Retracts of monounary algebras corresponding to groupoids

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Abstract. M. Novotný [9] defined the monounary algebra $\mathrm{un}(G, \circ)$ corresponding to a groupoid $(G, \circ)$. The aim of this paper is to prove that each monounary algebra is up to isomorphism a retract of $\mathrm{un}(G, \circ)$ for some groupoid $(G, \circ)$.

1 Introduction and preliminaries

Monounary algebras play a significant role in the study of algebraic and relational structures, especially in the case of finite structures (cf., e.g., Jónsson [1], Skornjakov [12], Chvalina [2]). Further, there exists a close connection between monounary algebras and some types of automata (cf. e.g., Bartol [1], Salij [11]).

M. Novotný [10] proved that all homomorphisms of groupoids can be constructed by means of homomorphisms of monounary algebras. In this construction he defined and investigated the notion of a monounary algebra denoted by $\mathrm{un}(G, \circ)$ which corresponds to a groupoid $(G, \circ)$.

In [9] cyclic monounary algebras of the form $\mathrm{un}(G, \circ)$ were studied.

The aim of the present paper is to prove that each monounary algebra is up to isomorphism a retract of some $\mathrm{un}(G, \circ)$ for a groupoid $(G, \circ)$.

On the other hand, there exists a paper class of monounary algebras which are not isomorphic to any $\mathrm{un}(G, \circ)$.

Retracts of monounary algebras were investigated by the author [3]-[7].

We recall some basic definitions.

A *monounary algebra* is a pair (A, f), where A is a non-empty set and f is a unary operation on A.

Let (A, f) be a monounary algebra. For $a \in A$ we put $f^{\circ}(a) = a$ and by induction, $f^n(a) = f(f^{n-1}(a))$ for each $n \in N$.

A monounary algebra (A, f) is said to be *connected* if for each $x, y \in A$ there are $m, n \in N \cup \{0\}$ such that $f^m(x) = f^n(y)$.

[1]Suported by grant VEGA 1/0423/03

A maximal connected subalgebra (B, f) of (A, f) is called a *connected component* of (A, f); we will say also that B is a connected component of (A, f).

An element $a \in A$ is *cyclic* if $f^n(a) = a$ for some $n \in N$. Let (B, f) be a connected component of (A, f). If each element of B is cyclic, then B is a *cycle* of (A, f).

Let (A, F) be an algebra. A subalgebra (B, F) of (A, F) is a *retract* of (A, F) if there is an endomorphism φ of (A, F) such that φ is a mapping of A onto B and $\varphi(b) = b$ for each $b \in B$; in this case φ is said to be a *retraction endomorphism.*

Let $(G, \circ)$ be a groupoid. A monounary algebra $\mathrm{un}(G, \circ)$ corresponding to $(G, \circ)$ is defined as follows: $\mathrm{un}(G, \circ) = (G \times G, g)$, where g is a unary operation on $G \times G$, such that if $(x, y) \in G \times G$, then $g((x, y)) = (y, x \circ y)$.

2 Underlying set of the groupoid $(G, \circ)$

In where follows let (A, f) be a monounary algebra.

As we already announced in Section 1, our aim is to construct a groupoid $(G, \circ)$ such that (A, f) is isomorphic to a retract of $\mathrm{un}(G, \circ)$. In the present section we construct the underlying set G of the groupoid under consideration; the operation $\circ$ will be dealt with in Section 3.

We will apply the following notation. Let α be a an ordinal and let a system of sets $\{B_\beta\}_{\beta<\alpha}$ be such that $B_\beta \subseteq B_\gamma$ for each $\beta \leq \gamma < \alpha$. Further assume that $\{\varphi_\beta\}_{\beta<\alpha}$ is a system of mappings $\varphi_\beta : B_\beta \to C$ for some set C such that if $\beta \leq \gamma < \alpha$, $b \in B_\beta$, then $\varphi_\gamma(b) = \varphi_\beta(b)$. By a union $\bigcup\limits_{\beta<\alpha} \varphi_\beta$ we understand the mapping φ such that whenever $b \in B_\beta$, $\beta < \alpha$, then $\varphi(b) = \varphi_\beta(b)$.

