On complete tripartite graphs arbitrarily decomposable into closed trails

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Abstract

In the paper it is proved that any complete tripartite graph $K_{r,r,r}$, where $r = 5 \cdot 2^n$ and $n$ is a nonnegative integer, has the following property: Whenever $(l_1, \ldots, l_p)$ is a sequence of integers $\geq 3$ adding up to $|E(K_{r,r,r})|$, there is a sequence $(T_1, \ldots, T_p)$ of edge-disjoint closed trails in $K_{r,r,r}$ such that $T_i$ is of length $l_i$, $i = 1, \ldots, p$.

1 Introduction

In any simple finite nonoriented graph $G$ with $\delta(G) \geq 2$ there is a cycle. Therefore, if $G$ is even (if all vertices of $G$ are of even degrees), it is an edge-disjoint union of cycles. Several authors investigated edge decompositions of complete multipartite graphs into cycles of equal lengths. The bipartite case has been completely solved by Sotteau [8]. For complete tripartite graphs some partial results are known, see Billington and Cavenagh [3], [4], Cavenagh [5]. In the general case a reader can consult Cockayne and Hartnell [6].

A connected edge-disjoint union of cycles is an Eulerian graph and has a closed Eulerian trail. So, an even graph can be expressed as an edge-disjoint union of closed trails, and there are many possibilities how to do it. Balister [1] has proved that if $n$ is odd and $l_1, \ldots, l_p$ are integers $\geq 3$ adding up to $|E(K_n)|$, there are edge-disjoint closed trails $T_1, \ldots, T_p$ in $K_n$ such that $T_i$ is

*The authors wish to thank for the support of the Slovak grant VEGA 1/0424/03.
of length \( l_i, i = 1, \ldots, p \). In the same paper a similar result has been reached for the graph \( K_n - M_n \), where \( n \) is even and \( M_n \) is a perfect matching in \( K_n \). Balister [2] has shown that there are positive constants \( n \) and \( \varepsilon \) such that any even graph \( G \) with \( |V(G)| \geq n \) and \( \delta(G) \geq (1 - \varepsilon)|V(G)| \) can be (edge-)decomposed in the above manner. Another graphs with analogous properties concerning closed trails are complete bipartite graphs \( K_m,n \) with \( m,n \) even, as proved by Horňák and Woźniak [7] (note that in that case all \( l_i \)'s have to be even). In this paper we do concentrate on complete tripartite graphs.

Let us now precise the problem we are going to deal with. For integers \( p,q \) we use the notation \([p,q] := \{z \in \mathbb{Z} : p \leq z \leq q \}\) and \([p,\infty) := \{z \in \mathbb{Z} : p \leq z \}\). The concatenation of finite sequences \( A = (a_1, \ldots, a_m) \) and \( B = (b_1, \ldots, b_n) \) is the sequence \( AB = (a_1, \ldots, a_m, b_1, \ldots, b_n) \). The concatenation is associative, and so we can use the symbol \( \prod_{i=1}^{k} A_i \) for the result of concatenation of finite sequences \( A_1, \ldots, A_k \) (independently from the order in which “partial” concatenations are realised). If \( k \in [0,\infty) \) and \( A_i = A \) for any \( i \in [1,k] \), we write \( A^k \) instead of \( \prod_{i=1}^{k} A_i \) (so that \( A^0 = ( ) \) is the empty sequence).

A closed trail of length \( n \in [3,|E(G)|] \) (an \( n \)-trail for short) in a graph \( G \) is a sequence \( T = \prod_{i=0}^{n} (v_i) \) of vertices of \( G \) such that \( v_0 = v_n, v_i v_{i+1} \in E(G) \) and \( v_i v_{i+1} \neq v_j v_{j+1} \) for any \( i, j \in [0, n-1], i \neq j \). The set \( \{v_i v_{i+1} : i \in [0, n-1]\} \) of edges of \( T \) induces an Eulerian subgraph of \( G \) and throughout the whole paper we shall identify \( T \) with this subgraph. Deleting the edges of a closed trail from an even graph \( G \) results in an even graph with a smaller number of edges. This process can be continued until the edgeless graph \( (V(G), \emptyset) \) is reached. If \( (T_1, \ldots, T_p) \) is the sequence of (closed) trails occurring in the process, then \( 1 \leq p \leq \lfloor |E(G)|/3 \rfloor \), \( \{E(T_i) : i \in [1,p]\} \) is a decomposition of \( E(G) \) and \( \sum_{i=1}^{p} |E(T_i)| = |E(G)| \). Let \( \text{Lct}(G) \) be the set of all lengths of closed trails in \( G \) and let

\[
\text{Sct}(G) := \bigcup_{p=1}^{\lfloor |E(G)|/3 \rfloor} \{(l_1, \ldots, l_p) \in (\text{Lct}(G))^p : \sum_{i=1}^{p} l_i = |E(G)|\}.
\]

A sequence \( L = (l_1, \ldots, l_p) \in \text{Sct}(G) \) is said to be \( G \)-realisable if there is a \( G \)-realisation of \( L \), a sequence \( (T_1, \ldots, T_p) \) of edge-disjoint closed trails in \( G \) (so that \( \bigcup_{i=1}^{p} E(T_i) = E(G) \)). A graph \( G \) is said to be arbitrarily decomposable into closed trails (ADCT for short) if \( \text{Sct}(G) \neq \emptyset \) and any sequence from \( \text{Sct}(G) \) is \( G \)-realisable. Evidently, if a graph is ADCT, it is even.

A sequence \( A = (a_1, \ldots, a_m) \) is said to be changeable to a sequence \( B = (b_1, \ldots, b_m) \) if there is a bijection \( \pi : [1, m] \rightarrow [1, m] \) such that \( b_i = a_{\pi(i)} \) for any \( i \in [1, m] \); if \( A \) is changeable to \( B \), we write \( A \sim B \). For \( I \subseteq [1, m] \)
Let $A(I)$ be the subsequence of $A$ created by deleting from $A$ all $a_i$'s with $i \in [1,m] - I$. If $A \in \mathbb{R}^m$ and $r \in \mathbb{R}$, we denote by $\text{nd}(A)$ the (unique) nondecreasing sequence that is changeable to $A$ and by $f_r(A)$ the frequency of $r$ in $A$. For $l \in \mathbb{Z}$ let $(l)_4$ be the unique $m \in [0,3]$ such that $l \equiv m$ (mod 4).

Let $G,H$ be isomorphic graphs and let $\varphi : V(G) \rightarrow V(H)$ be an isomorphism from $G$ onto $H$. If $T = \prod_{i=0}^{n}(v_i)$ is a closed trail in $G$, then $\varphi(T) := \prod_{i=0}^{n}(\varphi(v_i))$ is a closed trail in $H$. Further, if $T = \prod_{i=1}^{q}(T_i)$ is a sequence of edge-disjoint closed trails in $G$, then $\varphi(T) := \prod_{i=1}^{q}(\varphi(T_i))$ is a sequence of edge-disjoint closed trails in $H$.

Consider edge-disjoint closed trails $T_1, T_2$ in a graph $G$ and let $T_1 + T_2$ denote the set of all closed trails $T$ in $G$ with $E(T) = E(T_1) \cup E(T_2)$. Clearly, $T_1 + T_2$ is nonempty if and only if $V(T_1) \cap V(T_2) \neq \emptyset$.

A sequence is said to be simple if no two its terms at distinct positions are the same. Let $G$ be a graph and let $m \in [1,|V(G)|]$. A simple sequence $(v_1, \ldots, v_m)$ of vertices of $G$ is similar to a simple sequence $(w_1, \ldots, w_m)$ of vertices of $G$ provided that there is an automorphism $\varphi$ of $G$, such that $\varphi(v_i) = w_i$ for any $i \in [1,m]$; if $m = 1$, we say for short that a vertex $v_1$ is similar to a vertex $w_1$. The relation of similarity of vertices of a graph $G$ is an equivalence and a similarity class of $G$ is a class of this equivalence.

A set $S$ of edges of a graph $G$ is said to be complementary bipartite in $G$ provided that the graph $G - S$ is bipartite. Let $\text{Cb}(G)$ denote the system of all sets $S \subseteq E(G)$ that are complementary bipartite in $G$ and let $\text{mcb}(G)$ be the minimum cardinality of a set $S \in \text{Cb}(G)$.

Let $r \in [1,\infty)$ and $(n_1, \ldots, n_r) \in [1,\infty)^r$. Throughout the whole paper we shall suppose that the $r$-partition of the complete $r$-partite graph $K_{n_1,\ldots,n_r}$ is \{${v_{i,j} : i \in [1,n_j]} : j \in [1,r]$\}. If $(n_1, \ldots, n_r) = (n)^r$, we write for short $K_{(n)_r}$ instead of $K_{n_1,\ldots,n_r}$.

2 Some preparatory results

Lemma 1 If $G$ is a graph, $L_1, L_2 \in \text{Sct}(G)$ and $L_1 \sim L_2$, then $L_1$ is $G$-realisable if and only if $L_2$ is $G$-realisable.

Proof. If $L_1 = (l_1, \ldots, l_P)$ has a $G$-realisation $(T_1, \ldots, T_p)$ and $\pi : [1,p] \rightarrow [1,p]$ is such a bijection that $L_2 = (l_{\pi(1)}, \ldots, l_{\pi(p)})$, then $(T_{\pi(1)}, \ldots, T_{\pi(p)})$ is a $G$-realisation of $L_2$.

Proposition 2 If $G$ is a graph, $S \in \text{Cb}(G)$ and $T$ is a closed trail in $G$ of an odd length, then $E(T) \cap S \neq \emptyset$.

Proof. $T$ is a non-bipartite graph, so it cannot be a subgraph of the bipartite graph $G - S$. 


Proposition 3  If $G$ is a graph and a sequence $L = (l_1, \ldots, l_p) \in \text{Sct}(G)$ is $G$-realisable, then $L$ contains at most $\text{mcb}(G)$ odd terms.

Proof. Let $S \in \text{Cb}(G)$. Suppose that $(T_1, \ldots, T_p)$ is a $G$-realisation of the sequence $L$ in $G$ and put $I := \{i \in [1, p] : l_i \equiv 1 \mod 2\}$. If $i \in I$, by Proposition 2 there exists $e_i \in E(T_i) \cap S$. Since trails in $(T_1, \ldots, T_p)$ are edge-disjoint, we have $|I| = |\bigcup_{i \in I} \{e_i\}| \leq |\bigcup_{i \in I} (E(T_i) \cap S)| \leq |S| = \text{mcb}(G)$.

Let $\text{Sct}^*(G)$ be the subset of $\text{Sct}(G)$ consisting of sequences with at most $\text{mcb}(G)$ odd terms. From Proposition 3 it follows that if a sequence $L \in \text{Sct}(G)$ is $G$-realisable, then $L \in \text{Sct}^*(G)$. So, if $\text{Sct}(G) - \text{Sct}^*(G) \neq \emptyset$, the graph $G$ is not $\text{ADCT}$.

Proposition 4  If $n \in [1, \infty)$, $\{G^{(i)} : i \in [1, n]\}$ is a set of pairwise edge-disjoint graphs and $G = \bigcup_{i=1}^n G^{(i)}$, then $\text{mcb}(G) \geq \sum_{i=1}^n \text{mcb}(G^{(i)})$.

Proof. Suppose that $S \subseteq E(G)$ and put $S^{(i)} := S \cap E(G^{(i)})$ for $i \in [1, n]$. If $|S| < s := \sum_{i=1}^n \text{mcb}(G^{(i)})$, there is $j \in [1, n]$ such that $|S^{(j)}| < \text{mcb}(G^{(j)})$. The graph $G - S$ is a supergraph of the graph $G^{(j)} - S^{(j)}$ that is not bipartite, hence $S \not\in \text{Cb}(G)$. Thus, $\text{mcb}(G)$ cannot be smaller then $s$.

Proposition 5  If a sequence $(n_1, n_2, n_3) \in [1, \infty)^3$ is nondecreasing, then $\text{mcb}(K_{n_1,n_2,n_3}) = n_1n_2$.

Proof. Let $V_1, V_2, V_3$ be parts of $G := K_{n_1,n_2,n_3}$ with $|V_i| = n_i$, $i = 1, 2, 3$. Then $E(G) = E_{1,2} \cup E_{2,3} \cup E_{3,1}$, where $E_{i,j} := \{xy : x \in V_i, y \in V_j\}$, $i, j \in [1, 3]$, $i \neq j$. The graph $G - E_{1,2}$ is bipartite, hence $\text{mcb}(G) \leq |E_{1,2}| = n_1n_2$. Assume there is $S \in \text{Cb}(G)$ with $|S| < n_1n_2$. Then $S = S_{1,2} \cup S_{2,3} \cup S_{3,1}$, where $S_{i,j} := S \cap E_{i,j}$. If $i, j, k \in [1, 3]$ and $\{i, j, k\} = [1, 3]$, then the deletion of $e \in S_{i,j}$ from $G$ destroys exactly $n_k$ triangles of $G$. Therefore the deletion of $S$ from $G$ destroys at most $|S_{1,2}|n_3 + |S_{2,3}|n_1 + |S_{3,1}|n_2 \leq (|S_{1,2}| + |S_{2,3}| + |S_{3,1}|)n_3 = |S|n_3 < n_1n_2n_3$ triangles of $G$ and $G - S$ has a triangle in contradiction with the fact that $G - S$ is bipartite.

Let $p_{(m)} : V(K_{n_1}) \to [1, r]$ be the function defined by $p_{(m)}(v_{i,j}) = j$ for any $i \in [1, n]$ and $j \in [1, r]$ (a vertex $x$ of $K_{n_1}$ is assigned the number of the part containing $x$).

Proposition 6  Let $n, r \in [1, \infty)$, $m \in [1, rn]$ and let $v = (v_1, \ldots, v_m)$, $w = (w_1, \ldots, w_m)$ be simple sequences of vertices of $K_{n_1}$ such that there is a permutation $\pi : [1, r] \to [1, r]$ satisfying $p_{(m)}(v_{i,j}) = p_{(m)}(w_{i,j})$ for any $i \in [1, m]$. Then $v$ is similar to $w$.

Proof. Consider a bijection $\varphi : V(K_{n_1}) \to V(K_{n_1})$ such that $\varphi\{v_{i,j} : i \in [1, n]\} = \{v_{\pi(j), i} : i \in [1, n]\}$ for any $j \in [1, r]$ and $\varphi(v) = w_i$ for any $i \in [1, m]$. Clearly, $\varphi$ is an automorphism of $K_{n_1}$.

Theorem 7  If a graph $K_{n_1,n_2,n_3}$ with $n_1 \leq n_2 \leq n_3$ is $\text{ADCT}$, then either $(n_1, n_2, n_3) \in \{(1, 1, 3), (1, 1, 5)\}$ or $n_1 = n_2 = n_3$. 
Proof. Let $G := K_{n_1,n_2,n_3}$ and $e := |E(G)|$. Vertices of $G$ are of even degrees $n_1 + n_2$, $n_2 + n_3$ and $n_3 + n_1$, hence $n_1, n_2$ and $n_3$ are of the same parity.

1. Suppose that $n_1 = n_2 = 1$ and $n_3 = 6n + k$, where $n \geq 1$ and $k \in \{1, 3, 5\}$. The set $\text{Lct}(G)$ contains 3, 4, 8, since the following sequences are closed trails in $G$: $(v_{1,1}, v_{1,2}, v_{1,3}, v_{1,1})$, $(v_{1,1}, v_{1,3}, v_{1,2}, v_{2,3}, v_{1,1})$ and $(v_{1,1}, v_{1,3}, v_{1,2}, v_{2,3}, v_{1,1})$. As $e = 2(6n + k) + 1 = 3 \cdot 4n + 2k + 1$ and $2k + 1 \in \{3, 7, 11\}$, the set $\text{Sct}(G)$ contains one of the sequences $(3)^{4n+1}(4), (3)^{4n+1}(8)$, having $4n + 1 \geq 5$ odd terms. By Propositions 3 and 5 then $G$ is not ADCT.

