a, b, c such that $a < b < c$ denote the lengths of the sides of a right triangle and let r be the inradius of this triangle. Let m be an odd integer, $m \geq 3$. Then $H_m(c) = H_m(a) + H_m(b) + r$.*

The Figure 4 shows the case where $a = 3, b = 4, c = 5, (r = 1)$ and $m = 5$. The total length of the dashed line segments is exactly one inradius longer then the total length of the dotted line segments.

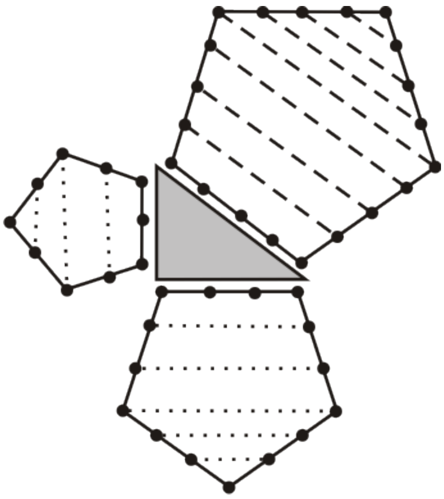

Figure 4: The hatching length in the case $a = 3, b = 4, c = 5, (r = 1)$ and $m = 5$

Proof. Firstly, we determine $H_m(n)$ for a regular m-gon where m is an odd integer, i.e. $m = 2k + 1, k \geq 1$, and n is a positive integer which denotes the length of a side of the m-gon.

Let us denote $A_0, A_n, A_{2n}, \ldots, A_{(m-1)n}$ the vertices of the regular m-gon (as in the Figure 5) and divide every its side $A_{in}A_{(i+1)n}$, $i \in \{0, 1, \ldots, m-1\}$, into n segment lines $A_{in}A_{in+1}$,$A_{in+1}A_{in+2}$,. . ., $A_{in+(n-1)}A_{(i+1)n}$. The Figure 5 shows the case for $m = 9$ and $n = 3$.

Now, we dissect the given m-gon into one triangle $A_{(k-1)n}A_{kn}A_{(k+1)n}$ and (if $k \geq 2$) $k-1$ trapezoids $A_0A_nA_{(2k-1)n}A_{2kn}$, $A_nA_{2n}A_{(2k-2)n}A_{(2k-1)n}$, $\ldots$, $A_{(k-2)n}A_{(k-1)n}A_{(k+1)n}A_{(k+2)n}$ and the value of $H_m(n)$ will be found as the sum of lengths of the line segments lying in these sections. The sum of lengths of the dotted segment lines in the triangle $A_{(k-1)n}A_{kn}A_{(k+1)n}$ is $(n-1)+\ldots+1 = \frac{n^2-n}{2}$. Thus, if $k = 1$ $(m = 3)$, then $H_3(n) = \frac{n^2-n}{2}$.

Let us denote by $z_1, z_2, \ldots z_{k-1}$ the lengths of the diagonals of the trapezoids, approaching alternatively from one and from the other side to the center of the polygon; e.g. $z_1 = \left|A_nA_{(m-1)n}\right|$, $z_2 = \left|A_{(k-1)n}A_{(k+2)n}\right|$, $z_3 = \left|A_{2n}A_{(m-2)n}\right|, \ldots$ Let z_{k-1} be the length of the diagonal belonging to the trapezoid containing the center of the m-gon.

Using the suitable central angles and the right triangles, we obtain the values $z_1, z_2, \ldots, z_{k-1}$. If we denote by R the circumradius of m-gon $A_0A_nA_{2n}\ldots A_{(m-1)n}$ and by γ one half of the central angle corresponding to one side of the m-gon, i.e. $\gamma = \frac{1}{2}\cdot\frac{360^\circ}{m}$, then $z_i = 2R\cdot\sin(i+1)\gamma = \frac{n}{\sin\gamma}\cdot\sin(i+1)\gamma$ for every $i \in \{1, \ldots, k-1\}$.

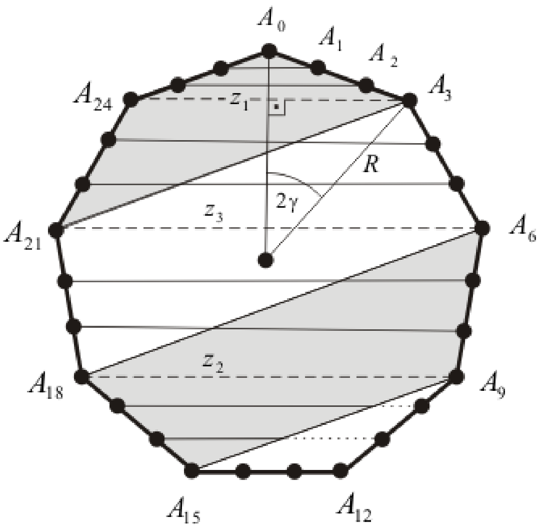

Figure 5: The dissection for $m = 9, n = 3$

To compute the lengths of the segments lying in a trapezoid with the diagonal of length z_i, we apply the similarity of triangles. Since for every trapezoid the total length of segments lying in it is nz_i, we obtain

$$H_m(n) = \sum_{i=1}^{kn-1} \left|A_i A_{mn-i}\right| = \frac{n^2 - n}{2} + \sum_{i=1}^{k-1} nz_i = \frac{n^2 - n}{2} + 2nR \sum_{i=2}^{k} \sin i\gamma.$$