First we are going to define by induction a set Λ of ordinal numbers.

For a set Γ of ordinals let Γ^+ be the smallest ordinal which greater than all $\gamma \in \Gamma$.

Applying the Axiom of Choice we can suppose that the set A is well-ordered, i.e.,

$$A = \{a_\mu : \mu < \mu_0\}, \ \mu_0 \in \mathrm{Ord},$$

and also that the system of all connected components of (A, f) is well-ordered, i.e., (A, f) possesses the system $\{K_\iota\}_{\iota<\iota_0}$ of connected components, $\iota_0 \in \mathrm{Ord}$.

For each $\iota < \iota_0$ let x_ι be a fixed element of K_ι such that if K_ι contains a cycle, then x_ι is cyclic. Further we define certain subsets P_n^ι, $n \in N \cup \{0\}$ of K_ι which we call folds generated by x_ι; they are defined as follows: $P_0^\iota = \{f^i(x_\iota) : i \in N \cup \{0\}\}$, $P_1^\iota = f^{-1}(P_0^\iota) - P_0^\iota$, $P_{n+1}^\iota = f^{-1}(P_n^\iota)$ for each $n \in N$.

Now we will proceed by induction and define, for each ordinal $\eta < \mu_0$,

- a set $D_\eta \subseteq A$,

- a set $\Lambda_\eta \subset \mathrm{Ord}$,

- a mapping $\varphi_\eta : D_\eta \to \Lambda_\eta \times \Lambda_\eta$ such that

  ($*1$) if $\eta' \le \eta'' < \mu_0$, then $D_{\eta'} \subseteq D_{\eta''}$, $\Lambda_{\eta'} \subseteq \Lambda_{\eta''}$,

  ($*2$) if $\eta' \le \eta'' < \mu_0$, $d \in D_{\eta'}$, then $\varphi_{\eta'}(d) = \varphi_{\eta''}(d)$,

  ($*3$) if $\eta' < \mu_0$, $\lambda_1 \in \Lambda_{\eta'}$, then there are $\lambda_2 \in \Lambda_{\eta'}$, $d \in D_{\eta'}$, such that either $\varphi_{\eta'}(d) = (\lambda_1, \lambda_2)$ or $\varphi_{\eta'}(d) = (\lambda_2, \lambda_1)$,

  ($*4$) if $\eta' < \mu_0$, $d, e \in D_{\eta'}$, $\varphi_{\eta'}(d) = (\lambda_1, \lambda_2)$, $\varphi_{\eta'}(e) = (\lambda_1, \lambda_3)$, then $d = e$.

I. For $\eta = 0$ put $D_\eta = \emptyset$, $\Lambda_\eta = \emptyset$.

II. Let $\eta \in \text{Ord}$, $\eta > 0$. Suppose that for all ordinals $\eta' < \eta$ sets $D_{\eta'}, \Lambda_{\eta'}$ and an injective mapping $\varphi_{\eta'} : D_{\eta'} \to \Lambda_{\eta'} \times \Lambda_{\eta'}$ are defined such that the conditions analogous to ($*1$)-($*4$) are valid, with the distinction that we take η instead of μ_0.

If $A \ne \bigcup\limits_{\eta'<\eta} D_{\eta'}$, then there is the smallest $\iota < \iota_0$ such that $K_\iota \not\subseteq \bigcup\limits_{\eta'<\eta} D_{\eta'}$ and there is the smallest $n \in N \cup \{0\}$ such that $P_n^\iota \not\subseteq \bigcup\limits_{\eta'<\eta} D_{\eta'}$. Denote $\beta = \big(\bigcup\limits_{\eta'<\eta} \Lambda_\eta\big)^+$.

a) Assume that $n = 0$ and $x_\iota \notin \bigcup\limits_{\eta'<\eta} D_{\eta'}$.