2. Suppose $2 \leq n_1 < n_3$. The set $\text{Lct}(G)$ contains 3, 4, 5, since the following sequences are closed trails in $G$: $(v_{1,1}, v_{1,2}, v_{1,3}, v_{1,1})$, $(v_{1,1}, v_{1,2}, v_{2,1}, v_{1,3}, v_{1,1})$ and $(v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, v_{1,3}, v_{1,1})$. If $k := \lfloor \frac{e-3}{3} \rfloor$, then $e = 3k + r$, where $r \in [3, 5]$, and the sequence $(3)^k(r) \in \text{Sct}(G)$ has at least $k$ odd terms. Since $\text{mcb}(G) = n_1n_2$ (Proposition 5), the inequality $n_1n_2 < k$ implies that $G$ is not ADCT. Let us show that $n_1n_2 < k$. Indeed, this inequality is a consequence of the first from the following two equivalent inequalities: $3n_1n_2 + 5 < e = n_1n_2 + n_1n_3 + n_2n_3, 5 < n_1(n_3 - n_2) + n_2(n_3 - n_1)$. Now, since $n_3 - n_1 \geq 2$ and $n_3 - n_2 \geq 0$, the latter inequality is true; note that if $n_2 = 2$, then $n_1 = 2$ and on the righthand side we have $4(n_3 - n_1) \geq 8$.

Lemma 8 If $G$ is an even graph, then $\text{Lct}(G) \subseteq [3, |E(G)| - 3] \cup \{|E(G)|\}$.

Proof. Let $T$ be a closed trail in $G$. Clearly, $T$ has at least three edges (a subgraph of $G$ induced by $m$ edges, $m \in [1, 2]$, has at least two vertices of degree one). If $E(T) \neq E(G)$, then $G - T$ is a nonempty even subgraph of $G$. As any component of $G - T$ has at least three edges, we have $|E(T)| = |E(G)| - |E(G - T)| \leq |E(G)| - 3$.

Proposition 9 Let $G$ be a tripartite graph with tripartition $\{V_1, V_2, V_3\}$ and let $T$ be a closed trail in $G$ such that there are $i, j \in [1, 3], i \neq j$, and $x, y \in V(T)$, satisfying $V(T) \cap V_i = \{x\}$ and $V(T) \cap V_j = \{y\}$. Then either $(|E(T)|)_4 = 3$ and $xy \notin E(T)$ or $(|E(T)|)_4 = 0$.

Proof. With $k := 6 - i - j$ we have $\{i, j, k\} = [1, 3]$. If $z \in V(T) \cap V_k$, then $xz, yz \in E(T)$.

1. If $xy \notin E(T)$, then $T$ is a closed trail in the bipartite graph with bipartition $\{\{x, y\}, V_k\}$ and $(|E(T)|)_4 = 0$ since the vertices $x$ and $y$ must alternate in $T$.

2. If $xy \in E(T)$, then $T' := T - \{xy, yz, zx\}$ is an even graph and either $E(T') = \emptyset$ or $T'$ is connected. In the latter case we can proceed as in (1) with $T'$ instead of $T$. Therefore, in both cases $(|E(T')|)_4 = 0$ and $|E(T)| = |E(T')| + 3 \equiv 3 \pmod{4}$.

Proposition 10 The graphs $K_{1,1,3}$ and $K_{1,1,5}$ are ADCT.
Proof. Let \( n \in \{3, 5\} \) and let \( V_1, V_2, V_3 \) be parts of \( K_{1,1,n} \) with \( |V_1| = |V_2| = 1 \). If \( T \) is a closed trail in \( K_{1,1,n} \), then clearly \( V(T) \cap V_i = V_i, i = 1, 2 \). Therefore, by Proposition 9, \( (|E(T)|)_1 \in \{0, 3\} \). Further, if \( T' \) is a closed trail in \( K_{1,1,n} \) that is edge-disjoint with \( T \), then \( T + T' \neq \emptyset \). Thus, using edge-disjoint closed trails \((v_{11}, v_{12}, v_{13}, v_{11}), (v_{11}, v_{23}, v_{12}, v_{33}, v_{11}) \) in \( K_{1,1,3} \subseteq K_{1,1,5} \) and \((v_{11}, v_{43}, v_{12}, v_{53}, v_{11}) \) in \( K_{1,1,5} \), we can see that \( \text{Lct}(K_{1,1,3}) = \{3, 4, 7\} \), \( \text{Lct}(K_{1,1,5}) = \{3, 4, 7, 8, 11\} \) and, for both \( n = 3, 5 \), any sequence from \( \text{Sct}(K_{1,1,n}) \) is \( K_{1,1,n} \)-realisable.

3 The graph \( K_{5,5,5} \) is ADCT

In the main theorem of our paper we shall prove by induction on \( n \) that the graph \( G_n:=K_{(5,2^n), n} \in [0, \infty) \), is ADCT. For brevity we denote the part \( \{v_{i,j} : i \in [1, 5 \cdot 2^n], j \in [1, 3]\} \) of \( K_{(5,2^n), n} \), by \( V_{n,j} \). Consider three mutually isomorphic 16-edge graphs \( F_i \) (full lines in Fig. 1), \( F_2 \) (short-dashed lines), \( F_3 \) (long-dashed lines), and put \( F^3 := \bigcup_{i=1}^{3} F_i, j = 1, 2, 3 \); then \( F^3 = K_{5,5,5} - K_{3,3,3} \). Let \( i, j \in [1, 3] \). The tripartition of \( F_i \) is \( \{V_i^3, V_{i+1}^3 \cup V_{i+1}^2 \cup V_{i+2}^2\} \), where lower indices are taken modulo 3 in the set \( [1, 3] \) (such a convention will be used throughout the whole paper without explicitly mentioning it), \( V_i^1 \subseteq V_i^3, V_i^2 \cup V_i^3 = V_0, n \) and \( |V_i^2| = j \). The mapping \( \iota_i : V(F_i) \to V(F_i) \), determined by \( \iota_i(v_{j,k}) = v_{j,k-1+i} \) is a natural isomorphism from \( F_1 \) onto \( F_i \) with \( \iota_i(V_i^m) = V_i^m \) (for all four meaningful pairs \( (l, m) \)).

Proposition 11 If \( j \in [1, 3] \), then \( \text{mcb}(F^j) = 4j \).

Proof. Putting \( E_i := \{xy \in E(F_i) : x \in V_{i+1}^2, y \in V_{i+2}^2\} \), \( i = 1, 2, 3 \), it is easy to see that \( \bigcup_{i=1}^{3} E_k \subseteq \text{Cb}(F^3) \), and so \( \text{mcb}(F^3) \leq 4j \). On the other hand, the sets \( \{v_{2,1}, v_{4,2}, v_{4,3}\} \cup \{v_{2,1}, v_{5,2}, v_{5,3}\} \cup \{v_{3,1}, v_{4,2}, v_{5,3}\} \cup \{v_{3,1}, v_{4,3}, v_{5,2}\} \) induce in \( F_1 \) four pairwise edge-disjoint \( K_3 \)’s. Therefore, by Proposition 4, \( \text{mcb}(F^j) \geq j \cdot \text{mcb}(F_i) \geq j \cdot 4 \text{mcb}(K_3) = 4j \). A closed trail \( T \) in \( F^3 \) is said to be \( F^3 \)-extendable if \( V(T) \cap V^3 \neq \emptyset \), where \( V^3 := V_1^3 \cup V_2^3 \cup V_3^3 \). If \( i \in [1, 3] \), the graph \( F_i - V^3 \) is isomorphic to \( C_4 \), hence any closed trail in \( F_i \) of length \( \neq 4 \) is \( F^3 \)-extendable. An \( F^3 \)-extendable closed trail \( T \) is said to be \( F^3 \)-good if \( V(T) \cap V_{0,j} \neq \emptyset, j = 1, 2, 3 \). Since a graph, induced in \( G_0 \) by two of its parts \( V_{0,1}, V_{0,2}, V_{0,3} \), is bipartite, a closed trail in \( F^3 \) of an odd length is \( F^3 \)-good. Moreover, from the structure of the graph \( F_i, i \in [1, 3] \), it is easy to see that any closed trail in \( F_i \) of length \( \neq 4 \) is \( F^3 \)-good.

For \( i \in [1, 3] \), a closed trail \( T \) in \( F_i \) is said to be \( F_i \)-good if \( V(T) \cap V_i^3 \neq \emptyset \), \( V(T) \cap V_{i+1}^3 \neq \emptyset \) and \( V(T) \cap V_{i+2}^3 \neq \emptyset \). Evidently, an \( F_i \)-good closed trail in \( F_i \) is also \( F^3 \)-good.
Let \( i \in [1, 3] \), \( l \in \text{Lct}(F^3) \) and let \( T \) be a sequence of closed trails in \( F^3 \). Consider the following five conditions:

1. Any trail of \( T \) length \( \neq 4 \) is \( F_i \)-good.
2. Any trail of \( T \) is \( F^3 \)-extendable.
3. \( T \) has a \( l \)-trail \( T \) such that \( V(T) \cap V^1_{i+1} \neq \emptyset \), \( V(T) \cap V^2_{i+1} \neq \emptyset \), and \( V(T) \cap V^2_{i+2} \neq \emptyset \).
4. Any trail of \( T \) of length \( \neq 4 \) is \( F^3 \)-good.
5. \( T \) has an \( l \)-trail \( T \) such that either \( V(T) \cap V^2_i \neq \emptyset \) and \( V(T) \cap V^3_i \neq \emptyset \) or \( V(T) \cap V^2_{i+1} \neq \emptyset \).

For \( i \in [1, 3] \), an \( F_i \)-realisation \( T \) of a sequence \( L \in \text{Sct}(F_i) \) is said to be good \([l\text{-good}] \) if \( T \) satisfies (C1.i) and (C2), [(C1.i), (C2) and (C3.i.l)].

**Theorem 12** Let \( i \in [1, 3] \), \( L = (l_1, \ldots, l_p) \in \text{Sct}(F_i) \) and \( j \in [1, p] \). Then the following hold:

1. There is a good \( F_i \)-realisation of \( L \).
2. If \( \text{nd}(L) \neq (3)^4(4) \) and \( l_j \geq 4 \), there is an \( l_j \)-good \( F_i \)-realisation of \( L \).

**Proof.** We are able to prove the statement of our Theorem for any sequence \( L = (l_1, \ldots, l_p) \) of integers from \([3,13] \cup \{16\}\) such that \( \sum_{k=1}^p l_k = 16 \). This in turn implies that \( \text{Lct}(F_i) = [3,13] \cup \{16\} \). We proceed in this way throughout the whole paper: when working with a graph \( G \), we do not take care of the structure of \( \text{Lct}(G) \), but in all cases it turns out that \( \text{Lct}(G) \) is of maximal extent, i.e., \( \text{Lct}(G) = [3,|E(G)|-3] \cup \{|E(G)|\} \) (cf. Lemma 8).

Using Lemma 1 we may suppose without loss of generality that \( L \) is non-decreasing (i.e., \( \text{nd}(L) = L \)). Examples of appropriate \( F_i \)-realisations of \( L \) are presented at http://umv.science.upjs.sk/adct and can be seen by clicking on Graph \( F_i \). To pass to \( F_i \)-realisations, \( i \in [2,3] \), consider the isomorphism \( \iota_i \).

The next proposition shows that the exclusion of the sequence \((3)^4(4)\) in Theorem 12.2 is unavoidable.

**Proposition 13** If \( i \in [1,3] \) and \( T \) is a \( 4 \)-trail of an \( F_i \)-realisation of \((3)^4(4)\), then \( V(T) = V_1^1 \cup V_{i+1}^1 \cup V_{i+2}^2 \).

**Proof.** The statement follows from the fact that both \( v_{1,i} \) and \( v_{1,i+1} \) belong in \( F_i \) only to trails of length \( \geq 4 \).

Let \( L \in \text{Sct}^*(F^2) \) and let \( l \) be a term of \( L \). An \( F^2 \)-realisation \( T \) of \( L \) is said to be \( l \)-good if \( T \) satisfies (C2), (C4) and (C5).

**Theorem 14** If \( L = (l_1, \ldots, l_p) \in \text{Sct}^*(F^2) \) and \( i \in [1,p] \), there exists an \( l_i \)-good \( F^2 \)-realisation of \( L \).

**Proof.** (1) Assume there is a decomposition \( \{I^1, I^2\} \) of \([1,p]\) such that \( \sum_{i \in I^1} l_i = 16, j = 1,2 \). We may suppose without loss of generality that \( i \in I^1 \Leftrightarrow l_i = 3 \). Let \( T^1 \) be a good \( F_1 \)-realisation of \( L(I^1) \) and, provided that \( l_i \geq 4 \) and \( \text{nd}(L(I^1)) \neq (3)^4(4) \) \( [l_i = 3 \text{ or } \text{nd}(L(I^1)) = (3)^4(4)] \), let \( T^2 \) be an \( l_i \)-good \( F_2 \)-realisation of \( L(I^2) \); both realisations do exist by Theorem 12. Then \( T := T^1 T^2 \) is an \( l_i \)-good \( F^2 \)-realisation of \( L \). First note that any trail of \( T^j, j = 1,2 \), satisfies (C2) and (C4) (as a consequence of (C1,j)). Further, \( T \) has an \( l_i \)-trail \( T \) such that \( V(T) \cap V_1^1 \neq \emptyset \) and \( V(T) \cap V_2^2 \neq \emptyset \) (if \( l_i \geq 4 \), see (C3.2,4) or Proposition 13) or \( V(T) \cap V_2^2 \neq \emptyset \) (if \( l_i = 3 \), see (C1,j)).

(2) Now assume there is a decomposition \( \{\{r\}, I^1, I^2\} \) of \([1,p]\) such that \( \sum_{i \in I} l_i \leq 13, j = 1,2 \). Putting \( l_r^1 := 16 - \sum_{i \in I} l_i \) and \( L^j := L(I^j)(l_r^j) \), we have \( L^j \in \text{Sct}(F_j), j = 1,2 \), and \( l_r^1 + l_r^2 = 32 - \sum_{i \in I} l_i = \theta_r \). We may suppose without loss of generality that \( i \neq r \Rightarrow (i \in I^1 \Leftrightarrow l_i = 3) \).

If \( l_r^1 \geq 4 \) and \( \text{nd}(L^1) \neq (3)^4(4) \) \( [l_r^1 = 3 \text{ or } \text{nd}(L^1) = (3)^4(4)] \), let \( T^1 \) be an \( l_r^1 \)-good \( F_1 \)-realisation of \( L^1 \).

If \( i \in I^2 \) and \( \text{nd}(L^2) \neq (3)^4(4) \) \( [i = r, l_r^2 \geq 4 \text{ and } \text{nd}(L^2) \neq (3)^4(4)] \) \{otherwise\}, let \( T^2 \) be an \( l_i \)-good \( \{l_r \text{-good}\} \{\text{a good}\} F_2 \)-realisation of \( L^2 \).
If \( i \neq r \) and \( l_i = 3 \), let \( T_i \) be any \( l_i \)-trail of \( T^1 \); it satisfies \( V(T_i) \cap V^2 \neq \emptyset \) (see (C1.1)). If \( i \neq r \) and \( l_i \geq 4 \), let \( T_i \) be an \( l_i \)-trail of \( T^2 \) satisfying \( V(T_i) \cap V^2 \neq \emptyset \) and \( V(T_i) \cap V^3 \neq \emptyset \) (see (C3.2.l_i) or Proposition 13). Further, \( T^3 \) contains an \( l^j_i \)-trail \( T^j_i \) (if \( T_i \) is defined at all, i.e., if \( i \neq r \), \( j = 1, 2 \)), with the following two properties: either \( V(T^j_i) \cap V^2 \neq \emptyset \) and \( V(T^j_i) \cap V^3 \neq \emptyset \) (if \( l^j_i \geq 4 \), see (C3.1.l^j_i) or Proposition 13) or \( V(T^j_i) \cap V^2 \neq \emptyset \) (if \( l^j_i = 3 \), see (C1.1)); either \( V(T^j_i) \cap V^3 \neq \emptyset \) (if \( nd(L) = (3)^4(4) \), see Proposition 13) or \( V(T^j_i) \cap V^3 \neq \emptyset \) (otherwise, see (C1.2) or (C3.2.l^j_i)).

We may suppose without loss of generality that \( l^1_i = 3 \) \( \Rightarrow \) \( nd(L^2) \neq (3)^4(4) \) (otherwise, if \( m \in I^2 \) and \( l_m = 3 \), then \( l_m + \sum_{k \in I^1} l_k = 16 \) and the case (1) applies). Therefore, we have either \( V(T^1_i) \cap V^2 \supseteq V^2 \neq \emptyset \) or \( V(T^1_i) \cap V^3 \neq \emptyset , j = 1, 2 \). In the latter case, since \( V^3 \) is a similarity class in \( F \), we may suppose without loss of generality that \( V(T^1_i) \cap V^3 \neq \emptyset \).

