Numbers of Edges in Supermagic Graphs

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Abstract

A graph is called supermagic if it admits a labelling of the edges by pairwise different consecutive integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. In the paper we establish some bounds for the number of edges in supermagic graphs.

Keywords: magic graph, supermagic graph, size of graph

1 Introduction

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index-mapping* of f is the mapping $f^\star$ from $V(G)$ into positive integers defined by

$$f^\star(v) = \sum_{e \in E(G)} \eta(v,e) f(e) \quad \text{for every } v \in V(G),$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labelling* of G for an *index* λ if its index-mapping $f^\star$ satisfies

$$f^\star(v) = \lambda \quad \text{for all } v \in V(G).$$

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A magic labelling f of G is called a *supermagic labelling* of G if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) if and only if there exists a supermagic (magic) labelling of G. Note that any supermagic regular graph G admits a supermagic labelling into the set $\{1, \ldots, |E(G)|\}$.

The concept of magic graphs was introduced by Sedláček [11]. The regular magic graphs are characterized in [2]. Two different characterizations of all magic graphs are given in [9] and [8].

Supermagic graphs were introduced by M. B. Stewart [12]. It is easy to see that the classical concept of a magic square of n^2 boxes corresponds to the fact that the complete bipartite graph $K_{n,n}$ is supermagic for every positive integer $n \neq 2$ (see also [12]). Stewart [13] characterized supermagic complete graphs. In [6] supermagic regular complete multipartite graphs and supermagic cubes are characterized. In [7] there are given characterizations of magic line graphs of general graphs and supermagic line graphs of regular bipartite graphs. In [10] and [1] supermagic labellings of the Möbius ladders and two special classes of 4-regular graphs are constructed. Some constructions of supermagic labellings of various classes of regular graphs are described in [5] and [6]. More comprehensive information on magic and supermagic graphs can be found in [3].

In this paper we deal with the number of edges in supermagic graphs.

2 Number of edges

For the number of edges in magic graphs it holds

Proposition 1. [14] *A connected magic graph with n vertices and ε edges exists if and only if $n = 2$ and $\varepsilon = 1$ or $n \geq 5$ and $\frac{5n}{4} < \varepsilon \leq \frac{n(n-1)}{2}$.*

It is easy to see that all components of a magic graph are magic graphs and at most one of them is isomorphic to the complete graph K_2. Thus we have immediately

Corollary 1. *A magic graph of order n and size ε exists if and only if $n = 2$ and $\varepsilon = 1$ or $n \in \{5, 6\}$ and $\frac{5n}{4} < \varepsilon \leq \frac{n(n-1)}{2}$ or $n \geq 7$ and $\frac{5n-6}{4} < \varepsilon \leq \frac{n(n-1)}{2}$. Moreover, any magic graph with at most $\frac{5}{4}n$ edges contains a component isomorphic to K_2.*

The previous assertions imply the following interpolation theorem: *Let G_1 and G_2 be magic graphs of order n. Then there exists a magic graph of order n and size ε for each integer ε satisfying $|E(G_1)| \leq \varepsilon \leq |E(G_2)|$.* A similar result for supermagic graphs is not valid.

Theorem 1. *Let d be the greatest common divisor of integers n and ε, and let $n_1 = \frac{n}{d}$. If n_1 and ε are both even, then there exists no supermagic graph of order n and size ε.*

Proof. Let d denote the greatest common divisor of n and ε and let $n_1 = \frac{n}{d}$, $\varepsilon_1 = \frac{\varepsilon}{d}$. Suppose that G is a supermagic graph of order n and size ε. Then it admits a supermagic labelling $f : E(G) \longrightarrow \{a, \ldots, a + \varepsilon - 1\}$ for an index λ. It holds

$$\begin{aligned} n\lambda = & \sum_{v\in V(G)} \sum_{e\in E(G)} \eta(v,e)f(e) = 2 \sum_{e\in E(G)} f(e) \\ = & 2[a + \ldots + (a + \varepsilon - 1)] = (2a + \varepsilon - 1)\varepsilon. \end{aligned}$$

As ε and n_1 are both even, the index $\lambda = \frac{(2a+\varepsilon-1)\varepsilon}{n} = \frac{(2a+\varepsilon-1)\varepsilon_1}{n_1}$ is not an integer, a contradiction. □

For example there exists no supermagic graph of order 8 and size $\varepsilon \equiv 2, 4, 6 \pmod 8$ (i.e., with 10, 12, 14, 18, 20, 22, 26, 28 edges). Thus the problem to characterize the numbers of edges in supermagic graphs seems to be difficult.