a1) If $f(x_\iota) = x_\iota$, then we set $D_\eta = \bigcup_{\eta'<\eta} D_{\eta'} \cup \{x_\iota\}$, $\Lambda_\eta = \bigcup\limits_{\eta'<\eta} \Lambda_{\eta'} \cup \{\beta\}$ and

$$\varphi_\eta(a) = \begin{cases} \big(\bigcup\limits_{\eta'<\eta} \varphi_{\eta'}\big)(a) & \text{if } a \in \bigcup\limits_{\eta'<\eta} D_{\eta'}, \\ (\beta, \beta) & \text{if } a = x_\iota. \end{cases}$$

The induction assumption yields that ($*1$)-($*4$) are satisfied if we take η^+ instead of μ_0.

a2) If $f(x_\iota) \ne x_\iota$, then either x_ι belongs to a k-element cycle, $k > 1$, or all elements $f^i(x_\iota)$, $i \in N \cup \{0\}$ are mutually distinct. We put $D_\eta = \bigcup\limits_{\eta'<\eta} D_{\eta'} \cup P_0^\iota$. In the first case $\Lambda_\eta = \bigcup\limits_{\eta'<\eta} \Lambda_{\eta'} \cup \{\beta, \beta+1, \dots, \beta+(k-1)\}$ and φ_η is an extension of $\bigcup\limits_{\eta'<\eta} \varphi_{\eta'}$ such that

$$\varphi_\eta(f^i(x_\iota)) = \begin{cases} (\beta+i, \beta+i+1) & \text{if } i = 0, \dots, k-1, \\ (\beta+k, \beta) & \text{if } i = k. \end{cases}$$

In the second case we set $\Lambda_\eta = \bigcup_{\eta'<\eta} \Lambda_{\eta'} \cup \{\beta + n : n < \omega\}$ and φ_η is an extension of $\bigcup_{\eta'<\eta} \varphi_{\eta'}$ such that

$$\varphi_\eta(f^i(x_\iota)) = (\beta + i, \beta + i + 1) \text{ for each } i < \omega.$$

Also in this case $(*1)$-$(*4)$ are satisfied (with η^+ instead of μ_0).

b) Assume that $x_\iota \in \bigcup_{\eta'<\eta} D_{\eta'}$. In view of a1) and a2) also $P_0^\iota \subseteq \bigcup_{\eta'<\eta} D_{\eta'}$, thus $n > 0$. There is the smallest element $y \in P_n^\iota - \bigcup_{\eta'<\eta} D_{\eta'}$. Then $f(y) \in P_{n-1}^\iota \subseteq \bigcup_{\eta'<\eta} D_{\eta'}$, i.e., there are $\eta' < \eta$ and $\alpha_1, \alpha_2 \in \Lambda_{\eta'}$ such that $\varphi_{\eta'}(f(y)) = (\alpha_1, \alpha_2)$. We put

$$D_\eta = \bigcup_{\eta'<\eta} D_{\eta'} \cup \{y\}, \ \Lambda_\eta = \bigcup_{\eta'<\eta} \Lambda_{\eta'} \cup \{\beta\}.$$

Further, let φ_η be an extension of $\bigcup_{\eta'<\eta} \varphi_{\eta'}$ such that $\varphi_\eta(y) = (\beta, \alpha_1)$.

There exists $\eta_0 \leqq \mu_0$ such that $A = \bigcup_{\eta'<\eta_0} D_{\eta'}$. Thus for each η with $\eta_0 \leqq \eta < \mu_0$ we put $D_\eta = D_{\eta_0}$, $\Lambda_\eta = \Lambda_{\eta_0}$, $\varphi_\eta = \varphi_{\eta_0}$.