Thus, in both cases there is a trail \( T_r \in T^1_i + T^3_i \).

Denote as \( \hat{T}^j \) the sequence obtained by deleting \( T^j_i \) from \( T^j \), \( j = 1, 2 \). Then \( T := \hat{T}^1 \hat{T}^2(T_r) \) is an \( l_i \)-good \( F^2 \)-realisation of \( L(I^1)L(I^2)(l_r) \sim L \).

First, the only trail of \( T \), that is neither in \( T^1 \) nor in \( T^2 \), is an \( F^3 \)-extendable trail \( T_r \) of length \( \geq 6 \) with \( V(T_r) \supseteq V^2 \), \( j = 1, 2 \). If \( T^1_i \) is \( F^1 \)-good, it is also \( F^3 \)-good, hence so is \( T_r \). In the opposite case \( l^1_i = 4 \), \( nd(L^1) = (3)^4(4) \) and, by Proposition 13, both \( T^1_i \) and \( T_r \) are \( F^3 \)-good. If \( i \neq r \), the \( l_i \)-trail \( T_i \) satisfies (C5.i). Finally, if \( i = r \), the \( l_r \)-trail \( T_r \) satisfies (C5.l_r): with \( l^1_r \geq 4 \) we have \( V(T_r) \cap V^1 \supseteq V^2 \cap V^3 \neq \emptyset \) and \( V(T_r) \cap V^3 \supseteq V^2 \cap V^3 \neq \emptyset \) (by (C3.2.l_r) or Proposition 13), while \( l^2_i = 3 \) implies \( V(T_r) \cap V^2 \supseteq V^2 \cap V^3 \neq \emptyset \) (by (C1.1) or (C3.l_r)); note that here we may suppose without loss of generality that \( nd(L^1) = (3)^4(4) \).

(3) In what follows we suppose that the sequence \( L \) is nondecreasing and the assumptions of (1) and (2) are not fulfilled. Let \( q \in [1,p] \) be such that \( \sum_{i=1}^{q-1} l_i \leq 13 \) and \( \sum_{i=q}^{p} l_i > 13 \). Then \( \sum_{i=1}^{q-1} l_i < 19 \) then \( \sum_{i=q}^{p} l_i < 13 \) and (2) is fulfilled with \( I^1 := [1,q - 1] \) and \( I^2 := [q+1,p] \). Let \( M_k \) be the set of all nondecreasing sequences with terms from \([3, \infty)\) adding up to \( k \) and let \( S_k \) be the set of all nondecreasing sequences \( L = (l_1, \ldots, l_p) \in \text{Sct}^*(F^2) \) such that \( \sum_{i=1}^{p} l_i = k \) for some \( q \in [1,p - 1] \) and \( L \) violates the assumptions of both (1) and (2). We are going to determine the structure of \( S_k \), \( k = 14, 15, 17, 18 \).

(31) If \( L \in S_{14} \) then \( l_j \in [l_j, l_j + 3, l_j + 4] \) for any \( j \in [q+1,p] \neq \emptyset \). Indeed, if \( l_j = l_j + 2 \) for some \( i \in [1,q] \) and \( j \in [q + 1,p] \), then (1) is fulfilled with \( I^1 := [1,q] \{ i \} \cup \{ j \} \) and \( I^2 := [q + 1,p] \{ j \} \cup \{ i \} \). If \( l_j \geq l_j + 5 \) for some \( i \in [1,q] \) and \( j \in [q + 1,p] \), then with \( I^1 := [2,q] \), \( I^2 := [1] \cup \{ q + 1,p \} \{ j \} \). We have \( \sum_{k \in I^1} l_k \leq 11 \) and \( \sum_{k \in I^2} l_k = 18 - (l_j - l_1) \leq 13 \). By analysing all sequences \((l_1, \ldots, l_q)\) from \( M_{14} \) such that \( l_q - l_1 \leq 4 \) (the above
intersection must be nonempty), we obtain $S_{14} = \{(3)^3(5)(6)^3, (3)^2(4)^3(7)^2, (3, 5)(6)^4, (5)(9)^3, (6, 8)(9)^2, (7)^3(11), (7)^2(8, 10), (14, 18)\}$. As (somewhat typical) examples of this analysis consider the following:

(a) If $(l_1, l_2, l_3, l_4) = (3)^3(5)$, then $l_j \in \{3, 4, 6, 7\} \cap \{5, 6, 8, 9\} = \{6\}$ for any $j \in [5, p]$, hence $p = 7$ and $L = (3)^3(5)(6)^3$.

(b) If $(l_1, l_2, l_3) = (4)^2(6)$, then $l_j \in \{4, 5, 7, 8\} \cap \{6, 7, 9, 10\} = \{7\}$ for any $j \in [4, p]$, and, since $7 \not\in 18$, there is no such $L$.

(c) If $(l_1, l_2) = (7)^2$, then $l_j \in \{7, 8, 10, 11\}$ for any $j \in [3, p]$, hence $p = 4$ and $(l_3, l_4)$ is either $(7, 11)$ or $(8, 10)$, which yields $L = (7)^3(11)$ or $L = (7)^2(8, 10)$.

(32) If $L \in S_{15}$, then $l_j \in \bigcap_{i=1}^q \{l_i, l_i + 2, l_i + 3\}$ for any $j \in [q + 1, p]$, hence $l_q - l_1 \leq 3$. In the same way as above we obtain the set $S_{15} = \{(3)^3(5), (3)^3(5)(6)^2, (5)(7)^3(15, 17)\}$; notice that $(3)^3(5) \not\in \text{Sct}^*(F^2)$ and, consequently, $(3)^3(5) \not\in S_{15}$.

(33) If $L \in S_{17}$, then $l_q = 17 - \sum_{i=1}^{q-1} l_i \geq 4$. Further, $l_j \in \{l_q, l_q + 1\}$ for any $j \in [q + 1, p]$; otherwise with $I^1 := [1, q - 1]$ and $I^2 := [q, p] - \{j\}$ we have $\sum_{i \in I^1} l_i \leq 13$ and $\sum_{i \in I^2} l_i = 15 - (l_j - l_q) \leq 13$. We can easily find that $S_{17} = \{(3)^3(5)^4, (3)^2(4)(7)^3(8), (3, 4)(5)^5, (4)(5)^3, (3)^3(7)^3(8), (4)(6)(7)^2(8), (5)^2(7)^3(8), (3, 14, 15)\}$.

(34) If $L \in S_{18}$, then $l_q \geq 5$ and $l_j = l_q$ for any $j \in [q + 1, p]$, otherwise $\sum_{i=q}^{p-1} l_i \leq 13$. Thus $S_{18} = \{(3)^3(5)(7)^3, (3)(4)(7)(3)^3, (4)(7)^4, (5)(6)(7)^3, (4)(14)^3\}$.

Examples of $l_r$-good $F^2$-realisations of sequences $L \in S_{14} \cup S_{15} \cup S_{17} \cup S_{18}$ can be seen by clicking on Graph $F^2$ (see the address presented in the proof of Theorem 12).

An $F^3$-realisation $T$ of a sequence $L \in \text{Sct}^*(F^3)$ is said to be good if $T$ satisfies (C2) and (C4).

**Theorem 15** If $L \in \text{Sct}^*(F^3)$, there exists a good $F^3$-realisation of $L$.

**Proof.** Let $L = (l_1, \ldots, l_p)$.

(1) Assume there is a decomposition $\{I^1, I^2\}$ of $[1, p]$ such that $\sum_{i \in I^1} l_i = 16$, $\sum_{i \in I^2} l_i = 32$, and $\text{nd}(L(I^2)) \neq (3)^9(5)$ (so that $L(I^2) \in \text{Sct}^*(F^2)$).

Consider an arbitrary $i \in I^2$. By Theorems 12 and 14 there exists a good $F_3$-realisation $T^1$ of $L(I^1)$ and an $l_r$-good $F^2$-realisation $T^2$ of $L(I^2)$. Then $T^1T^2$ is a good $F^3$-realisation of $L(I^1)L(I^2) \sim L$.

(2) Let there exist a decomposition $\{\{r\}, I^1, I^2\}$ of $[1, p]$ such that $\sum_{i \in I^1} l_i \leq 13$, $\sum_{i \in I^2} l_i \leq 29$ and $\text{nd}(L(I^2)) \neq (3)^8(5), (3)^9$. If $l_r^j := 16j - \sum_{i \in I^1} l_i$, $j = 1, 2$, then $l_r^1 + l_r^2 = 48 - \sum_{i \in I^1 \cup I^2} l_i = l_r$. Put $L^2 := L(I^1)(l_r^j)$, $j = 1, 2$. If $l_r^1 \geq 4$ and $\text{nd}(L^1) \neq (3)^4(4)$ [$l_r^2 = 3$ or $\text{nd}(L^1) = (3)^4(4)$], let $T^1$ be an $l_r^1$-good [a good] $F_3$-realisation of $L^1$. Then $T^1$ has an $l_r^1$-trail $T^1_i$ such that $V(T^1_i) \cap V_2 \neq \emptyset$ and either $V(T^1_i) \cap V_3 \neq \emptyset$ (if $\text{nd}(L^1) = (3)^4(4)$) or
\( V(T_1^1) \cap V_1^2 \neq \emptyset \) (otherwise). Further, let \( T^2 \) be an \( \ell^2 \)-good \( F^2 \)-realisation of \( L^2 \) and let \( T_2^2 \) be an \( \ell^2 \)-trail of \( T^2 \) satisfying (C5.\( \ell^2 \)). Then either \( V(T_1^1) \cap V(T_2^2) \supseteq V_3^1 \neq \emptyset \) or there is \( k \in [1,2] \) such that \( V(T_2^2) \cap V_k^2 \neq \emptyset \), \( j = 1,2 \). In the latter case, as both \( V_1^2 \) and \( V_2^2 \) are similarity classes in \( F_3 \), we may suppose without loss of generality that \( V(T_1^1) \cap V(T_2^2) \neq \emptyset \). Thus, in any case there is a trail \( T_r \in T_1^3 \cap T_2^2 \). If \( \hat{T}^3 \) results from \( T^2 \) by deleting \( T_r \), then \( \hat{T}^1 \hat{T}^2(T_r) \) is a good \( F^3 \)-realisation of \( L(I^1) L(I^2)(l_r) \sim L \).

(3) Suppose that \( L \) is nondecreasing and violates the assumptions of both (1) and (2). Let \( q \in [1,\rho] \) be defined by the inequalities \( \sum_{i=1}^{q} l_i \leq 13 \) and \( s := \sum_{i=1}^{q} l_i > 13 \). Then \( s \neq 16 \), for otherwise \( l_i \geq l_q \geq 4 \) for any \( i \in [q+1,\rho] \), and with \( I^1 := [1,q] \) and \( I^2 := [q+1,\rho] \) we are in the case (1). Also, \( s \leq 18 \), since \( s \geq 19 \) leads to \( l_i \geq l_q \geq 6 \) for any \( i \in [q+1,\rho] \), and then with \( I_1 := [1,q-1] \) and \( I_2 := [q+1,\rho] \) we are in the case (2). Thus, \( s \in \{14,15,17,18\} \) and, similarly as in the proof of Theorem 14, we conclude that \( L \) is one of the following sequences: \((3)^{12}(6)^2, (3)^{10}(6)^3, (3)^6(6)^4, (3)^6(6)^5, (3)^4(6)^6, (3)^2(6)^7, (3)(5)^9, (4)^2(5)^8, (5)^8(8), (6)^8, (14)(17)^2\); for good \( F^3 \)-realisations of these sequences a reader is referred to Graph \( F^3 \).

Put \( P := \{(0,0), (0,3), (3,0)\} \).

**Proposition 16** Let \( n \in [2,\infty) \) and let \( T_1, T_2 \) be edge-disjoint closed trails in the graph \( K_{n,n,n} \) with the tripartition \( \{V_1,V_2,V_3\} \). Then the following hold:

1. There is \( p \in [1,3] \) such that \( \{V(T_1),V(T_2)\} \) has a system of distinct representatives in \( V_p \).

2. If there is \( m \in [1,2] \) such that \( V(T_m) \cap V_i \neq \emptyset \), \( i = 1,2,3 \), and \( \{\{E(T_1)\}_i \cup \{E(T_2)\}_i\} \notin P \), then there are \( q, r \in [1,3] \), \( q \neq r \), such that \( \{V(T_1),V(T_2)\} \) has a system of distinct representatives in both \( V_q \) and \( V_r \).

**Proof.** Set \( c_i := (|E(T_i)|)_i \) and let \( \rho_i : [1,3] \to [1,3] \) be a bijection satisfying \( x(i,1) \geq x(i,2) \geq x(i,3) \), where \( x(i,j) := |V(T_i) \cap V_{\rho(i,j)}|, i = 1,2,3 \). Clearly, \( x(i,2) \geq 1, i = 1,2 \).

1. If there is \( i \in [1,2] \) such that \( x(i,2) \geq 2 \), then there exists \( j \in [1,2] \) such that \( V(T_{3-i}) \cap V_{\rho(i,j)} \neq \emptyset \), and we are done with \( p := \rho_i(j) \). In the opposite case \( x(i,2) = x(i,3) = 1, i = 1,2 \). Let \( i \in [1,2] \) be such that \( x(i,1) = \max(x(1,1), x(2,1)) \). If \( x(i,1) \geq 2 \), then, since \( V(T_{3-i}) \cap V_{\rho(i,1)} \neq \emptyset \), we can take \( p := \rho(i,1) \). So, suppose that \( x(k,j) = 1, k = 1,2, j = 1,2,3 \). As \( E(T_1) \cap E(T_2) = \emptyset \), we have \( V(T_1) \neq V(T_2) \), and the existence of \( p \) follows.

2. Here we have \( x(m,3) \geq 1 \). If \( x(3-m,3) = 0 \), then \( T_{3-m} \) is a bipartite graph, \( x(3-m,2) \geq 3 \) and we are done with \( q := \rho_{3-m}(1) \) and \( r := \rho_{3-m}(2) \). So, suppose that \( x(3-m,3) \geq 1 \) and let \( \rho : [1,3] \to [1,3] \) be a bijection satisfying \( x(1) \geq x(2) \geq x(3) \), where \( x(j) := |X_j| \) and \( X_j := (V(T_1) \cup V(T_2)) \cap V_{\rho(j)}, j = 1,2,3 \). If \( x(2) = x(3) = 1 \), then \( X_j = V(T_1) \cap V_{\rho(j)} = \)}
\( V(T_2) \cap V_{\rho(j)} = \{x_j\}, j = 2, 3 \). From Proposition 9 it follows that \( c_i \in \{0, 3\} \) and \( c_i = 3 \Rightarrow x_2x_3 \in E(T_i), i = 1, 2 \). Since \( E(T_1) \cap E(T_2) = \emptyset \), we obtain \((c_1, c_2) \in P\), a contradiction. Thus, \( x(2) \geq 2 \), and it suffices to take \( q := \rho(1) \) and \( r := \rho(2) \).

If \( l \in [3, 5] \), then any closed trail of length \( l \) is in fact a cycle. Let us identify all \( 2l \) closed trails of length \( l \) in \( K_{3,3,3} \) having the same edge set and denote by \( \mathcal{C}_l \) the set of all (representatives of) closed trails of length \( l \) in \( K_{3,3,3} \). Then \( c_l := |\mathcal{C}_l| \) is equal to the number of \( l \)-element subsets of \( E(K_{3,3,3}) \) inducing a cycle of length \( l \).