Let $M(n)$ ($m(n)$) denote the maximal (minimal) number of edges in a supermagic graph of order n. Evidently, $M(n)$ and $m(n)$ are not defined for $n = 1, 3, 4$ and $M(2) = m(2) = 1$. In the next sections we determine $M(n)$ and establish some bounds of $m(n)$ for $n \geq 5$.

3 Upper bound

Finding the maximum number $M(n)$ of edges in a supermagic graph is closely associated with the characterization of supermagic complete graphs.

Proposition 2. [13] *A complete graph of order n is supermagic if and only if $n = 2$ or $5 < n \not\equiv 0 \pmod 4$.*

Next we prove that by deleting an edge from a complete graph K_n, $n \geq 6$, we obtain a supermagic graph.

Theorem 2. *For every positive integer $n \geq 6$, the complete graph K_n without an edge is supermagic.*

Proof. We will consider the following cases

A. $6 \leq n \not\equiv 0 \pmod 4$. By Proposition 2 the complete graph K_n is supermagic. Thus there exists a supermagic labelling $f : E(K_n) \longrightarrow$

$\{1, \ldots, \frac{n(n-1)}{2}\}$ for an index λ. Let $\hat{e}$ be an edge of K_n such that $f(\hat{e}) = 1$. Define a labelling $g : E(K_n - \hat{e}) \longrightarrow \{1, \ldots, \frac{n(n-1)}{2} - 1\}$ by

$$g(e) = f(e) - 1 \quad \text{for all } e \in E(K_n - \hat{e}).$$

Since K_n is an $(n-1)$-regular graph we have

$$g^\star(v) = f^\star(v) - (n-1) \quad \text{for all } v \in V(K_n).$$

Therefore, g is a supermagic labelling of $K_n - \hat{e}$.

B. $8 \le n \equiv 0 \pmod 4$. Let $\hat{e}$ be an arbitrary edge of the complete graph K_n. Denote the vertices of K_n by $v_1, \ldots, v_n$ in such a way that $\hat{e} = v_{n-1}v_n$.

Let G be a subgraph of K_n induced by the set $\{v_1, \ldots, v_{n-2}\}$. The graph G is isomorphic to K_{n-2} and by Proposition 2 there exists a supermagic labelling f from $E(G)$ into $\{1, \ldots, \binom{n-2}{2}\}$. Clearly, $f^\star(v_i) = (\binom{n-2}{2} + 1)\frac{n-3}{2}$ for all i, $1 \le i \le n-2$.

Put a positive integer $a := \frac{1}{4}[n^3 - 6n^2 + 7n + 4]$ and define a mapping $g : E(K_n - \hat{e}) \longrightarrow \{a, \ldots, a + \frac{n(n-1)}{2} - 2\}$ by

$$g(v_iv_j) = \begin{cases} a - 1 + f(v_iv_j) & \text{for } 1 \le j \le n-2 \text{ and } 1 \le i \le n-2, \\ a + \binom{n-2}{2} - 1 + i & \text{for } j = n-1 \text{ and } n-8 \ge i \equiv 0, 3 \pmod 4, \\ & \text{or } j = n-1 \text{ and } i = n-7, n-5, \\ & \text{or } j = n \text{ and } n-8 \ge i \equiv 1, 2 \pmod 4, \\ & \text{or } j = n \text{ and } i = n-6, n-4, n-3, n-2, \\ a + \frac{(n-2)(n+1)}{2} - i & \text{otherwise.} \end{cases}$$

A part of definition of g was inspired by [4]. It is easy to see that the mapping g is a bijection and for its index-mapping we get

$$g^\star(v_i) = \frac{1}{4}(n^4 - 6n^3 + 9n^2 + 4n - 12) \qquad \text{for } 1 \le i \le n.$$

Thus, g is a supermagic labelling and $K_n - \hat{e}$ is a supermagic graph. □

It is not difficult to check that $M(5) = 8$. Therefore, using previous results we get the following theorem

Theorem 3. *Let $n \ge 5$ be a positive integer. Then*

$$M(n) = \begin{cases} 8 & \textit{for } \ n = 5, \\ \frac{n(n-1)}{2} & \textit{for } \ 6 \le n \not\equiv 0 \pmod 4, \\ \frac{n(n-1)}{2} - 1 & \textit{for } \ 8 \le n \equiv 0 \pmod 4. \end{cases}$$