Notation 2.1. *Now we have* $A = \bigcup_{\eta<\mu_0} D_\eta$. *Put*

$$\begin{aligned} &\Lambda = \bigcup_{\eta<\mu_0} \Lambda_\eta, \quad \varphi = \bigcup_{\eta<\mu_0} \varphi_\eta, \\ &G = \Lambda \cup \{\Lambda^+\}, \\ &\Omega = \varphi(A). \end{aligned}$$

3 Operation $\circ$ of the groupoid $(G, \circ)$

Using 2.1, in this section a binary operation $\circ$ on G will be defined.

First we define $\alpha * \beta$ for $(\alpha, \beta) \in \Omega$ as follows. Let $(\alpha, \beta) \in \Omega$. There is $x \in A$ with $\varphi(x) = (\alpha, \beta)$. The definition of φ implies that $\varphi(f(x)) = (\beta, \gamma)$ for some $\gamma \in \Lambda$; put $\alpha * \beta = \gamma$.

Lemma 3.1. *Let* $\Box$ *be a binary operation on* G *such that if* $(\alpha, \beta) \in \Omega$ *then* $\alpha \Box \beta = \alpha * \beta$. *Further let* $\operatorname{un}(G, \Box) = (G \times G, h)$. *Then* Ω *is closed with respect to* h.

Proof. Let $(\alpha, \beta) \in \Omega$. Then $h((\alpha, \beta)) = (\beta, \alpha \Box \beta) = (\beta, \alpha * \beta) \in \Omega$. $\Box$

Lemma 3.2. *Let the assumption of 3.1 hold. Then φ is an isomorphism of (A, f) onto (Ω, h).*

Proof. By 2.1, the mapping φ is surjective. From the construction in Section 2 it follows that φ is injective.

Let $x \in A$, $\varphi(x) = (\alpha, \beta) \in \Omega$. Then $\varphi(f(x)) = (\beta, \gamma)$ and $\gamma = \alpha * \beta$, which yields

$$\varphi(f(x)) = (\beta, \gamma) = (\beta, \alpha * \beta) = (\beta, \alpha \Box \beta) = h((\alpha, \beta)) = h(\varphi(x)).$$

Thus φ is an isomorphism of (A, f) onto (Ω, h). $\Box$

Now we are going to define the operation $\circ$ on G. In A there exist (not necessarily distinct) elements a, a', a'', a''' such that $f(a''') = a''$, $f(a'') = a'$, $f(a') = a$; we take fixed elements with this property. Then there are ordinals $\delta, \tau, \tau', \tau'', \tau''' \in \Lambda$ such that

$$\varphi(a) = (\tau, \delta), \varphi(a') = (\tau', \tau), \varphi(a'') = (\tau'', \tau'), \varphi(a''') = (\tau''', \tau''). \tag{1}$$

By the definition of $*$ we obtain

$$\tau''' * \tau'' = \tau', \tau'' * \tau' = \tau, \tau' * \tau = \delta. \tag{2}$$

Further denote $\lambda = \Lambda^+$; notice that $\lambda \notin \Lambda$, thus we have

(†) $\qquad (\alpha, \lambda) \notin \Omega$ for any $\alpha \in \Lambda$.

Notation 3.3. *Let $\circ$ be a binary operation on G defined as follows:*

$$\alpha \circ \beta = \begin{cases} \alpha * \beta & \text{if } (\alpha, \beta) \in \Omega, \\ \delta & \text{if } \alpha = \lambda, \beta = \tau, \\ \tau & \text{if } \beta = \lambda, \\ \lambda & \text{otherwise.} \end{cases}$$

Put $(B, g) = \mathrm{un}(G, \circ)$.

In view of (†), $\alpha \circ \beta$ is correctly defined.

Lemma 3.4. *(Ω, g) is a retract of (B, g).*

Proof. Let us define a retraction endomorphism $h : B \to \Omega$. For $(\alpha, \beta) \in B = G \times G$ we define

$$h((\alpha, \beta)) = \begin{cases} (\alpha, \beta) & \text{if } (\alpha, \beta) \in \Omega, \\ (\tau', \tau) & \text{if } \alpha = \lambda, \beta = \tau, \\ (\tau'', \tau') & \text{if } \beta = \lambda, \\ (\tau''', \tau'') & \text{otherwise.} \end{cases}$$

The mapping is correctly defined according to (†).