Let \( T \) be a closed trail of length \( l \in [3, 5] \), let \( V_i := V(T) \cap \{v_{1,i}, v_{2,i}, v_{3,i}\} \) and \( v_i := |V_i|, i = 1, 2, 3 \). Let \( i, j, k \in [1, 3] \) be such that \( \{i, j, k\} = [1, 3] \) and \( v_i \leq v_j \leq v_k \). If \( l \in \{3, 5\} \), \( T \) is a non-bipartite subgraph of \( K_{3,3,3} \), therefore \( v_i \geq 1 \). If \( l = 3 \), then \( v_1 = v_2 = v_3 = 1 \), and so \( c_3 = \binom{3}{1} = 3 \). If \( l = 4 \), then either \( v_1 = 0 \) and \( v_j = v_k = 2 \) or \( v_i = v_j = 1 \) and \( v_k = 2 \), so that \( c_4 = 3 \cdot \binom{3}{2} + 3 \cdot \binom{3}{1} \cdot \binom{3}{1} = 108 \). If \( l = 5 \), then from Proposition 9 it follows that \( v_1 = 1, v_j = v_k = 2 \). The edge set of \( T \) is uniquely determined by a simple sequence \( \prod_{i=1}^{5} (x_i) \) such that \( x_1, x_3 \in V_j, x_2, x_4 \in V_k \) and \( x_5 \in V_i \). Therefore, \( c_5 = 3 \cdot (3 \cdot 2 \cdot 2 \cdot 3) = 324 \).

A \( K_{3,3,3} \)-realisation \((T_1, \ldots, T_p)\) of a sequence \((l_1, \ldots, l_p) \in \text{Sct}(K_{3,3,3})\) is said to be \textit{good} if, for any \( i \in [1, p] \), \( E(T_i) \) is a union of edge sets of some (edge-disjoint) trails from \( \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5 \).

**Theorem 17** If \( L = (l_1, \ldots, l_p) \in \text{Sct}(K_{3,3,3}) \), there exists a good \( K_{3,3,3} \)-realisation of \( L \).

**Proof.** Good \( K_{3,3,3} \)-realisations of sequences from \( \text{Sct}(K_{3,3,3}) \) have been found by a computer; to see them, click on Graph \( K_{3,3,3} \). If \( n \in [0, \infty) \), we have \( c_n := |E(G_n)| = 3 \cdot (5 \cdot 2^n)^2 \). A closed trail \( T \) in \( G_n \) is said to be \( G_n \)-\textit{good} if \( V(T) \cap V_{\alpha} \neq \emptyset \), \( \alpha \in [1, 5] \). Clearly, if \( T \) is of an odd length, it is \( G_n \)-\textit{good}. A \( G_n \)-realisation \( T \) of a sequence from \( \text{Sct}(G_n) \) is said to be \textit{good} if any trail of \( T \) of length \( \neq 0 \) (mod 4) is \( G_n \)-\textit{good}. The graph \( G_n \) is said to be \textit{strongly} ADCT provided that (i) for any \( L \in \text{Sct}(G_n) \) there is a good \( G_n \)-realisation of \( L \) and (ii) if \((t, 1), \ t \in [1, 5] \), is a position of Table 1 (regarded as a \( 5 \times 2 \) matrix) containing a sequence \( L = (l_1, \ldots, l_p) \in \text{Sct}(G_n) \) \( n \) has to be in a specified congruence class modulo 3 such that the exponent of \((7) \) is an integer), there is a good \( G_n \)-realisation \((T_1, \ldots, T_p)\) of \( L \) satisfying the conditions presented in the position \((t, 2)\) of Table 1.

**Theorem 18** The graph \( G_0 \) is strongly ADCT.

**Proof.** Consider a sequence \( L = (l_1, \ldots, l_p) \in \text{Sct}(G_0) \).

1. Suppose there is a decomposition \( \{I^1, I^2\} \) of \([1, p]\) such that \( \sum_{i \in I^1} l_i = 48 \) and \( \sum_{i \in I^2} l_i = 27 \). If, moreover, \( L(I^1) \in \text{Sct}(F^3) \), then, by Theorem 15,
there exists a good $F^3$-realisation $T^1$ of $L(I^1)$ and, by Theorem 17, there exists a good $K_{3,3,3}$-realisation $T^2$ of $L(I^2)$. The above additional assumption is true, if $f_3(L) \leq 8$, since in such a case $L(I^1)$ contains at most twelve odd terms (note that $\frac{48-8}{3} = 4$ and, by Proposition 11, $mcb(F^3) = 12$). Then $T^1T^2$ is a good $G_0$-realisation of the sequence $L(I^1)L(I^2)$. Indeed, trails of $T^1$ of length $\neq 4$ are $G_0$-good since they are $F^3$-good, and trails of $T^2$ of length $\neq 0$ (mod 4) are $G_0$-good as they are composed of at least one trail of length 3 or 5. So, we may suppose that $f_3(L) \geq 9$ and choose $I^2$ so that $L(I^2) = (3)^9$. If, additionally $L(I^1) \in \text{Sct}(F_3) - \text{Sct}^*(F_3)$, then $L(I^1)$ has more than 12 odd terms and it is easy to see that for $\text{nd}(L)$ we have only six possibilities, namely $(3)_{25}$, $(3)_{23}(6)$, $(3)_{22}(4, 5)$, $(3)_{22}(9)$, $(3)_{21}(5, 7)$ and $(3)_{20}(5, 3)$. Examples of corresponding good $G_0$-realisations are available by clicking on Graph $G_0$.

(2) Let there be a decomposition $\{\{r\}, I^1, I^2\}$ of $[1, p]$ such that $\sum_{i \in I^1} l_i \leq 45$, $\sum_{i \in I^2} l_i \leq 24$. If $l^1_r := 48 - \sum_{i \in I^1} l_i$ and $l^2_r := 27 - \sum_{i \in I^2} l_i$, then $l_r = l^1_r + l^2_r$. We may suppose that $f_3(L) \leq 8$ (otherwise we are in the case (1)), and then $L^1 := L(I^1)(l^1_r) \in \text{Sct}^*(F^3)$. By Theorem 15 there exists a good $F^3$-realisation $T^1(T^1_r)$ of $L^1$ with $V(T^1_r) \cap V(K_{3,3,3}) = V(T^1_r) \cap V^3 \neq \emptyset$ ($T^1_r$ is $F^3$-extendable). Further, by Theorem 17 there is a $K_{3,3,3}$-realisation $T^2(T^2_r)$ of the sequence $L^2 := \langle l^2_r \rangle \in \text{Sct}(K_{3,3,3})$. By Proposition 6 we may suppose without loss of generality that $V(T^1_r) \cap V(T^2_r) \neq \emptyset$, hence there is a trail $T_r \in T^1_r + T^2_r$. If $(l_r)_4 \neq 0$, then $(l_r)_j \neq 0$ for some $j \in [1, 2]$, and so $V(T_r) \cap V_{0,i} \supseteq V(T^1_r) \cap V_{0,i} \neq \emptyset$, $i = 1, 2, 3$. Thus, $T^1T^2(T_r)$ is a good $G_0$-realisation of $L(I^1)L(I^2)(l_r) \sim L$.

(3) Now suppose that $L$ is a nondecreasing sequence violating the assumptions of both (1) and (2). Let $q \in [1, p]$ be defined by the inequalities $\sum_{i=1}^{q-1} l_i \leq 45$ and $\sum_{i=1}^q l_i > 45$. Then $\sum_{i=1}^q l_i \in \{46, 47, 49, 50\}$. Proceeding analogously as in the proof of Theorem 14 we exhibit five possibilities for $L$, namely $(4)_{3}(7)^9$, $(4)_{7}(9)(8)$, $(5)_{15}$, $(5)_{7}(10)$, $(25)^3$. For examples of remaining appropriate $G_0$-realisations see again Graph $G_0$ (note that for $n = 0$ the

| $\begin{array}{ccc} (3)^4(7)_{2-12} & v_{i,1} \in V(T_i), & i = 1, 2, 3 \\ (3)^3(4)(7)_{2-12} & v_{i,1} \in V(T_i), & i = 1, 2, 3 \\ (4)^6(7)_{2-24} & v_{i,1} \in V(T_i), & i = 1, 2, 3, v_{i,2} \in V(T_{3+i}), & i = 1, 2, v_{i,3} \in V(T_6) \\ (4)^5(7)_{2-20} & v_{i,1} \in V(T_i), & i = 1, 2, 3, v_{i,2} \in V(T_{3+i}), & i = 1, 2 \\ (4)^3(7)_{2-12} & v_{i,1} \in V(T_i), & i = 1, 2, 3 \\ \end{array}$ |
“real” sequences of Table 1 are \((4)^3(7)^9\) and \((3)^4(7)^9\). ■

4 The main theorem

Our main Theorem will be proved by induction on \(n\). Therefore, we need to know how to construct \(G_{n+1}\) from \(G_n\). Consider two copies \(G_1^n, G_2^n\) of the graph \(G_n\) such that the tripartition of \(G_n^k\) is \(\{V_{n,j}^k : j \in [1,3]\}\) with \(V_{n,j}^k = \{v_{n,j}^k : i \in [1,5 \cdot 2^n]\}\), where \(v_{i}^1 = v_{i}^2, v_{i}^1 = v_{i}^2\), for any \(i \in [1,5 \cdot 2^n]\) and \(V_{n,j_1}^1 \cap V_{n,j_2}^2 = \emptyset\) for any \(j_1, j_2 \in [2,3]\). Then \(\varphi_n^k : V(G_n^k) \rightarrow V(G_n^k)\), with \(\varphi_n^k(v_{i,j}) = v_{i,j}^k\), for any \(i \in [1,5 \cdot 2^n]\) and \(k \in [1,3]\), is a natural isomorphism from \(G_n\) onto \(G_n^k\), \(k = 1, 2\). Let \(H_n := G_1^n \cup G_2^n\) (see Fig. 2); the subsets \(V_{n,j}^k, j \in [2,3], k \in [1,2]\), are called eccentric parts of \(H_n\), while \(V_{n,1}^1 = V_{n,1}^2\) is the central part of \(H_n\).

The graph \(H_n\) is tripartite with one possible tripartition \(\{W_{n,1}, W_{n,2}, W_{n,3}\}\), where \(W_{n,1} := V_{n,1}^1\) and \(W_{n,j} := V_{n,1}^1 \cup V_{n,j}^2, j = 2, 3\).

Suppose that \(L = (l_1, \ldots, l_p) \in \text{Sct}(H_n)\) is an \(H_n\)-realisable sequence and \(T = (T_1, \ldots, T_p)\) is an \(H_n\)-realisation of \(L\). Consider two terms \(l, l'\) of \(L\). The \(H_n\)-realisation \(T\) of \(L\) is said to be \((l, l')\)-global if there is an \(l\)-trail \(T_i\) and an \(l'\)-trail \(T_j\) of \(T\), \(i \neq j\), such that \(V(T_i) \cap (V_{n,2}^k \cup V_{n,3}^k) \neq \emptyset\) and \(V(T_j) \cap (V_{n,2}^{3-k} \cup V_{n,3}^{3-k}) \neq \emptyset\) for some \(k \in [1,2]\) (or, equivalently, \(V(T_i) \cup V(T_j) \subseteq V(G_n^k)\), \(k = 1, 2\)).

Proposition 19 Let \(n \in [0, \infty), m \in [1,5 \cdot 2^n]\), and let \(v = (v_1, \ldots, v_m), w = (w_1, \ldots, w_m)\) be simple sequences of vertices of the graph \(H_n\). Then \(v\) is similar to \(w\) whenever one of the following conditions is fulfilled:

1. \(\{v_i : i \in [1, m]\} \subseteq V_{n,j_1}^k\) and \(\{w_i : i \in [1, m]\} \subseteq V_{n,j_2}^k\) for some \(j_1, j_2 \in [2,3]\) and \(k_1, k_2 \in [1,2]\);
2. \(m = 2\), \(v_1 \in V_{n,j_1}^k\), \(v_2 \in V_{n,\overline{j_1}}^{3-k}\), \(w_1 \in V_{n,j_2}^{3-k}\), \(w_2 \in V_{n,\overline{j_2}}^{3-k}\) for some \(j_1, j_2 \in [2,3]\), \(k_1, k_2 \in [1,2]\);
3. \(m = 2\), \(v_1 \in V_{n,j_1}^k\), \(v_2 \in V_{n,\overline{j_1}}^{3-k}\), \(w_1 \in V_{n,j_2}^{3-k}\), \(w_2 \in V_{n,\overline{j_2}}^{3-k}\) for some \(j_1, j_2 \in [2,3]\), \(k_1, k_2 \in [1,2]\);
4. \(m = 2\), \(v_1 \in V_{n,j_1}^k\), \(v_2 \in V_{n,\overline{j_1}}^{3-k}\), \(w_1 \in V_{n,j_2}^{3-k}\), \(w_2 \in V_{n,\overline{j_2}}^{3-k}\) for some \(j_1, j_2 \in [2,3]\), \(k_1, k_2 \in [1,2]\).

Proof. There is a bijection \(\varphi : V(H_n) \rightarrow V(H_n)\) such that \(\varphi(V_{n,1}^1) = V_{n,1}^1\), any eccentric part of \(H_n\) is mapped under \(\varphi\) to an eccentric part of \(H_n\), \(\varphi(V_{n,2}^k \cup V_{n,3}^k) \subseteq \{V_{n,2}^k \cup V_{n,3}^k, V_{n,2}^3 \cup V_{n,3}^{3-k}\}, k = 1, 2\), and \(\varphi(v_i) = w_i\) for any \(i \in [1, m]\); clearly, \(\varphi\) is an automorphism of \(H\). ■

Now consider two copies \(H_1^n, H_2^n\) of the graph \(H_n\) such that, for both \(l = 1, 2\), \(H_l^n\) has parts \(V_{n,j}^{k,l} = \{v_{i,j}^{k,l} : i \in [1,5 \cdot 2^n]\}\), \(j = 1, 2, 3, k = 1, 2\), where \(V_{n,1}^{1,l} = V_{n,2}^{2,l}\), with \(v_{i,1}^{1,l} = v_{i,2}^{2,l}\) for any \(i \in [1,5 \cdot 2^n]\), is the central part, \(V_{n,1}^{1,l} \cap V_{n,2}^{1,l} = \emptyset\), and eccentric parts are chosen so that \(v_{i,2}^{1,l} = v_{i,2}^{2,l}\) and...
$v_{i,3}^{k,1} = v_{i,3}^{3-k,2}$ for any $i \in [1, 5 \cdot 2^n]$ and $k \in [1, 2]$. Then \(\psi_n^l : V(H_n) \to V(H_n^l)\), with \(\psi_n^l(v_{i,j}^{k,l}) = v_{i,j}^{k,l}\) for any $i \in [1, 5 \cdot 2^n]$, $j \in [1, 3]$ and $k \in [1, 2]$, is a natural isomorphism from $H_n$ onto $H_n^l$, $l = 1, 2$. In the graph $H_n^l$ the eccentric part $V_{n,2}^{k,l}$ is “joined” to the eccentric part $V_{n,3}^{k,l}$, $k = 1, 2$ (see Fig. 3). Clearly, the graph $H_n^1 \cup H_n^2$ is isomorphic to $G_{n+1}$. We shall suppose that, if $i \in [1, 5 \cdot 2^n]$, then $v_{i,1}^{1,1} = v_{i,j}^{1,2}$ and $v_{i,j}^{2,1} = v_{5 \cdot 2^n + i,j}$ for any $j \in [1, 3]$. For any $j \in [1, 3]$, $v_{i,1}^{1,2} = v_{5 \cdot 2^n + i,1}$ and $v_{i,j}^{2,1} = v_{5 \cdot 2^n + i,j}$ for any $j \in [2, 3]$.

Let $n \in [0, \infty)$. A closed trail $T$ in $H_n$ is said to be $H_n$-good if $V(T) \cap W_{n,i} \neq \emptyset$, $i = 1, 2, 3$. Evidently, if $T$ is of an odd length, it is $H_n$-good. An $H_n$-realisation $T$ of a sequence from $\text{Sct}(H_n)$ is said to be good if any trail of $T$ of length $\not\equiv 0 \pmod{4}$ is $H_n$-good. The graph $H_n$ is said to be strongly ADCT provided that (i) for any $L \in \text{Sct}(H_n)$ there is a good $H_n$-realisation of $L$ and (ii) if $t \in [1, 8]$ and $L = (l_1, \ldots, l_p) \in \text{Sct}(H_n)$ is “a real $(t, 1)$-sequence” of Table 2 (so that the exponent of (7) is an integer), there is a good $H_n$-realisation $(T_1, \ldots, T_p)$ of $L$ satisfying “$(t, 2)$-conditions” of Table 2.