4 Lower bound

In this section we establish some bounds for $m(n)$. The main result is

Theorem 4. *Let $n \geq 5$ be a positive integer. Then*

$$m(n) \geq 3n - \frac{1}{2} - \sqrt{3n^2 - 2n + \frac{1}{4}}.$$

Proof. Suppose that G is a supermagic graph of order n with $\varepsilon = m(n)$ edges. It admits a supermagic labelling $f : E(G) \longrightarrow \{a, \dots, a+\varepsilon-1\}$ for an index

$$\lambda = \frac{(2a + \varepsilon - 1)\varepsilon}{n}. \tag{1}$$

Let V_3 denote the set of vertices of degree at least 3, the cardinality of this set is denoted by n_3. By n_2 denote the number of 2-vertices (i.e., vertices of degree 2). As every vertex of a supermagic graph G has degree at least 2, $n = n_2 + n_3$. For the number of edges we have

$$2\varepsilon = \sum_{v \in V(G)} deg(v) = 2n_2 + \sum_{v \in V_3} deg(v) \geq 2n_2 + 3n_3 = 3n - n_2,$$

thus

$$\varepsilon \geq \frac{3n}{2} - \frac{n_2}{2}. \tag{2}$$

If G contains no 2-vertex, then $\varepsilon \geq \frac{3n}{2}$ and the assertion is satisfied. So we can assume that $n_2 \geq 1$.

In any supermagic graph there exists no edge joining the vertices of degree 2, i.e. every vertex of degree 2 is adjacent to two distinct vertices of degree at least 3. It means all edges incident with the n_2 vertices of degree 2 are mutually distinct and their number is $2n_2$. The sum of the labels of edges incident with 2-vertices has to be less or equal to the sum of maximal values which can be assigned to any $2n_2$ edges in the supermagic labelling f

$$n_2\lambda \leq (a + \varepsilon - 1) + \ldots + (a + \varepsilon - 2n_2) = (2a + 2\varepsilon - 2n_2 - 1)n_2.$$

As $n_2 \neq 0$, by (1) we get

$$\frac{(2a + \varepsilon - 1)\varepsilon}{n} = \lambda \leq 2a + 2\varepsilon - 2n_2 - 1.$$

From this inequality we yield

$$2n_2 \leq 2a\left(1 - \frac{\varepsilon}{n}\right) - \frac{\varepsilon^2}{n} + \frac{\varepsilon}{n} + 2\varepsilon - 1. \tag{3}$$

Any supermagic graph of order $n > 2$ has more edges than vertices and so $1 - \frac{\varepsilon}{n} < 0$. Since $a \geq 1$,

$$2a\Big(1 - \frac{\varepsilon}{n}\Big) \leq 2\Big(1 - \frac{\varepsilon}{n}\Big).$$

Appointing this in (3) we have

$$2n_2 \leq 1 - \frac{\varepsilon^2}{n} - \frac{\varepsilon}{n} + 2\varepsilon.$$

From (2) we get $n_2 \geq 3n - 2\varepsilon$, and combining it we get

$$0 \geq \varepsilon^2 + \varepsilon(1 - 6n) + 6n^2 - n.$$

This inequality immediately implies

$$\varepsilon \geq 3n - \frac{1}{2} - \sqrt{3n^2 - 2n + \frac{1}{4}},$$

which is the desired lower bound for $m(n)$. □

The previous theorem implies $m(n) \geq \lceil (3 - \sqrt{3}\,)n \rceil$. It seems that it is not possible to reach this bound. The best bound we know is $\frac{9n}{7}$ for $n = 14$, 42, 70. (We suppose that there exists an infinite family of supermagic graphs of size $\frac{9n}{7}$.) The corresponding supermagic graph for $n = 14$ is illustrated on Figure 1.

Figure 1.

Figure 2.

Now we deal with an upper bound for $m(n)$.

Let d and p be non-negative integers such that $k := d + p \geq 2$. By $M_{d,p}$ denote the graph with the vertex set $\{u_1, \dots, u_{2k}, v_1, \dots, v_p\}$ and the edge set consisting of edges

$$\begin{aligned}
&e_1 = u_1u_2,\ e_2 = u_2u_3,\ \dots,\ e_{2k-1} = u_{2k-1}u_{2k},\ e_{2k} = u_{2k}u_1,\\
&f_1 = u_1u_{k+1},\ f_2 = u_2u_{k+2},\ \dots,\ f_d = u_du_{k+d},\\
&l_1 = v_1u_{d+1},\ l_2 = v_2u_{d+2},\ \dots,\ l_p = v_pu_{d+p},\\
&r_1 = v_1u_{2k},\ r_2 = v_2u_{2k-1},\ \dots,\ r_p = v_pu_{2k-p+1}.
\end{aligned}$$