Finally, for $m = 3$ $(k = 1)$ it holds

$$H_3(a) + H_3(b) = \frac{a^2 - a}{2} + \frac{b^2 - b}{2} =$$

$$= \frac{a^2 + b^2 - c - a - b + c}{2} + \frac{c^2 - c - 2r}{2} = H_3(c) - r.$$

If $m > 3$ $(k > 1)$ we have

$$H_m(a) + H_m(b) = \frac{a^2 - a}{2} + 2a \frac{a}{2 \sin \gamma} \sum_{i=2}^{k} \sin i\gamma + \frac{b^2 - b}{2} + 2b \frac{b}{2 \sin \gamma} \sum_{i=2}^{k} \sin i\gamma =$$

$$= a^2 \left(\frac{1}{2} + \frac{1}{\sin \gamma} \sum_{i=2}^{k} \sin i\gamma \right) + b^2 \left(\frac{1}{2} + \frac{1}{\sin \gamma} \sum_{i=2}^{k} \sin i\gamma \right) - \frac{a+b}{2} =$$

$$= c^2 \left(\frac{1}{2} + \frac{1}{\sin \gamma} \sum_{i=2}^{k} \sin i\gamma \right) - \frac{a+b}{2},$$

and on the other side

$$H_m(c) - r = \frac{c^2 - c}{2} + 2c\frac{c}{2\sin\gamma}\sum_{i=2}^{k}\sin i\gamma - \frac{a+b-c}{2} =$$

$$= c^2\left(\frac{1}{2} + \frac{1}{\sin\gamma}\sum_{i=2}^{k}\sin i\gamma\right) - \frac{a+b}{2}.$$

□

Now, let us define **the hatching length** of the regular m-gon with the side of length n where m is an even integer, $m = 2k, k \geq 2$. Again, let a regular m-gon have all its sides divided by points into n line segments of the length 1. The dividing points together with the vertices of the polygon will be denoted by B_i as the Figure 6 shows. Then we define the hatching length of the regular m-gon with the side of length n as $H_m(n) = \sum_{i=1}^{kn-1} \left|B_i B_{mn-i}\right|$. The Figure 6 shows the particular case for $m = 6$ and $n = 4$.

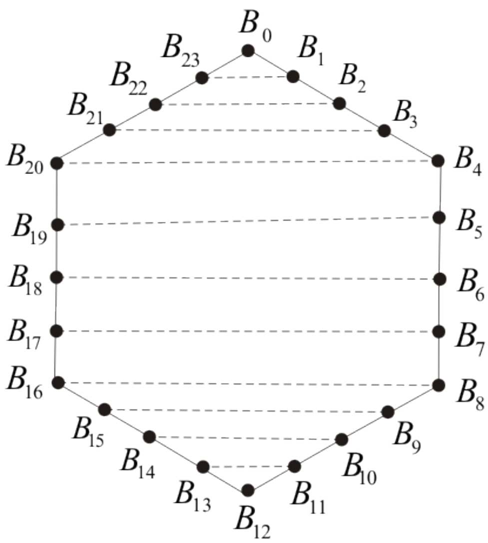

Figure 6: The hatching length in the regular 6-gon

Proposition 2.5. *Let the positive integers a, b, c such that $a < b < c$ denote the lengths of the sides of a right triangle and let m be an even integer, $m \geq 4$. Then $H_m(c) = H_m(a) + H_m(b)$.*

The Figure 7 shows the case where $a = 3, b = 4, c = 5, (r = 1)$ and $m = 6$. Now, the total length of the dashed line segments is exactly the sum of the total length of the dotted line segments.

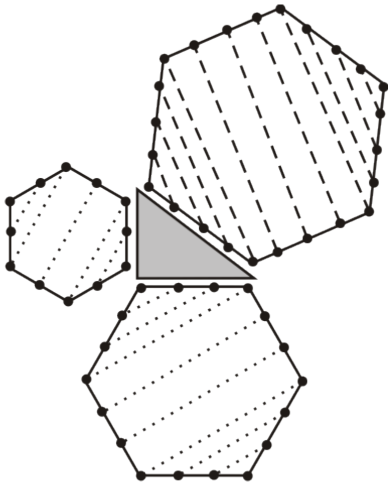

Figure 7: The hatching length in the case $a = 3, b = 4, c = 5, (r = 1)$ and $m = 6$.