**Theorem 20** For any $n \in [0, \infty)$, the graph $G_n$ is strongly ADCT.

**Proof.** For $n \in [0, \infty)$ and $X \in \{G, H\}$ let $S(X_n)$ denote the following statement: The graph $X_n$ is strongly ADCT. We are going to prove the statement $\forall n \in [0, \infty) S(G_n)$ by induction on $n$. By Theorem 18, $S(G_0)$ is true.

So, suppose that $n \in [0, \infty)$ and $S(G_n)$ is true. We prove that the implication $S(G_n) \Rightarrow S(G_{n+1})$ is true by proving that both implications $S(G_n) \Rightarrow S(H_n)$ and $(S(G_n) \land S(H_n)) \Rightarrow S(G_{n+1})$ are true.
<table>
<thead>
<tr>
<th>Case</th>
<th>Statement</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3)^4(7)</td>
<td>$\frac{2m-12}{7}$</td>
<td>$v_{i,2} \in V(T_i)$, $i = 1, 2, 3$</td>
</tr>
<tr>
<td>(3)^2(4)(7)</td>
<td>$\frac{2m-10}{7}$</td>
<td>$v_{i,2} \in V(T_i)$, $i = 1, 2$</td>
</tr>
<tr>
<td>(3)^2(6)(7)</td>
<td>$\frac{2m-12}{7}$</td>
<td>$v_{i,2} \in V(T_i)$, $i = 1, 2$</td>
</tr>
<tr>
<td>(3)^2(7)</td>
<td>$\frac{2m-6}{7}$</td>
<td>$v_{i,2} \in V(T_i)$, $i = 1, 2$</td>
</tr>
<tr>
<td>(4)^6(7)</td>
<td>$\frac{2m-24}{7}$</td>
<td>$v_{i,2} \in V(T_i)$, $v_{i,3}^2 \in V(T_{3+i})$, $i = 1, 2, 3$</td>
</tr>
<tr>
<td>(4)^5(7)</td>
<td>$\frac{2m-20}{7}$</td>
<td>$v_{i,2} \in V(T_i)$, $i = 1, 2, 3$, $v_{i,3}^1 \in V(T_{3+i})$, $i = 1, 2$</td>
</tr>
<tr>
<td>(4)^3(7)</td>
<td>$\frac{2m-12}{7}$</td>
<td>$v_{i,2} \in V(T_i)$, $i = 1, 2, 3$</td>
</tr>
<tr>
<td>(4)^3(7)</td>
<td>$\frac{2m-12}{7}$</td>
<td>$v_{i,2} \in V(T_i)$, $i = 1, 2$, $v_{i,1}^1 \in V(T_3)$</td>
</tr>
</tbody>
</table>

Table 2: Required properties of $H_n$-realisations

Figure 3: The graph $G_n$
(a) $S(G_n) \Rightarrow S(H_n)$

**Claim 1** A sequence $L = (l_1, \ldots, l_p) \in \text{Sct}(H_n)$ has a good $H_n$-realisation whenever one of the following conditions is fulfilled:

1. There is a decomposition $\{1^1, I^2\}$ of $[1, p]$ such that $\sum_{i \in I^1} l_i = e_n$.
2. There is $m \in [1, p]$ and a decomposition $\{i_1, \ldots, i_m, 1^1, I^2\}$ of $[1, p]$ such that, for both $j = 1, 2$, there is a sequence $L^j = (l^j_1, \ldots, l^j_m)$ such that $L^j L(I^j) \subseteq \text{Sct}(G_n)$ and a good $G_n$-realisation $(T^j_1, \ldots, T^j_m) T^j$ of $L^j L(I^j)$ satisfying $l^j_i = l^j_{i-1} + l^j_i$ and $V(T^j_1) \cap V(T^j_2) \cap V_{n,1} \neq \emptyset$ for any $k \in [1, m]$.

**Proof.**

1. With $I^1 := [1, p] - I^2$ we have $\sum_{i \in I^2} l_i = e_n$. By $S(G_n)$ there is a good $G_n$-realisation $T^1$ of $L(I^1)$, $j = 1, 2$. If $T$ is a trail of $T^j$, $j \in [1, 2]$, and $i \in [1, 3]$, then $V(l(T)) \cap W_{n,i} \supseteq V(l(T)) \cap V_{n,i} = \varphi_n(V(T) \cap V_{n,i})$. As $Y \neq \emptyset \Rightarrow \varphi_n(Y) \neq \emptyset$ for any $Y \subseteq V(G_n)$, it is clear that $\varphi_n(T^1) \varphi_n(T^2)$ is a good $H_n$-realisation of the sequence $(l_1, \ldots, l_m) L(I^1) L(I^2) \sim L$.

2. We have $V(\varphi_n(T^1)) \cap V(\varphi_n(T^2)) = \varphi_n(V(T^1)) \cap V(T^2) \cap V_{n,1} \neq \emptyset$, hence there is a trail $T_{ik} \subseteq \varphi_n(T^1) + \varphi_n(T^2)$ for any $k \in [1, m]$. If $l_{ik} \neq 0$, there exists $j \in [1, 2]$ such that $(l^j_{ik})_a \neq 0$, and then $V(T_{ik}) \cap W_{n,i} \supseteq V(T^j_{ik}) \cap V_{n,i} = \varphi_n(V(T^j_{ik}) \cap V_{n,i}) \neq \emptyset$ (as $T^j_{ik}$ is $G_n$-good), $i = 1, 2, 3$, so that $T_{ik}$ is $H_n$-good. Therefore, $(T_{i1}, \ldots, T_{im}) \varphi_n(T^1) \varphi_n(T^2)$ is a good $H_n$-realisation of the sequence $(l_1, \ldots, l_m) L(I^1) L(I^2) \sim L$.

**Claim 2** A sequence $L = (l_1, \ldots, l_p) \in \text{Sct}(H_n)$ has a good $H_n$-realisation whenever one of the following conditions is fulfilled:

1. There is a decomposition $\{i_1, I^1, I^2\}$ of $[1, p]$ such that $\sum_{i \in I} l_i \leq 2e_n - 3$, $j = 1, 2$.
2. There is a decomposition $\{i_1, i_2, I^1, I^2\}$ of $[1, p]$ and $l_{i_1} \in [3, l_{i_2} - 3]$, $k = 1, 2$, such that $l_{i_1}^1 + l_{i_2}^1 \sum_{i \in I} l_i = e_n$.

**Proof.**

1. Put $l_{i_1}^1 := e_n - \sum_{i \in I} l_i \in [3, l_{i_1} - 3]$, $j = 1, 2$. By $S(G_n)$ there is a good $G_n$-realisation $(T^1_{i_1}) T^j$ of $(l_{i_1}^1) L(I^j)$, $j = 1, 2$. By Proposition 6 we may suppose without loss of generality that $V(T^1_{i_1}) \cap V(T^2_{i_2}) \cap V_{n,1} \neq \emptyset$, and then it suffices to use Claim 1.2.

2. If $l_{i_2}^2 := l_{i_2} - l_{i_2}^1$, then $l_{i_2}^2 \in [3, l_{i_2} - 3]$, $k = 1, 2$, and $l_{i_1}^2 + l_{i_2}^2 + \sum_{i \in I} l_i = e_n$. By $S(G_n)$ there is a good $G_n$-realisation $(T^j_{i_1}, T^j_{i_2}) T^j$ of the sequence $(l_{i_1}^1, l_{i_2}^2) L(I^j)$, $j = 1, 2$. Because of Propositions 6 and 16.1 we may suppose without loss of generality that $V(T^1_{i_1}) \cap V(T^2_{i_2}) \cap V_{n,1} \neq \emptyset$, $k = 1, 2$, and we are done by Claim 1.2 again.

Consider $L \in \text{Sct}(H_n)$ and suppose that $\text{nd}(L) = (l_1, \ldots, l_p)$. Let $q \in [1, p]$ be defined by the inequalities $\sum_{i=1}^{q-1} l_i \leq e_n - 3$ and $\sum_{i=1}^{q-1} l_i + l_p > e_n - 3$.

Since $e_n = 75 \cdot 4^n$, the following statements are easy to checked:

<table>
<thead>
<tr>
<th>Condition</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall m \in {3, 5, 6}$</td>
<td>$e_n \equiv 0 \pmod{m}$</td>
</tr>
<tr>
<td>$n = 0 \Rightarrow e_n \equiv 3 \pmod{4}$</td>
<td></td>
</tr>
<tr>
<td>$n \in [1, \infty) \Rightarrow e_n \equiv 0 \pmod{4}$</td>
<td></td>
</tr>
<tr>
<td>$\exists r \in {3, 5, 6}$</td>
<td>$e_n \equiv r \pmod{7}$</td>
</tr>
</tbody>
</table>

(1) $S(G_n) \Rightarrow S(H_n)$

(2) $\forall m \in \{3, 5, 6\}$, $e_n \equiv 0 \pmod{m}$

(3) $n \equiv 3 \pmod{4}$

(4) $n \equiv 0 \pmod{4}$

(5) $r \equiv e_n \pmod{7}$
(1) If \( \sum_{i=1}^{q-1} l_i + l_p = e_n \), use Claim 1.1. (In order to simplify the presentation, \( \rightarrow i,j \) will mean that Claim \( i,j \) guarantees the existence of a good \( H_n \)-realisation of a sequence \( \sim L \). Moreover, we write for short \( f_i(L) \).

(2) If \( \sum_{i=1}^{q-1} l_i + l_p \geq e_n + 3 \), then \( \sum_{i=q}^{p-1} l_i = 2e_n - (\sum_{i=1}^{q-1} l_i + l_p) \leq e_n - 3 \rightarrow 2.1 \) (with \( T^1 := [1, q - 1] \) and \( T^2 := [q, p - 1] \)).

(3) \( \exists \delta \in \{-2, -1, 1, 2\}, \sum_{i=q}^{p-1} l_i + l_p = e_n + \delta \)

(31) If \( l_{p-1} \geq 8 \), then, with \( m := \max(3, 3 - \delta) \), we have \( m + (l_p - m - \delta) + \sum_{i=q}^{p-1} l_i = e_n = (l_{p-1} - m) + (m + \delta) + \sum_{i=q}^{p-2} l_i \rightarrow 2.2. \)

(32) If \( l_{p-1} \leq 7 \), let \( r \in [q, p] \) be defined by the inequalities \( \sum_{i=1}^{r-1} l_i \leq e_n - 3 \) and \( \sum_{i=1}^{r} l_i > e_n - 3 \).

(321) \( \sum_{i=1}^{r} l_i = e_n \rightarrow 1.1. \)

(322) If \( \sum_{i=1}^{r} l_i \geq e_n + 3 \), then \( \sum_{i=r+1}^{p} l_i \leq e_n - 3 \rightarrow 2.1. \)

(323) \( \sum_{i=1}^{r} l_i = e_n + \varepsilon, \varepsilon \in \{-2, -1, 1, 2\} \)

(3231) \( \varepsilon \in [-2, -1] \)

(32311) If \( l_p \geq l_1 + 3 - \varepsilon \), then \( \sum_{i=1}^{p-1} l_i = e_n + \varepsilon - l_1 \leq e_n - 4 \) and \( l_1 + \sum_{i=r+1}^{p} l_i = l_1 + e_n - \varepsilon - l_p \leq e_n - 3 \rightarrow 2.1. \)

(32312) \( l_p \leq l_1 + 2 - \varepsilon \)

(323121) If there is \( j \in [1, r] \) and \( k \in [r + 1, p] \) such that \( l_k = l_j - \varepsilon \), then \( \sum_{i=1}^{r} l_i + \sum_{i=j+1}^{r} l_i + k = e_n \rightarrow 1.1. \)

(323122) \( \forall j \in [1, r] \) \( \forall k \in [r + 1, p] l_k \neq l_j - \varepsilon \)

(3231221) If \( \varepsilon = -2 \), then \( l_i \in [l_1, l_1 + 4] \) for any \( i \in [1, p] \).

(32312211) If \( l_r = l_1 \), then \( e_n - 2 \equiv 0 \mod(l_1) \) in contradiction with \( l_1 \leq 7 \) and (G1)-(G4).

(32312212) If \( l_r = l_1 + 1 \), then \( f_{l_1+2} = f_{l_1+3} = 0. \)

(323122121) If \( l_{p-1} = l_1 + 1 \), then \( e_n + 2 \leq (p - r)(l_1 + 1) + 3, p - r \geq \frac{e_n - 1}{l_1 + 1} \geq \left\lfloor \frac{e_n}{l_1+1} \right\rfloor = 10 \) and \( l_{r+1} = l_{r+2} = l_1 + 1. \)

(3231221211) If \( l_2 = l_1 \), then \( \sum_{i=3}^{r+2} l_i = e_n \rightarrow 1.1. \)

(3231221212) If \( l_2 = l_1 + 1 \), then \( e_n - 2 = r(l_1 + 1) - 1 \) and \( e_n \equiv 1 \mod (l_1 + 1) \) in contradiction with (G1)-(G4).

(323122122) If \( l_{p-1} = l_1 + 4 \), then \( l_1 = 3 \), there is \( I \subseteq [1, r] \) such that \( \sum_{i \in I} l_i = 12 \) and \( \sum_{i \in [r+1, r]} l_i + l_{p-1} + l_p = e_n \rightarrow 1.1. \)

(323122123) If \( l_r = l_1 + 2 \), then \( l_k = l_1 + 3 \) for any \( k \in [r + 1, p] \), \( e_n + 2 \equiv 0 \mod (l_1 + 3) \), hence, because of \( l_1 \leq 4 \) and (G1), \( l_1 = 4 \). As \( f_5 = 0 \) (a consequence of \( l_p = 7 \)), from \( \sum_{i=1}^{r} l_i = e_n - 2 \geq 73 \) it follows that there is \( I \subseteq [1, r] \) such that \( \sum_{i \in I} l_i = 12 \). Therefore, \( \sum_{i \in [r+1, r+2]} l_i = e_n \rightarrow 1.1. \)

(32312214) \( l_r = l_1 + 3 \)

(323122141) If \( l_{p-1} = l_1 + 3 \), there is \( m \in [0, 1] \) such that \( e_n + 2 \equiv m \mod (l_1 + 3) \), and then \( l_1 \leq 4 \) together with (G1) imply \( l_1 = 4 \). We have also \( f_5 = 0 \) and \( 77 \leq e_n + 2 = 7(p - r - 1) + l_p \), so that \( l_{r+1} = l_{r+2} = 7. \)
(3231221411) If \( f_4 \geq 3 \) or \( f_6 \geq 2 \), there is \( I \subseteq [1, r] \) such that \( \sum_{i \in [1, r+2] - I} l_i = e_n \rightarrow 1.1 \).

(3231221412) \( f_4 \leq 2 \land f_6 \leq 1 \)

(32312214121) If \( l_p = 7 \), then \( e_n + 2 = 7(p - r) \), \( e_n - 2 \equiv 3 \pmod{7} \), \( \text{nd}(L) = (4, 6)(7) ^{2r-2} \), \( l_1^1 := 3 \leq l_i - 3, i = 3, 4, \) and \( 3 + 3 + 6 + 7(r - 2) = e_n \rightarrow 2.2 \) (with \( i_1 := 3, i_2 := 4, I^1 := \{2\} \cup [5, r+2] \) and \( I^2 := \{1\} \cup [r+3, 2r] \)).

(32312214122) If \( l_p = 8 \), then \( e_n + 2 = 7(p - r) + 1 \), \( e_n - 2 \equiv 4 \pmod{7} \), \( \text{nd}(L) = (4)(7) ^{2r-2} \), \( l_1^p := 5 \leq l_p - 3, l_2 := 4 \leq l_2 - 3 \) and \( 5 + 4 + 4 + 7(r - 2) = e_n \rightarrow 2.2 \).

(3231221414) If \( l_p - 1 = l_1 + 4 \), then \( l_1 = 3 \), \( f_5 = 0 \), there is \( I \subseteq [1, r] \) such that \( \sum_{i \in I} l_i = 12 \) and \( \sum_{i \in [1, r] - I} l_i + l_p - 1 + l_p = e_n \rightarrow 1.1 \).

(323122151) If \( l_p = 1 + 4 \), then \( l_1 = 3 \), \( f_5 = 0 \), \( e_n + 2 = 7(p - r) \) and \( e_n - 2 \equiv 3 \pmod{7} \).

(323122152) If \( f_3 \geq 4 \), \( f_4 \geq 3 \) or \( f_6 \geq 2 \), there is \( I \subseteq [1, r] \) such that \( \sum_{i \in [1, r+2] - I} l_i = e_n \rightarrow 1.1 \).