Lemma 1. *The graph $M_{d,p}$ is supermagic for every odd positive integer d.*

Proof. For every odd positive integer d there exists a positive integer s such that $d = 2s - 1$. Put $a = p + s$. By g denote the mapping from the edge set of $M_{d,p}$ to the set $\{a, \dots, a + |E(M_{d,p})| - 1\}$ defined by

$$g(e) = \begin{cases} a + \frac{i-1}{2} & \text{for } e = e_i \text{ and } i = 1, 3, \dots, 2k-1, \\ a + k + \frac{i-1-d}{2} & \text{for } e = e_i \text{ and } i = d+1, d+3, \dots, 2k, \\ a + k + \frac{2k+i-1-d}{2} & \text{for } e = e_i \text{ and } i = 2, 4, \dots, d-1, \\ a + 2k - 1 + i & \text{for } e = r_i \text{ and } i = 1, 2, \dots, p, \\ a + 3k - i & \text{for } e = f_i \text{ and } i = 1, 2, \dots, d, \\ a + 3k + p - i & \text{for } e = l_i \text{ and } i = 1, 2, \dots, p. \end{cases}$$

It is easy to see that g is a bijection and its index-mapping $g^\star$ satisfies

$$g^\star(v) = 8p + 12s - 6 \quad \text{for every } v \in V(M_{d,p}).$$

Thus g is a supermagic labelling and $M_{d,p}$ is a supermagic graph. □

Lemma 2. *For every positive integer $k \geq 2$ there exists a supermagic graph of order $3k$ and size $4k$.*

Proof. Consider a cycle C_{2k} with vertices u_1, u_2, …, u_{2k} and edges $e_1 = u_1u_2$, …, $e_{2k-1} = u_{2k-1}u_{2k}$, $e_{2k} = u_{2k}u_1$. Let f be a mapping from $E(C_{2k})$ to the set of positive integers defined by

$$f(e) = \begin{cases} k - 1 + \frac{i-1}{2} & \text{for } i = 1, 3, \dots, k, k+4, \dots, 2k-1, \\ 4k - 2 & \text{for } i = 2, \\ k - 1 + \frac{i}{2} & \text{for } i = k+1, \\ 2k - 2 + \frac{i-3}{2} & \text{for } i = k+2, \\ 2k - 2 + \frac{i-2}{2} & \text{for } i = 4, 6, \dots, k-1, k+3, \dots, 2k, \end{cases}$$

for k odd, and

$$f(e) = \begin{cases} k - 1 + \frac{i-1}{2} & \text{for } i = 1, 3, k+1, \\ 4k - 2 & \text{for } i = 2, \\ 2k - 2 + \frac{i-2}{2} & \text{for } i = k+2, 2k, \\ 2k - \frac{i}{2} & \text{for } i = 4, 6, \dots, k, k+4, \dots, 2k-2, \\ 3k - 3 - \frac{i-3}{2} & \text{for } i = 5, 7 \dots, k-1, k+3, \dots, 2k-1, \end{cases}$$

for k even.

Let S_k be a graph with vertex set $V(C_{2k}) \cup \{v_1, \dots, v_k\}$ and edge set $E(C_{2k}) \cup \bigcup_{i=1}^{k} \{v_i u_{i1}, v_i u_{i2}\}$, where u_{i1}, u_{i2} are vertices of C_{2k} such that

$$f^\star(u_{i1}) = 3k - 3 + i \quad \text{and} \quad f^\star(u_{i2}) = 5k - 1 - i.$$

Consider a mapping $g : E(S_k) \longrightarrow \{k-1, \dots, 5k-2\}$ defined by

$$g(e) = \begin{cases} f(e) & \text{for } e \in E(C_{2k}), \\ 5k - 1 - i & \text{for } e = v_i u_{i1}, \\ 3k - 3 + i & \text{for } e = v_i u_{i2}. \end{cases}$$

It is easy to check that g is a bijection. Moreover its index-mapping $g^\star$ satisfies

$$g^\star(v) = 8k - 4 \quad \text{for every } v \in V(S_k).$$

Thus g is a supermagic labelling and S_k is a desired supermagic graph.

□

On Figure 3 there are illustrated the graphs $M_{3,0}$, $M_{1,3}$ and S_4 and their supermagic labellings.

Figure 3.