Let $(\alpha,\beta)\in\Omega$. Then $g((\alpha,\beta))\in\Omega$ in view of 3.1, thus

$$h(g((\alpha,\beta))) = g((\alpha,\beta)) = g(h((\alpha,\beta))).$$

For $(\alpha,\beta)=(\lambda,\tau)$ we obtain

$$\begin{aligned}h(g((\alpha,\beta))) &= h((\beta,\alpha\circ\beta)) = h((\tau,\delta)) = (\tau,\delta) = \\ &= (\tau,\tau'*\tau) = (\tau,\tau'\circ\tau) = g((\tau',\tau)) = g(h((\alpha,\beta))).\end{aligned}$$

Let $(\alpha,\beta)\in B$, $\beta=\lambda$. Then

$$\begin{aligned}h(g((\alpha,\beta))) &= h((\beta,\alpha\circ\beta)) = h((\lambda,\tau)) = (\tau',\tau) = \\ &= (\tau',\tau''*\tau') = (\tau',\tau''\circ\tau') = g((\tau'',\tau')) = g(h((\alpha,\beta))).\end{aligned}$$

Finally, consider the remaining case for (α,β). Then

$$\begin{aligned}h(g((\alpha,\beta))) &= h((\beta,\alpha\circ\beta)) = h((\beta,\lambda)) = (\tau'',\tau') = \\ &= (\tau'',\tau'''*\tau'') = (\tau'',\tau'''\circ\tau'') = g((\tau''',\tau'')) = g(h((\alpha,\beta))).\end{aligned}$$

Therefore h is a retraction endomorphism onto (Ω,g), thus (Ω,g) is a retract of (B,g). □

Theorem 3.5. *Let (A,f) be a monounary algebra. There exists a groupoid $(G,\circ)$ such that (A,f) is isomorphic to a retract of the monounary algebra* $\mathrm{un}(G,\circ)$ *corresponding to the groupoid $(G,\circ)$.*

Proof. The assertion follows from 3.2 and 3.4. □

We conclude by giving an example which shows that there exists a proper class of monounary algebras which are not isomorphic to any $\mathrm{un}(G,\circ)$ for a groupoid $(G,\circ)$.

Example 3.6. *Let (A,f) be a monounary algebra such that $|A|>1$ and there is $a\in A$ with $f(x)=a$ for each $x\in A$. We will show that $(A,f)\not\cong \mathrm{un}(G,\circ)$ for any groupoid $(G,\circ)$.*

By way of contradiction, suppose that there are a groupoid $(G,\circ)$ and an isomorphism φ of (A,f) onto $\mathrm{un}(G,\circ)=(G\times G,g)$. Denote $\varphi(a)=(a_1,a_2)$. Then

$$\begin{aligned}(a_1,a_2) &= \varphi(a) = \varphi(f(a)) = g(\varphi(a)) = \\ &= g((a_1,a_2)) = (a_2,a_1\circ a_2),\end{aligned}$$

which implies $a_1=a_2=a_1\circ a_2$. If $b\in A-\{a\}$, $\varphi(b)=(b_1,b_2)$, then

$$\begin{aligned}(a_1,a_2) &= \varphi(a) = \varphi(f(b)) = g(\varphi(b)) = \\ &= g((b_1,b_2)) = (b_2,b_1\circ b_2),\end{aligned}$$

thus $a_1 = b_2$. Therefore

$$\varphi(A) \subseteq \{(x, a_1) : x \in G\}.$$

Since $|A| > 1$, we obtain that $\varphi(A) \neq G \times G$, which is a contradiction.

We have constructed (A, f) for each cardinality $|A| > 1$, therefore there is a proper class of (A, f) with $(A, f) \not\cong \mathrm{un}(G, \circ)$ for any groupoid $(G, \circ)$.

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