(32312215152) If \( f_4 = 2 \) and \( l_j = l_{j+1} = 4 \) for some \( j \in [1, r - 1] \), then \( \sum_{i=1}^{j} l_i = 4 \) and \( l_j + l_{j+1} + \sum_{i=r+2}^{p-1} l_i = e_n - 4 \rightarrow 2.1 \).

(32312215221) If \( \text{nd}(L) = (3)^{2}(4)(7) ^{2r-4} \), then \( l_1^I := 4 \leq l_i - 3, i = 4, 5, \) and \( 4 + 4 + 4 + 7(r - 2) = e_n \rightarrow 2.2 \).

(32312215222) If \( \text{nd}(L) = (4, 6)(7) ^{2r-4} \), then \( l_1^I := 3 \leq l_i - 3, i = 3, 4, \) and \( 3 + 3 + 6 + 7(r - 2) = e_n \rightarrow 2.2 \).

(3231221523) If \( f_4 = 0 \), then \( \text{nd}(L) = (3)(7) ^{2r-3} \) and we can use Claim 1.2 with \( m := 3, l_k := k + 1 \), \( l_1^I = 3, l_2^I = 4, k = 1, 2, 3, I^1 := \{1\} \cup [5, r+2] \) and \( I^2 := \{r+3, 2r\} \) (notice that by \( \text{St}(G_n) \) there are good \( G_n \)-realisations of the sequences \( (3)^{I}(7) ^{\alpha r-3} \) and \( (4)^{3}(7) ^{\alpha r-12} \) corresponding to the rows 1 and 5 of Table 1).

(32312222) If \( \varepsilon = -1 \), then \( l_i \in [l_1^I, l_1 + 3] \) for any \( i \in [1, p] \).

(32312221) If \( l_r = l_1 \), then \( e_n - 1 = rl_1 \) in contradiction with (G1)–(G4).

(32312222) If \( l_r = l_1 + 1 \), then \( l_k \notin [l_1 + 1, l_1 + 2] \) for any \( k \in [r + 1, p] \), \( e_n + 1 = (p - r)l_1 + 3 \), \( l_i \leq 4 \) and, because of (G1), \( l_1 = 4 \). Since \( 4l_1 + 5f_5 = e_n \geq 75 \), there is \( I \subseteq [1, r] \) such that \( \sum_{i \in I} l_i = 20 \), and then \( \sum_{i \in [1, r+3] - I} l_i = e_n \rightarrow 1.1 \).

(32312223) If \( l_p = l_1 + 2 \), then \( f_{l_1+3} = 0 \), hence \( e_n + 1 = (p - r)(l_1 + 2), \) \( l_1 \leq 5 \) and, because of (G1), \( l_1 = 5 \) and \( e_n = 1 \equiv 5 \pmod{7} \).

(323122231) If \( f_5 \geq 4 \), then \( \sum_{i=5}^{r+3} l_i = e_n \rightarrow 1.1 \).

(323122232) If \( f_5 \leq 3 \), then \( \text{nd}(L) = (5)(7) ^{2r-1} \), \( l_1^I := 3 \leq l_i - 3, i = 2, 3, \) and \( 3 + 3 + 7(r - 1) = e_n \rightarrow 2.2 \).
If \( l_r = l_1 + 3 \), then \( e_n + 1 = (p - r)(l_1 + 3) \), \( l_1 \leq 4 \) and, using (G1), \( l_1 = 4 \), so that \( f_6 = 0 \) and \( e_n - 1 \equiv 5 \mod 7 \).

(323122241) \( \exists j \in [2, r - 1] \) \( l_j = 5 \)

(3231222411) If \( f_4 \geq 2 \), then \( \sum_{j=3}^{r-1} l_i + \sum_{i=j+1}^{r+2} l_i = e_n \rightarrow 1.1. \)

(3231222412) \( f_4 = 1 \)

(32312224121) If \( f_5 \geq 4 \), then \( l_1 + \sum_{i=j+1}^{r+2} l_i = e_n \rightarrow 1.1. \)

(32312224122) If \( f_5 \leq 3 \), then \( nd(L) = (4)(5)^3(7)^{2r-5}, l_1 + l_2 + \sum_{i=5}^{r+1} l_i = e_n - 4 \) and \( l_3 + l_4 + \sum_{i=r+2}^{p-1} l_i = e_n - 3 \rightarrow 2.1. \)

(323122242) \( f_5 = 0 \)

(3231222421) If \( f_4 \geq 5 \), then \( \sum_{i=3}^{r+3} l_i = e_n \rightarrow 1.1. \)

(3231222422) If \( f_4 \leq 4 \), then \( nd(L) = (4)^3(7)^{2r-12} \). By \( S(G_n) \) there exists a good \( G_n \)-realisation \((T_1^1, T_1^2, T_2^1, T_2^2) \) of the sequence \((4)(5)^3(7)^{2r-20} \) such that \( v_{i,1} \in V(T_i^1) \), \( i = 1, 2, 3 \) and \( v_{i,2} \in V(T_{i+1}^1) \), \( i = 1, 2 \) (see the row 4 of Table 1). Therefore, by Proposition 6, there is also a good \( G_n \)-realisation \((T_1^2, T_2^2, T_3^2) \) of \((4)(5)^3(7)^{2r-20} \) such that \( v_{i,1} \in V(T_i^1) \), \( i = 1, 2 \), and \( v_{i,2} \in V(T_{i+1}^1), i = 1, 2, 3 \). Further, by \( S(G_n) \) and Propositions 6 and 16.1, there exist good \( G_n \)-realisations \((T_1^1, T_2^1, T_3^1) \) and \((T_1^2, T_2^2, T_3^2) \) of the sequence \((3)^3(7)^{2r-20} \) such that \( v_{i+1,1} \in V(T_i^2) \) and \( v_{i,1} \in V(T_i^1), i = 1, 2 \). Since \( V(\varphi_n^1(T_i^1)) \cap V(\varphi_n^2(T_{i-1}^2)) \supseteq \{v_{i,1}\}, i = 2, 3, \) and \( V(\varphi_n^1(T_i^1)) \cap V(\varphi_n^2(T_{i-1}^2)) \supseteq \{v_{i,1}\}, i = 1, 2 \), there are trails \( T_i \in \varphi_n^1(T_i^1) + \varphi_n^2(T_{i-1}^2), i = 2, 3, \) and \( T_i \in \varphi_n^1(T_i^2) + \varphi_n^2(T_{i-1}^2), i = 1, 2 \). Then \( \varphi_n^1(T_i^1), \varphi_n^1(T_i^2), \varphi_n^1(T_i^1), T_2, T_3, \varphi_n^1(T_i^1), \varphi_n^2(T^1), \varphi_n^2(T^2) \) are \( H_n \)-good realisations of \( nd(L) \); the former satisfies “(8,2)-conditions” and the latter one “(7,2)-conditions” of Table 2.

(3232) If \( \varepsilon \in [1, 2] \), then \( l_r = \sum_{i=1}^{p-1} l_i \geq \sum_{i=1}^{p-1} l_i \geq e_n + \varepsilon - (e_n - 3) = 3 + \varepsilon \).

With \( l := \min(l_i : i \in [1, r], l_i \geq 3 + \varepsilon) \) we have \( 3 + \varepsilon \leq l \leq l_r \).

(32321) If \( l_p \geq l + 3 - \varepsilon \), let \( j \in [1, r] \) be such that \( l_j = l \). Then \( \sum_{i=1}^{r-1} l_i + \sum_{i=j+1}^{r+2} l_i = e_n + \varepsilon - l \leq e_n + \varepsilon - (3 + \varepsilon) \) and \( l_j + \sum_{i=r+1}^{p} l_i = l + (e_n - \varepsilon) - l_p \leq e_n - 3 \rightarrow 2.1. \)

(32322) \( \forall i \in [1, p], l_i \in [3, 2 + \varepsilon] \cup [l, l + 2 - \varepsilon] \)

(323221) \( \varepsilon = 1 \)

(3232211) If \( l_r = l + 1 \), then \( e_n - 1 = (p - r)(l + 1) \) in contradiction with (G1)–(G4).

(3232212) \( l_r = l \)

(32322121) If \( l_{p-1} = l \), then \( e_n - 1 = (p - r)(l + m) \) for some \( m \in [0, 1] \) and \( e_n \equiv 1 + m \mod l \) in contradiction with (G1)–(G4).

(32322122) \( l_{p-1} = l + 1 \)

(323221221) If \( f_3 = 0 \), then \( e_n + 1 = rl \), hence from \( l \leq 6 \) and (G1)–(G3) it follows that \( l = 4 \) and \( n = 0 \).

(3232212211) If \( f_5 \geq 3 \), then \( \sum_{i=5}^{r} l_i + \sum_{i=p-2}^{p} l_i = e_n \rightarrow 1.1. \)
(3232212212) If \( f_5 = 2 \), then nd(\( L \)) = (4)^35(5)^2; for a good \( H_0 \)-realisation of nd(\( L \)) see Graph \( H_0 \).

(3232212222) \( f_3 \geq 1 \)

(3232212221) \( l = 4 \)

(32322122211) If \( f_3 = 1 \), then \( e_n + 1 = 3 + 4(r - 1) \) in contradiction with (G2) and (G3).

(32322122212) If \( f_3 \geq 2 \), then \( \sum_{i=3}^{r} l_i + l_p = e_n \to 1.1 \).

(3232212222) \( l \in [5, 6] \)

(32322122221) If \( l_{r-1} = 3 \), then \( e_n + 1 = 3(r - 1) + l \) in contradiction with (G1).

(32322122222) If \( l_{r-1} = l \), then \( \sum_{i=1}^{r-2} l_i + l_{p-1} = (e_n+1-2l)+(l+1) \leq e_n-3 \) and \( \sum_{i=r-1}^{p-2} l_i = e_n - 3 \to 2.1 \).

(323222) If \( e = 2 \), then \( e_n - 2 = (p - r)l \) in contradiction with (G1)–(G4).

To conclude the proof of the implication \( S(G_n) \Rightarrow S(H_n) \) we have to find good \( H_n \)-realisations of sequences satisfying “(t,2)-conditions” of Table 2, \( t \in [1, 6] \).

\( t = 1 \): If \( 7|e_n - 12 \), then \( 7|e_n - 6 \) and \( 7|e_n - 13 \). By \( S(G_n) \) (the row 2 of Table 1) and Proposition 6 there is a good \( G_n \)-realisation \( (T_1^n, T_2^n, T_3^n, T_1^1)T^1 \) of the sequence \( (3)^3(4)(7)^{2n-13} \) such that \( v_{i,1} \in V(T_1^n) \) and \( v_{i,2} \in V(T_1^1) \), \( i = 1, 2, 3 \). By \( S(G_n) \) and Proposition 6 there is a good \( G_n \)-realisation \( (T_1^n, T_2^n)T^2 \) of the sequence \( (3)^2(7)^{2n-15} \) with \( v_{i,1} \in V(T_1^n) \). As V(\( \varphi_n^1(T_1^n) \) ) \( \cup \) V(\( \varphi_n^2(T_1^n) \) ) \( \supseteq \) \( \{ v_{i,1} \} \), there is a trail \( T_4 \in \varphi_n^1(T_1^n) + \varphi_n^2(T_1^n) \) and \( (\varphi_n^1(T_1^n), \varphi_n^1(T_2^n), \varphi_n^1(T_3^n), \varphi_n^2(T_2^n), T_4, \varphi_n^1(T_1^n) \varphi_n^2(T_2^n) \) is an appropriate good \( H_n \)-realisation of the sequence \( (3)^3(4)(7)^{2n-13} \).

\( t = 2 \): By \( S(G_n) \), Propositions 6 and 16.2, there is a good \( G_n \)-realisation \( (T_1^n, T_2^n, T_3^n, T_1^1)T^1 \) of the sequence \( (3)^3(7)^{2n-15} \) such that \( v_{i,1} \in V(T_1^n) \), \( v_{i,2} \in V(T_1^1) \), \( i = 1, 2 \). By \( S(G_n) \) and Proposition 16.1 there exists a good \( G_n \)-realisation \( (T_1^n, T_2^n)T^2 \) of the sequence \( (4)^3(7)^{2n-12} \) such that \( v_{i,1} \in V(T_1^n) \), \( i = 1, 2 \). As V(\( \varphi_n^1(T_1^n) \) ) \( \cup \) V(\( \varphi_n^2(T_1^n) \) ) \( \supseteq \) \( \{ v_{i,1} \} \), there is a trail \( T_1 \in \varphi_n^1(T_1^n) + \varphi_n^2(T_1^n) \), \( i = 1, 2 \). Then \( (\varphi_n^1(T_1^n), \varphi_n^1(T_1^n)) \varphi_n^2(T_2^n) (T_1^n, T_2^n) \varphi_n^1(T_1^n) \) is a good \( H_n \)-realisation of the sequence \( (3)^2(4)(7)^{2n-15} \) having required properties.

\( t = 3 \): By \( S(G_n) \) there are good \( G_n \)-realisations \( (T_1^n, T_2^n)T^1 \) of \( (3)^2(7)^{2n-6} \) and \( T^2 \) of \( (6)(7)^{2n-6} \). By Proposition 16.1 we may suppose without loss of generality that \( v_{i,2} \in V(T_1^n), i = 1, 2 \). Then \( (\varphi_n^1(T_1^n), \varphi_n^2(T_2^n)) \varphi_n^1(T_2^n) \varphi_n^1(T_1^n) \) is a necessary good \( H_n \)-realisation of \( (3)^2(6)(7)^{2n-15} \).

\( t = 4 \): By \( S(G_n) \) and Propositions 6 and 16.1 there exist good \( G_n \)-realisations \( (T_1^n, T_2^n, T_3^n)T^1 \) of \( (3)^2(4)(7)^{2n-10} \) and \( T^2 \) of \( (3)(7)^{2n-3} \) such that \( v_{i,2} \in V(T_1^n), i = 1, 2 \), and \( v_{i,1} \in V(T_1^n) \) \( \cup \) V(\( T_2^n ) \) T(1) (4)|(7)|. There is a tail \( T_3 \in \varphi_n^1(T_3^n) + \varphi_n^2(T_2^n) \) and \( (\varphi_n^1(T_2^n), \varphi_n^1(T_2^n), T_3) \varphi_n^1(T_1^n) \varphi_n^2(T_2^n) \) is a good \( H_n \)-realisation of \( (3)^2(4)(7)^{2n-10} \).
realisation of \((3)^2(7)^{2\nu_3-6}\) we are seeking for.

t = 5: By \(S(G_n)\) (the row 5 of Table 1) and Proposition 6 there are good 
\(G_n\)-realisations \((T^1_i, T^2_i, T^3_i)\) of the sequence \((4)^3(7)^{2\nu_3-12}\) such that \(v_{i,1+j} \in V(T^1_i)\), \(i = 1, 2, 3, j = 1, 2\). Then \((\varphi^1_n(T^1_i), \varphi^2_n(T^2_i), \varphi^3_n(T^3_i))\) is a good 
\(H_n\)-realisation of \((4)^3(7)^{2\nu_3-12}\) having required properties.

t = 6: By \(S(G_n)\) (the row 3 of Table 1) and Proposition 6 there exists a good 
\(G_n\)-realisation \(\prod_{i=1}^6 (T^1_i)\) of the sequence \((4)^6(7)^{9\nu_3-24}\) with \(v_{i,2} \in V(T^1_i)\), \(i = 1, 2, 3, v_{i,3} \in V(T^1_{i+2})\), \(v_{i,1} \in V(T^2_i)\). By \(S(G_n)\) and Proposition 6 there exists also a good 
\(G_n\)-realisation \((T^3_i)\) of the sequence \((3)(7)^{9\nu_3-24}\) with \(v_{i,1} \in V(T^3_i)\). There is a trail \(T_6 \in \varphi^1_n(T^1_6) + \varphi^2_n(T^2_6)\) and 
\(\prod_{i=1}^6 (\varphi^1_n(T^1_i)) (T_6) \varphi^1_n(T^1_i) \varphi^2_n(T^2)\) is an appropriate good 
\(H_n\)-realisation of \((4)^5(7)^{8\nu_3-20}\).