From previous lemmas we immediately obtain

Theorem 5. *Let $n \geq 5$ be a positive integer. Then*

$$m(n) \leq \begin{cases} \frac{4n}{3} & \text{for } n \equiv 0 \pmod 3, \\ \frac{4n}{3} + \frac{5}{3} & \text{for } n \equiv 1 \pmod 3, \\ \frac{4n}{3} + \frac{1}{3} & \text{for } n \equiv 2 \pmod 3. \end{cases}$$

Proof. It is obvious that the graphs $S_{n/3}$ (for $n \equiv 0 \pmod 3$), the graph on Figure 2 and the graphs $M_{5,(n-10)/3}$ (for $n \equiv 1 \pmod 3$) and $M_{1,(n-2)/3}$ (for $n \equiv 2 \pmod 3$) are supermagic graphs of order n with the required number of edges. □

We conclude this paper with a determination of $m(n)$ for prime number n, but first we prove the following auxiliary result.

Lemma 3. *Let G be a supermagic graph of order $n \geq 5$ and size ε. If the greatest common divisor of the numbers n and ε is 1, then $\varepsilon > \frac{4n}{3}$. Moreover, if ε is an even integer, then $\varepsilon > \frac{4n+2}{3}$.*

Proof. Consider a supermagic labelling $f : E(G) \longrightarrow \{a, \dots, a+\varepsilon-1\}$ for an index $\lambda = \frac{(2a+\varepsilon-1)\varepsilon}{n}$. As n and ε are coprime and λ is a positive integer, then $\gamma := \frac{2a+\varepsilon-1}{n}$ is also a positive integer. From this we can express

$$\lambda = \gamma\varepsilon \tag{4}$$

$$a = \tfrac{1}{2}(\gamma n - \varepsilon + 1) \tag{5}$$

Let n_2 denote the number of 2-vertices in G. Values of the edges (mutually distinct) incident with the 2-vertices are at most $a+\varepsilon-1$, $a+\varepsilon-2$, $\dots$, $a+\varepsilon-2n_2$. Thus

$$\lambda \leq (a+\varepsilon-1) + (a+\varepsilon-2n_2) = 2a + 2\varepsilon - 2n_2 - 1.$$

Appointing (4), (5) in this inequality we get

$$n_2 \leq \tfrac{1}{2}((1-\gamma)\varepsilon + \gamma n).$$

As in the proof of Theorem 4 we get (2), and then $n_2 \geq 3n - 2\varepsilon$. Combining it we have

$$(5-\gamma)\varepsilon \geq (6-\gamma)n. \tag{6}$$

Since γ is a positive integer it is sufficient to consider the following cases.

A. $\gamma \geq 5$. According to Corollary 1 it yields

$$\varepsilon > \frac{5n}{4} = \Big(1 + \frac{1}{4}\Big)n \geq \Big(1 + \frac{1}{\gamma-1}\Big)n = \frac{\gamma}{\gamma-1}n.$$

Therefore, $\varepsilon(\gamma-1) > \gamma n > \gamma n - 2$. Hence $\varepsilon - 2 < \gamma(\varepsilon - n) = \frac{2a+\varepsilon-1}{n}(\varepsilon - n)$. After some manipulation we have

$$(a+\varepsilon-1) + (a+\varepsilon-2) < \frac{(2a+\varepsilon-1)\varepsilon}{n} = \lambda.$$

It means $n_2 = 0$, and then $\varepsilon \geq \frac{3n}{2} > \frac{4n+2}{3}$.

B. $\gamma \in \{3, 4\}$. By (6) we get $\varepsilon \geq \frac{6-\gamma}{5-\gamma}\, n \geq \frac{3n}{2} > \frac{4n+2}{3}$.

C. $\gamma = 2$. According to (6) we have $\varepsilon \geq \frac{4}{3}n$. Moreover, $\varepsilon \neq \frac{4}{3}n$. In opposite case we get $\varepsilon = 4k$ and $n = 3k$ for some integer $k > 1$. It means the greatest common divisor of n and ε is k, a contradiction. Note also that (5) implies $\varepsilon = 2(n-a)+1$, therefore ε is an odd integer in this case.

D. $\gamma = 1$. From (5) we get $\varepsilon = n - 2a + 1 < n$, contrary to Corollary 1. Thus, this case is impossible. □

If n is the prime number, then using Theorem 5 and Lemma 3 we immediately obtain

Theorem 6. *Let $n \geq 5$ be a prime number. Then*

$$m(n) = \begin{cases} \frac{4n}{3} + \frac{5}{3} & \text{for } n \equiv 1 \pmod 3, \\ \frac{4n}{3} + \frac{1}{3} & \text{for } n \equiv 2 \pmod 3. \end{cases}$$

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