Proof. Let $m = 2k$, where $k \geq 2$. It is obvious that $H_4(n) = n^2 \cdot \sqrt{2}, (k = 2)$. In next, let $k > 2$.

Let us denote $B_0, B_n, B_{2n}, \ldots, B_{(m-1)n}$ the vertices of the regular m-gon (as in the Figure 8) and divide every its side $B_{in}B_{(i+1)n}, i \in \{0, 1, \ldots, m-1\}$, into n segment lines $B_{in}B_{in+1}, B_{in+1}B_{in+2}, \ldots B_{in+(n-1)}B_{(i+1)n}$.

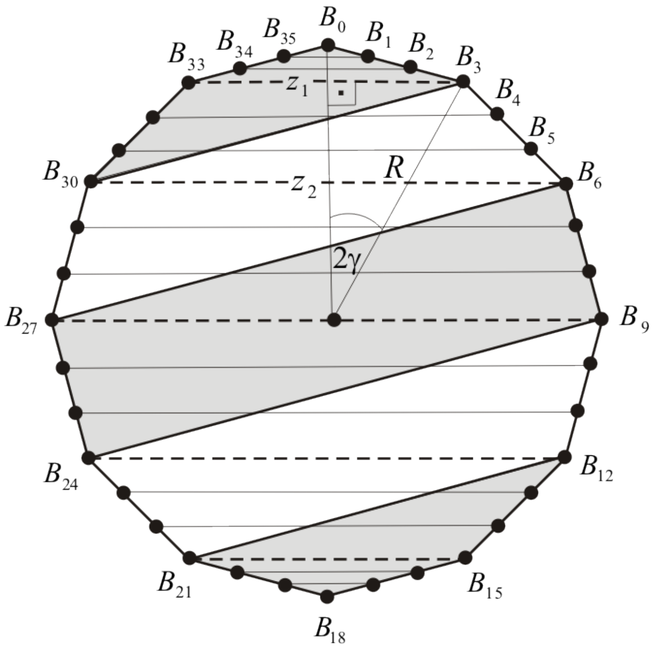

Figure 8: The dissection for $m = 12, n = 3$

Let k be even. The Figure 8 shows the case for $m = 12$ $(k = 6)$ and $n = 3$. We dissect the m-gon into $k - 1$ trapezoids $B_{(i-1)n}B_{in}B_{(m-i-1)n}B_{(m-i)n}$,

$i \in \{1, \ldots, k-1\}$. By using the central symmetry of the regular m-gon, the value of $H_m(n)$ is twice the sum of lengths of the line segments lying in the trapezoids $B_{(i-1)n}B_{in}B_{(m-i-1)n}B_{(m-i)n}$, for $i \in \{1, \ldots, \frac{k}{2}-1\}$ plus $2Rn$ (in the central rectangular section), where R is the circumradius of m-gon.

Let us denote $\gamma = \frac{180}{m}$. Then similarly to the previous proof, we obtain values $z_i = 2R \cdot \sin 2i\gamma = \frac{n}{\sin\gamma} \cdot \sin 2i\gamma$ for every $i \in \{1, \ldots, \frac{k}{2}-1\}$. Thus the sum of lengths of the line segments lying in the trapezoid $B_{(i-1)n}B_{in}B_{(m-i-1)n}B_{(m-i)n}$, as well as in $B_{(k+i-1)n}B_{(k+i)n}B_{(k-i-1)n}B_{(k-i)n}$ is nz_i for $i \in \{1, \ldots \frac{k}{2}-1\}$. Then

$$H_m(n) = \sum_{i=1}^{kn-1} \left|B_iB_{mn-i}\right| = \frac{n^2}{\sin\gamma} + \frac{2n^2}{\sin\gamma} \cdot \sum_{i=1}^{\frac{k}{2}-1} \sin 2i\gamma.$$

Now, let k be odd. Applying an analogical reasoning we obtain

$$H_m(n) = \sum_{i=1}^{kn-1} \left|B_iB_{mn-i}\right| = 2\sum_{i=1}^{\frac{k-1}{2}} nz_i = \frac{2n^2}{\sin\gamma} \cdot \sum_{i=1}^{\frac{k-1}{2}} \sin 2i\gamma.$$

Hence, if $k = 2$ $(m = 4)$, then

$$H_4(a) + H_4(b) = a^2 \cdot \sqrt{2} + b^2 \cdot \sqrt{2} = c^2 \cdot \sqrt{2} = H_4(c).$$