(b) \((S(G_n) \land S(H_n)) \Rightarrow S(G_{n+1})\)

Claim 3 A sequence \((l_1, \ldots, l_p) \in \text{Sct}(G_{n+1})\) has a good 
\(G_{n+1}\)-realisation whenever one of the following conditions is fulfilled:

1. There is \(I^1 \subseteq [1, p]\) such that \(\sum_{i \in I^1} l_i = 2e_n\).
2. There is \(m \in [1, p]\) and a decomposition \(\{ \{ i_1, \ldots, i_m \}, I^1, I^2 \}\) of \([1, p]\) 
such that, for both \(j = 1, 2\), there is a sequence \(L^j = (l_1^j, \ldots, l_m^j)\) such that 
\(L^j L(I^j) \subseteq \text{Sct}(H_n)\) and a good \(H_n\)-realisation \((T^j_{i_1}, \ldots, T^j_{i_m})\) of 
\(L^j L(I^j)\) satisfying \(l_{i_k} = l_{i_k}^1 + l_{i_k}^2\) and \(V(T^1_{i_k}) \cap V(T^2_{i_k}) \neq \emptyset\) for any \(k \in [1, m]\).
3. There is a decomposition \(\{ \{ i_1, i_2 \}, I^1, I^2 \}\) of \([1, p]\) and \(l_{i_k}^1 \in [3, l_{i_k} - 3]\), 
k = 1, 2, such that there is a good \((l_{i_1}^1, l_{i_2}^1)\)-global \(H_n\)-realisation of the sequence 
\((l_1^1, l_2^1) L(I^1)\).

Proof. 1. With \(I^2 := [1, p] - I^1\) we have \(\sum_{i \in I^2} l_i = e_{n+1} - \sum_{i \in I^1} l_i = 2e_n\). By 
\(S(H_n)\) there exists a good \(H_n\)-realisation \(T^j\) of \(L(I^j)\), \(j = 1, 2\). Let \(T\) be a 
trail of length \(\neq 4\) of \(T^j\), \(j \in [1, 2]\), and let \(i \in [1, 3]\). Since \(\psi^j_n(W_{n,i}) \subseteq V_{n,i+1}\), 
from \(V(T) \cap W_{n,i} \neq \emptyset\) it follows that \(V(\psi^j_n(T)) \cap V_{n,i+1} \neq \emptyset\). Therefore, 
\(\psi^j_n(T^j)\) is a good \(G_{n+1}\)-realisation of \((l_1^1, l_2^1) L(I^1)\).

2. As \(V(\psi^1_n(T_{i_k}^1)) \cap V(\psi^2_n(T_{i_k}^2)) \supseteq\psi^1_n(V(T_{i_k}^1)) \cap V(T_{i_k}^2) \cap V_{n,3} \neq \emptyset\), there is a 
trail \(T_{i_k} \in \psi^1_n(T_{i_k}^1) + \psi^2_n(T_{i_k}^2)\) for any \(k \in [1, m]\). If \(l_{i_k}^4 = 0\), then \(i \in [1, 2]\) 
such that \(l_{i_k}^4 \neq 0\). Clearly, the trail \(\psi^j_n(T_{i_k}^j)\) is \(G_{n+1}\)-good (as above), hence, 
because of \(V(T_{i_k}) \supseteq V(\psi^j_n(T_{i_k}^j))\), so is \(T_{i_k}\). Thus, \((T_{i_1}, \ldots, T_{i_m}) \psi^1_n(T^1) \psi^2_n(T^2)\) is 
a good \(G_{n+1}\)-realisation of \((l_{i_1}, \ldots, l_{i_m}) L(I^1)\) \((l_1^1, l_2^1) L(I^1)\).

3. Let \((T^1_{i_1}, T^1_{i_2})\) be a good \((l_{i_1}^1, l_{i_2}^1)\)-global \(H_n\)-realisation of \((l_{i_1}^1, l_{i_2}^1) L(I^1)\) 
such that \(|E(T^1_{i_1})| = l_{i_1}^1, k = 1, 2\); there is \(j \in [1, 2]\) with \(V(T^1_{i_1}) \cap (V_{n,2}^3 \cup 
V_{n,3}^3) \neq \emptyset\) and \(V(T^2_{i_1}) \cap (V_{n,2} \cup V_{n,3}^3) \neq \emptyset\). If \(L^2_{i_1} := l_{i_1}^1 - l_{i_1}, k = 1, 2\), 
from \(l_{i_1}^1 + l_{i_2}^1 + \sum_{i \in I^1} l_i = 4e_n\) and \(l_{i_1}^1 + l_{i_2}^1 + \sum_{i \in I^1} l_i = 2e_n\) it follows 
that \(L^2 := (l_{i_1}^1, l_{i_2}^1) L(I^2) \subseteq \text{Sct}(H_n)\). Hence, by \(S(H_n)\) there is a good \(H_n\)-
realisation \((T^1_{i_1}, T^1_{i_2}) T^2\) of \(L^2\).
If there is $k \in [1, 2]$ such that $V(T^1_n) \cap (V^{1}_{n, 2} \cup V^{1}_{n, 3}) \neq \emptyset$ and $V(T^2_n) \cap (V^{2}_{n, 2} \cup V^{2}_{n, 3}) \neq \emptyset$, by Proposition 19.3 we may suppose without loss of generality that $v^{n}_{1, 2} \in V(T^1_{m, 1}) \cap V(T^2_{m, 1})$, $m = 1, 2$. Since $V(v^{n}_{1, 2}) \cap V(\psi^2_n(T^1_{m, 1})) \subseteq \{v^{n}_{1, 2}\}$, there is a trail $T_m \in \psi^1_n(T^1_{m, 1}) + \psi^2_n(T^2_{m, 1})$, $m = 1, 2$, and $(T_1, T_2)\psi^1_n(T^1)\psi^2_n(T^2)$ is a good $G_{n, 1}$-realisation of $(l_{i_1}, l_{i_2})L(I^1)L(I^2) \sim L$.

If there is $k \in [1, 2]$ such that $V(T^1_n) \cup V(T^2_n) \subseteq V(G^k_n)$, by Proposition 19.2, 3 we may suppose without loss of generality that $v^{n}_{1, 2} \in V(T^1_{m, 1}) \cap V(T^2_{m, 1})$, $v^{n}_{1, 3} \in V(T^1_{m, 1}) \cup V(T^2_{m, 1})$ and $v^{n}_{1, 3} \in V(T^2_{m, 1})$. As above, there is a trail $T_1 \in \psi^1_n(T^1_{m, 1}) + \psi^2_n(T^2_{m, 1})$. Further, since $V(v^{n}_{1, 2}) \cap V(\psi^2_n(T^1_{m, 1})) \subseteq \{v^{n}_{1, 2}\}$, there exists a trail $T_2 \in \psi^1_n(T^1_{m, 1}) + \psi^2_n(T^2_{m, 1})$ and $(T_1, T_2)\psi^1_n(T^1)\psi^2_n(T^2)$ is a good $G_{n, 1}$-realisation of a sequence changeable to $L$.

**Claim 4** A sequence $\{l_1, \ldots, l_p\} \in \text{Sct}(G_{n, 1})$ has a good $G_{n, 1}$-realisation whenever one of the following conditions is fulfilled:

1. There is a decomposition $\{(i_1), I^1, I^2\}$ of $[1, p]$ such that $\sum_{i \in I^1} l_i \leq 2e_n - 3$, $j = 1, 2$.
2. There is a decomposition $\{(i_1, i_2), I^1, I^2\}$ of $[1, p]$ and $l_{i_1} \in [3, l_{i_2} - 3]$, $k = 1, 2$, such that $((l_{i_1} - 1, l_{i_2} - 1), (l_{i_1} - 1, l_{i_2} - 1)) \notin P$ and $l_{i_1} + l_{i_2} + \sum_{i \in I^1} l_i = 2e_n$.
3. There is a decomposition $\{(i_1, i_2, i_3), I^1, I^2, I^3\}$ of $[1, p]$ and $l_{i_1} \in [3, l_{i_2} - 3]$, $k = 1, 2, 3$, such that $l_{i_1} + l_{i_2} + \sum_{i \in I^1} l_i = e_n = l_{i_1} + (l_{i_2} - l_{i_3} - l_{i_3}) + \sum_{i \in I^1} l_i$.
4. There is a decomposition $\{(i_1, i_2), I^1, I^2, I^3\}$ of $[1, p]$ and $l_{i_1} \in [3, l_{i_2} - 3]$, $j = 1, 2$, such that $l_{i_1} + \sum_{i \in I^1} l_i = e_n$, $j = 1, 2$.

**Proof.**

1. If $l_{i_1} := 2e_n - \sum_{i \in I^1} l_i$, then $l_{i_1} \in [3, 2e_n]$, $j = 1, 2$, and $l_{i_1} = l_{i_1} + l_{i_1}$. By $S(H_n)$ there exists a good $H_n$-realisation $(T^j_{m, 1})T_j$ of $(l_{i_1})L(I^j)$, $j = 1, 2$.

2. By Proposition 19.1 we may suppose without loss of generality that $V(T^1_{m, 1}) \cap V(T^2_{m, 1}) \cap V_{n, 2}^1 \neq \emptyset$ (notice that a closed trail in $H_n$ necessarily contains a vertex of an eccentric part of $H_n$). Thus, we are done by Claim 3.2.

3. Put $l_{i_3} := l_{i_3} - l_{i_3} \in [3, l_{i_3} - 3]$. By $S(G_n)$ there is a good $G_n$-realisation $(T^j_{m, 1})T_j$ of $(l_{i_1}, l_{i_2})L(I^j)$, $j = 1, 2$; we may suppose without loss of generality that $v_{k, 2} \in V(T^1_{m, 1}) \cup V(T^2_{m, 1})$. As $V(\psi^1_n(T^1_{m, 1})) \cap V(\psi^2_n(T^2_{m, 1})) \subseteq \{v_{1, 1}\}$, there
is a trail $T_3 \in \varphi_n(T_{i_3}) + \varphi_n^2(T_{i_3})$ and $(\varphi_n^{T_3}(T_{i_3}^1), \varphi_n^2(T_{i_3}^2), T_3) \varphi_n(T_3) \varphi_n^2(T_3)$ is a good $(l_{i_3}, l_{i_2})$-global $H_n$-realisation of the sequence $(l_{i_3}^1, l_{i_2}^1, l_{i_2}^1 L \langle I^1 \rangle L \langle I^2 \rangle \sim (l_{i_3}^1, l_{i_2}^1 L \langle \{i_3 \} \cup I^1 \cup I^2 \rangle )$. Now, we are done by Claim 3.3 with the decomposition $\{ \{i_1, i_2\}, \{i_3\} \cup I^1 \cup I^2 \}$ of $T_3$ (and by Lemma 1).

4. By $S(G_n)$ there is a good $G_n$-realisation $(T_{i_3}^j) T^j$ of $(l_{i_3}^1) L \langle I^j \rangle$, $j = 1, 2$.

Since $(\varphi_n^1(T_{i_3}^1), \varphi_n^2(T_{i_3}^2)) \varphi_n(T_1) \varphi_n^2(T_2)$ is a good $(l_{i_3}^1, l_{i_2}^1)$-global $H_n$-realisation of $(l_{i_3}^1, l_{i_2}^1) L \langle I^1 \rangle L \langle I^2 \rangle \sim (l_{i_3}^1, l_{i_2}^1) L \langle I^1 \cup I^2 \rangle$, Claim 3.3 with the decomposition $\{ \{i_1, i_2\}, I^1 \cup I^2 \}$ of $T_3$ can be used.

Consider $L \in \text{Sct}(G_{n+1})$, assume that nd$(L) = (l_1, \ldots, l_p)$ and let $q \in [1, p]$ be defined by the inequalities $\sum_{i=1}^{q-1} l_i \leq 2e_n - 3$ and $\sum_{i=q}^{p-1} l_i + l_p > 2e_n - 3$.

Instead of (G1)–(G4) we are going to use the following assertions:

\begin{enumerate}
  \item $\forall m \in \{3, 5, 6\} \ 2e_n \equiv 0 \pmod{m}$, \quad (H1)
  \item $n = 0 \Rightarrow 2e_n \equiv 2 \pmod{4}$, \quad (H2)
  \item $n \in [1, \infty) \Rightarrow 2e_n \equiv 0 \pmod{4}$, \quad (H3)
  \item $\exists m \in \{3, 5, 6\} \ 2e_n \equiv m \pmod{7}$, \quad (H4)
  \item $n = 0 \Rightarrow 2e_n \equiv 6 \pmod{8}$, \quad (H5)
  \item $n \in [1, \infty) \Rightarrow 2e_n \equiv 0 \pmod{8}$. \quad (H6)
\end{enumerate}

(1) $\sum_{i=1}^{q-1} l_i + l_p = 2e_n \rightarrow 3.1$.

(2) If $\sum_{i=1}^{q-1} l_i + l_p \geq 2e_n + 3$, then $\sum_{i=q}^{p-2} l_i \leq 2e_n - 3 \rightarrow 4.1$.

(3) $\exists \delta \in \{-2, -1, 1, 2\} \sum_{i=1}^{q-1} l_i + l_p = 2e_n + \delta$

(31) $l_{p-1} \geq 9$

(311) $\delta = -2$

(3111) If $l_q \leq 4$, then $\sum_{i=1}^{q-1} l_i + l_p \leq 2e_n - 2 - 3$ and $l_1 + \sum_{i=q}^{p-2} l_i = l_1 + e_{n+1} - (2e_n - 2) - l_{p-1} \leq 4 + 2e_n + 2 - l_{p-1} \rightarrow 4.1$.

(3112) If $l_q \in [5, l_{p-1} - 1]$, then $\sum_{i=1}^{q-1} l_i \leq 2e_n - 3$ and $\sum_{i=q}^{p-1} l_i = 2e_n + 2 - l_{p-1} \leq 2e_n - 3 \rightarrow 4.1$.

(3113) If $l := l_q$ and $l_i = \text{for any } i \in [q, p]$, we obtain $2e_n + 2 = (p - q)l$ and $2e_n + 2 \geq l(2e_n + 2)$, so that $l \leq e_n + 1$ and $p - q \geq 2$. Using (H1)–(H3) we see that $3 \nmid l$, $4 \nmid l$ (because of (H3), $4 \nmid l$ implies $n = 0$, $l \mid 152$ and $l \leq 8$, while we have $l \geq 9$) and $5 \nmid l$, hence $l \geq 11$.

(31131) If $l_1 \leq l - 5$, then $\sum_{i=1}^{q-1} l_i + l_p \leq 2e_n - 2 - 3$ and $l_1 + \sum_{i=q}^{p-2} l_i = 2e_n + 2 - (l - 1) \leq 2e_n - 3 \rightarrow 4.1$.

(31132) If $l_1 = l - 2$ for some $j \in [1, q - 1]$, then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+1}^{q-1} l_i = 2e_n \rightarrow 3.1$.

(31133) $\forall i \in [1, q - 1] l_i \in \{l - 4, l - 3, l - 1, l\}$

(311331) If $f_{j-1} \geq 2$ and $l_j = l_{j+1} = l - 1$ for some $j \in [1, q - 1]$, then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+1}^{q-2} l_i = 2e_n \rightarrow 3.1$.