If k is an even integer, $k > 2$, then

$$H_4(a) + H_4(b) = \frac{a^2}{\sin\gamma} + \frac{2a^2}{\sin\gamma} \cdot \sum_{i=1}^{\frac{k-2}{2}} \sin 2i\gamma + \frac{b^2}{\sin\gamma} +$$

$$+ \frac{2b^2}{\sin\gamma} \cdot \sum_{i=1}^{\frac{k-2}{2}} \sin 2i\gamma = \frac{a^2+b^2}{\sin\gamma} + \frac{2a^2+2b^2}{\sin\gamma} \cdot \sum_{i=1}^{\frac{k-2}{2}} \sin 2i\gamma =$$

$$= \frac{c^2}{\sin\gamma} + \frac{2c^2}{\sin\gamma} \cdot \sum_{i=1}^{\frac{k-2}{2}} \sin 2i\gamma = H_m(c),$$

and if k is an odd integer, $k > 2$,then

$$H_m(a) + H_m(b) = \frac{2a^2}{\sin\gamma} \cdot \sum_{i=1}^{\frac{k-1}{2}} \sin 2i\gamma + \frac{2b^2}{\sin\gamma} \cdot \sum_{i=1}^{\frac{k-1}{2}} \sin 2i\gamma =$$

$$= \frac{2a^2+2b^2}{\sin\gamma} \cdot \sum_{i=1}^{\frac{k-1}{2}} \sin 2i\gamma = \frac{2c^2}{\sin\gamma} \cdot \sum_{i=1}^{\frac{k-2}{2}} \sin 2i\gamma = H_m(c).$$

□

Let us remark that for an even integer $m, m \geq 4$, there is just one other "natural" way how to hatch the regular m-gon (see Fig. 9). As above, let a regular m-gon have all its sides divided by points into n line segments of the length 1. The dividing points together with the vertices of the polygon will be denoted by B_i as the Figure 9 shows. Then we define **the longitudinal hatching length** of the regular m-gon with the side of length n as $LH_m(n) = \sum_{i=1}^{kn-n-1} \left|B_{n+i}B_{mn-i}\right|$. The Figure 9 shows the particular case for $m = 6$ and $n = 4$.

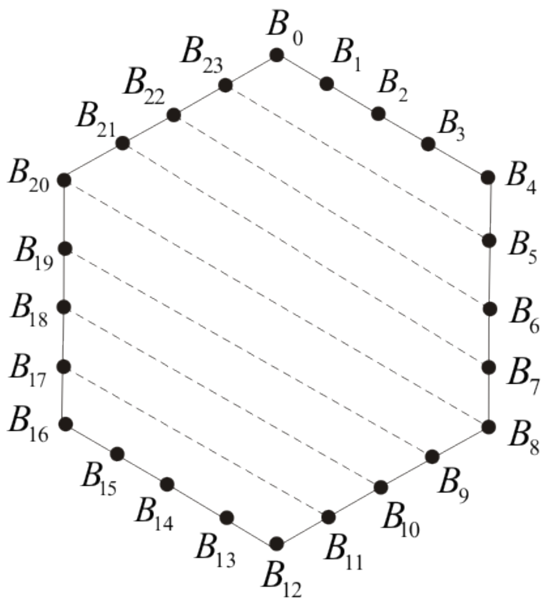

Figure 9: The longitudinal hatching length in the regular 6-gon

The next proposition can be proved by the same method as Proposition 2.5, so the proof is ommited. Again (see Propositions 2.1, 2.2 and 2.4 above), the difference between $LH_m(c)$ and the sum $LH_m(a) + LH_m(b)$ is expressed by the inradius of the right triangle.

Proposition 2.6. *Let the positive integers a, b, c such that $a < b < c$ denote the lengths of sides of a right triangle. Let m be an even integer, $m \geq 4$ and let r be the inradius of this triangle. Then $LH_m(c) = LH_m(a) + LH_m(b) + 2r$.*

The Figure 10 shows the case where $a = 3, b = 4, c = 5, (r = 1)$ and $m = 6$. Now, the total length of the dashed line segments in the largest hexagon is $2r$ longer than the sum of the total lengths of the dotted lines in the two smaller hexagons together.

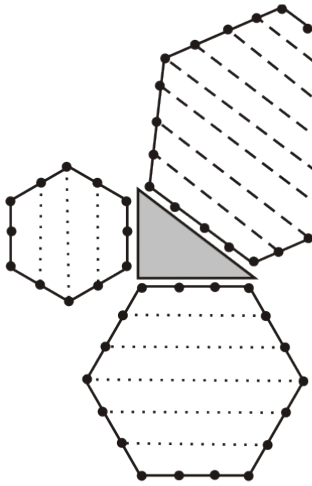

Figure 10: The longitudinal hatching length in the case $a = 3, b = 4, c = 5, (r = 1)$ and $m = 6$

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