(311332) If $f_{j-4} + f_{j-3} \geq 2$, then $l - 4 \leq l_1 \leq l_2 \leq l - 3$, $\sum_{i=q}^{p-1} l_i = 2e_n - 2 + l - (l_1 + l_2) \leq 2e_n + 6 - l \leq 2e_n - 3$ and $l_1 + l_2 + \sum_{i=q+2}^{p-1} l_i = 2e_n + 2 - (l - l_1) - (l - l_2) \leq 2e_n - 4 \rightarrow 4.1$. 
\[(311333) \text{ If } f_{l-4} + f_{l-3} + f_{l-1} \leq 2, \text{ then } 2e_n - 2 \equiv -4f_{l-4} - 3f_{l-3} - f_{l-1} \pmod{l}, \text{ } 2e_n + 2 \equiv 0 \pmod{l}, \text{ } -4f_{l-4} - 3f_{l-3} - f_{l-1} \equiv -4 \pmod{l}, \text{ nd}(L) = (l_1, l_2)(l)p^{2-}, \text{ } p = 2q \geq 4 \text{ and } (l_1, l_2) \in \{(l - 3, l - 1), (l - 4, l)\}. \text{ Since } pl - 4 = 4e_n, \text{ we have } pl \equiv 0 \pmod{4}.

\[(313331) \text{ If } q \text{ is even, } q = 2r, \text{ then } 3 \leq \frac{r_1^1}{2} := \left\lfloor \frac{l - 2}{2} \right\rfloor \leq l - 7 \leq l_1 - 3, \text{ } 3 \leq \frac{r_2^1}{2} := \left\lfloor \frac{l - 2}{2} \right\rfloor \leq l - 4 \leq l_2 - 3, \text{ } 3 \leq \frac{r_3^1}{3} := \left\lfloor \frac{l}{2} \right\rfloor \leq l - 3 = l_3 - 3 \text{ and } \left\lfloor \frac{l - 2}{2} \right\rfloor + \left\lceil \frac{l}{2} \right\rceil + (r - 1)l = e_n = \left\lfloor \frac{l - 2}{2} \right\rfloor + (l - \left\lfloor \frac{l}{2} \right\rfloor) + (r - 1)l \rightarrow 4.3 \text{ (with } i_1 := 1, \text{ } i_2 := 2, \text{ } i_3 := 3, \text{ } I^1 := [4, r + 2], \text{ } I^2 := [r + 3, 2r + 1] \text{ and } I^3 = [2r + 2, 4r]).

\[(313332) \text{ If } q \text{ is odd, } q = 2r + 1, \text{ then } pl = 2(2r + 1)l \equiv 0 \pmod{4} \text{ implies that } l \text{ is even, } l = 2m \geq 12, \text{ } 3 \leq \frac{r_1^1}{2} := m - 1 \leq 2m - 7 \leq l - 1, \text{ } 3 \leq \frac{r_2^1}{2} := m - 1 \leq 2m - 7 \leq l_2 - 3 \text{ and } m - 1 + r \cdot 2m = e_n \rightarrow 4.4 \text{ (with } i_1 := 1, \text{ } i_2 := 2, \text{ } I^1 := [3, r + 2], \text{ } I^2 := [r + 3, 2r + 2] \text{ and } I^3 = [2r + 3, 4r + 2]).

\[(3132) \text{ If } \delta = -1, \text{ then } 3 \leq \frac{r_1^1}{4} := 6 \leq l_1 - 3, \text{ } 3 \leq \frac{r_2^1}{2} := l_2 - 5 \leq l - 3, \text{ where } ((6)4, (l_1 - 6)4, (5)4) \not\in P \text{ and } 6 + (l_2 - 5) + \sum_{i=1}^{q-1} l_1 = 2e_n \rightarrow 4.2.

\[(3133) \text{ If } \delta = 1, \text{ then } 3 \leq \frac{r_1^1}{4} := 5 \leq l_1 - 3, \text{ } 3 \leq \frac{r_2^1}{2} := l_2 - 6 \leq l - 3, \text{ where } ((5)4, (l_1 - 6)4, (4)4) \not\in P \text{ and } 5 + (l_2 - 6) + \sum_{i=1}^{q-1} l_1 = 2e_n \rightarrow 4.2.

\[(314) \delta = 2

\[(3141) \text{ If there is } j \in [1, q - 1] \text{ such that } l_j \in [5, l_1 - 1], \text{ then } \sum_{i=3}^{q-1} l_1 + \sum_{i=3}^{q-1} l_1 + l_2 \leq 2e_n + 2 - 5 \text{ and } l_3 + \sum_{i=3}^{q-2} l_1 = 2e_n - 2 - (l_1 - l_2) \leq 2e_n - 3 \rightarrow 4.1.

\[(3142) \forall i \in [1, q - 1] \text{ } l_i \in \{3, 4, l_1 - 1\}

\[(31421) \text{ If } f_3 + f_4 \geq 2, \text{ then } \sum_{i=3}^{q-1} l_i + l_2 \leq (2e_n + 2) - 2 \cdot 3 \text{ and } l_1 + l_2 + \sum_{i=3}^{q-2} l_i \leq 2 \cdot 4 + (2e_n - 2) - 9 \rightarrow 4.1.

\[(31422) \text{ If } f_3 + f_4 \leq 1, \text{ then } l_i = l_1 - 1 = \text{ for any } i \in [2, p - 1], \text{ } 2e_n + 2 = \sum_{i=1}^{q-1} l_1 + l_2 \equiv j + l_2 \pmod{l} \text{ for some } j \in \{0, 3, 4\} \text{ and } 2e_n - 2 \equiv 0 \pmod{l}, \text{ so that } j + l_2 \equiv 4 \pmod{l} \text{ and, using (H1)-(H3) 3} \n \text{ l, } 4} \n \text{ l (as } l \geq 9), \text{ } 5} \n \text{ l \text{ and } l \geq 11.}

\[(314221) \text{ If } l_2 = l + 2, \text{ then } \sum_{i=1}^{q} l_1 = 2e_n - 3.1.

\[(314222) \text{ If } l_2 \geq l + 5, \text{ then } \sum_{i=1}^{q} l_1 = (2e_n + 2) - (l_2 - l) \leq 2e_n - 3 \text{ and } \sum_{i=1}^{q-1} l_1 = (2e_n - 2) - l \leq 2e_n - 11 \rightarrow 4.1.

\[(314223) \text{ If } l_2 = l + k \text{ for some } k \in \{0, 1, 3, 4\}, \text{ then } j + k \equiv 4 \pmod{l}. \text{ Consequently, from } l \geq 11 \text{ it follows that } j + k = 4 \text{ and } (j, k) \in \{(0, 4), (3, 1), (4, 0)\}.

\[(3142231) \text{ If } (j, k) = (0, 4), \text{ then } p = 2q, \text{ nd}(L) = (l)q-1(l + 4) \text{ and } 2q + 4 = 4e_n, \text{ so that } 2ql \equiv 0 \pmod{4}.

\[(31422311) \text{ If } q \text{ is even, } q = 2r, \text{ then } 3 \leq \frac{l_1^1}{2} := \left\lfloor \frac{l + 2}{2} \right\rfloor \leq l + 1 = l_p - 3, \text{ } 3 \leq \frac{l_1^1}{2} := \left\lfloor \frac{l + 2}{2} \right\rfloor \leq l - 3 = l_1 - 3, \text{ } 3 \leq \frac{l_1^1}{2} := \left\lfloor \frac{l}{2} \right\rfloor \leq l - 3 = l_2 - 3 \text{ and } \left\lfloor \frac{l + 2}{2} \right\rfloor + \left\lceil \frac{l}{2} \right\rceil + (r - 1)l = e_n = \left\lfloor \frac{l + 2}{2} \right\rfloor + (l - \left\lfloor \frac{l}{2} \right\rfloor) + (r - 1)l \rightarrow 4.3.

\[(31422312) \text{ If } q \text{ is odd, } q = 2r + 1, \text{ then } l \text{ must be even, } l = 2m, \text{ } 3 \leq l_1^1 := m + 1 \leq 2m + 1 = l_p - 3, \text{ } 3 \leq l_1^1 := m + 1 \leq 2m - 3 = l_1 - 3 \text{ and }
\[m + 1 + r \cdot 2m = e_n \rightarrow 4.4.\]

(3142232) If \((j, k) \in \{(3, 1), (4, 0)\}\), then \(p = 2q + 1, \text{nd}(L) = (j)(l)^{2r-1}(l+4-j)\) and \(2q! \equiv 0 \pmod{4}\).

(31422321) If \(q\) is even, \(q = 2r\), then \(3 \leq l^1_p := \lceil \frac{l+1-j}{2} \rceil \leq l - 3 \leq l_p - 3, 3 \leq l^2_p := \lceil \frac{l+1-j}{2} \rceil \leq l - 3 = l_3 - 3\) and \(\lceil \frac{l+1-j}{2} \rceil + \lceil \frac{l+1-j}{2} \rceil + j + (r-1)l = e_n = \lceil \frac{l+1-j}{2} \rceil + (l - \lceil \frac{l+1-j}{2} \rceil ) + (r-1)l \rightarrow 4.3.\)

(31422322) If \(q\) is odd, \(q = 2r + 1\), then \(l\) is even, \(l = 2m \geq 12, 3 \leq l^1_p := m + 1 - j \leq 2m + 4 - j = l_p, 3 \leq l^2_p := m + 1 \leq 2m - 3 = l_2 - 3\) and \((m + 1 - j) + j + r \cdot 2m = e_n = m + 1 + r \cdot 2m \rightarrow 4.4.\)

(32) If \(l_{p-1} \leq 8\), let \(r \in [g, p]\) be defined by the inequalities \(\sum_{i=1}^{r-1} l_i \leq 2e_n - 3\) and \(\sum_{i=r}^{p} l_i > 2e_n - 3\).

(321) \(\sum_{i=1}^{r} l_i = 2e_n + 3.1.\)

(322) If \(\sum_{i=1}^{r} l_i \geq 2e_n + 3, \text{then} \sum_{i=r+1}^{p} l_i \leq 2e_n - 3 \rightarrow 4.1.\)

(323) \(\exists \varepsilon \in \{-2, -1, 1, 2\} \sum_{i=1}^{r} l_i = 2e_n + \varepsilon\)

(3231) \(\varepsilon \in [-2, -1]\)

(32311) If \(l_p \geq l_1 + 3 - \varepsilon, \text{then} \sum_{i=2}^{r} l_i = 2e_n + \varepsilon - l_1 \leq 2e_n - 4 \text{and} l_1 + \sum_{i=r+1}^{p} l_i = l_1 + (2e_n - \varepsilon) - l_p \leq 2e_n - 3 \rightarrow 4.1.\)

(32312) \(l_p \leq l_1 + 2 - \varepsilon\)

(323121) If there are \(j \in [1, r]\) and \(k \in [r+1, p]\) such that \(l_k = l_j - \varepsilon, \text{then} \sum_{i=1}^{r} l_i + \sum_{i=j+1}^{r} l_i + l_k = 2e_n \rightarrow 3.1.\)

(323122) \(\forall j \in [1, r] \forall k \in [r+1, p] l_k \neq l_j - \varepsilon\)

(3231221) \(\varepsilon = -2\)

(32312211) If \(l_r = l_1, \text{then} 2e_n - 2 \equiv 0 \pmod{2}\) and from (H1)–(H6) it follows that \(n = 0, l_1 = 4\) and \(r = 37.\)

(323122111) If \(f_5 \geq 2\), then \(35 \cdot 4 + 2 \cdot 5 = 2e_0 \rightarrow 3.1.\)

(323122112) If \(f_7 \geq 2, \text{then} 34 \cdot 4 + 2 \cdot 7 = 2e_0 \rightarrow 3.1.\)

(323122113) If \(f_5 + f_7 \leq 2, \text{then} 2e_0 + 2 = \sum_{i=38}^{p} l_i \equiv 5f_5 + 7f_7 \mod{2}\) (recall that \(l_p \leq 8\), and so \(f_5 = f_7 \leq 1)\).

(3231221131) If \(l_{p-1} \geq 7, \text{then} 3 =: l^1_{p-1} \leq l_{p-1} - 3, 3 =: l^2_{p} \leq l_{p} - 3, ((3)_5, (3)_4, ((l_{p-1} - 3)_4, (l_{p} - 3)_4) \notin P \text{ (here we use the inequality} f_7 \leq 1)\) and \(3 + 3 + 36 \cdot 4 = 2e_0 \rightarrow 4.2.\)

(3231221132) If \(l_{p-1} \leq 6, \text{then} (l_{p-1}, l_{p}) \in \{(4, 4), (4, 8), (5, 7)\}\).

(32312211321) If \(\text{nd}(L) \in \{(4)^{75}, (4)^{73}(8)\}\), we can use the fact that the graph \(G_1 = K_{10,10,10} \text{ is an edge-disjoint union of graphs} K^i \cong K_{10,10,10}, i = 1,2,3. \text{ By [7], the graph} K_{10,10} \text{ is ADTC, there is a} K^i\text{-realisation} \mathcal{T}^i \text{ of} (4)^{25}, i = 1, 2, 3, \text{and a} K^3\text{-realisation} \mathcal{T}^3 \text{ of} (4)^{23}(8). \text{ Then} \mathcal{T}^1\mathcal{T}^2\mathcal{T}^3 \text{ is a good} H_1\text{-realisation of} (4)^{75} \text{ and} \mathcal{T}^1\mathcal{T}^2\mathcal{T}^3 \text{ is a good} H_1\text{-realisation of} (4)^{73}(8).\)

(32312211322) If \(\text{nd}(L) = (4)^{72}(5, 7), \text{consider again the above} H_1\text{-realisation of} (4)^{75}. \text{ By Lemma 1 and Proposition 6 we may suppose without loss of generality that} \mathcal{T}^1\mathcal{T}^2\mathcal{T}^3 = (T_1, T_2, T_3)\mathcal{T}, \text{where} V(T_i) = \{v_{1,i}, v_{2,i}, v_{1,i+1}, \ldots \}.\)
\(v_{2,i+1}\), \(i = 1, 2\), and \(V(T_3) = \{v_{1,1}, v_{2,1}, v_{1,3}, v_{2,3}\}\). Now \(T((v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, v_{1,3}, v_{1,1}), (v_{1,1}, v_{2,2}, v_{2,3}, v_{2,1}, v_{1,3}, v_{1,2}, v_{2,3}, v_{1,1}))\) is a good \(G_1\)-realisation of \(\text{nd}(L)\).

(32312212) If \(l_r = l_1 + 1\), then \(f_{l_1+2} = f_{l_1+3} = 0\).

(32312212) If \(l_{p-1} = l_1 + 1\), then \(2e_n + 2 \leq (p - r)(l_1 + 1) + 3\), \(p - r \geq \frac{2e_n - 1}{l_{r+1}} \geq \lceil \frac{4r}{3} \rceil\) is 18 and \(l_{r+1} = l_{r+2} = l_1 + 1\).

(323122111) If \(l_2 = l_1\), then \(\sum_{i=3}^{r+2} l_i = 2e_n \to 3.1\).

(3231221212) If \(l_2 = l_1 + 1\), then \(2e_n - 2 = r(l_1 + 1) - 1\) and \(2e_n \equiv 1 \mod (l_1 + 1)\) in contradiction with (H1)-(H6).

(323122122) If \(l_{p-1} = l_1 + 4\), then \(l_1 \leq 4\).

(3231221221) If \(l_1 = 3\), there is \(I \subseteq [1, r]\) such that \(\sum_{i \in I} l_i = 12\) and \(\sum_{i \in [1, r]-I} l_i + l_{p-1} + l_r = 2e_n \to 3.1\).

(32312222) If \(l_1 = 4\), then \(4f_5 + 5f_5 = 2e_n - 2 \equiv 0 \mod 2\), hence \(f_5 \geq 2\). \(\sum_{i=3}^{r+2} l_i + l_p = 2e_n - 4\) and \(\sum_{i=r+2}^{p} l_i \to 4.1\).

(32312213) If \(l_r = l_1+2\), then \(l_1 + 3\) for any \(j \in [r+1, p]\), \(l_i \in \{l_1, l_1+2\}\) for any \(i \in [1, r]\), \(2e_n + 2 \equiv 0 \mod (l_1 + 3)\) and, since \(l_1 \leq 5\), from (H1) it follows that \(l_1 \in [4, 5]\). We have also \(2e_n + 2 = (p - r)(l_1 + 3)\), hence \(l_{r+1} = l_{r+2} = l_1 + 3\).

(323122131) If \(l_1 = 4\), there is \(I \subseteq [1, r]\) such that \(\sum_{i \in I} l_i = 12\), hence \(\sum_{i \in [1, r+2]-I} l_i = 2e_n \to 3.1\).

(323122132) If \(l_1 = 5\), then \(n = 0\) and \(5f_5 + 7f_7 = 2e_0 - 2 = 148\).

(3231221321) If \(f_5 \geq 2\), then \(\sum_{i=3}^{r+1} l_i = 2e_0 - 4 = l_1 + l_2 + \sum_{i=r+2}^{p} l_i \to 4.1\).

(3231221322) If \(f_7 \geq 2\), then \(\sum_{i=1}^{r+2} l_i + l_{r+1} + l_{r+2} = 2e_0 \to 3.1\).

(32312214) \(l_r = l_1 + 3\)

(3231221411) If \(l_{p-1} = l_1 + 3\), then \(l_i \in \{l_1, l_1+2, l_1+3\}\) for any \(i \in [1, r]\), there is \(m \in [0, 1]\) such that \(2e_n + 2 \equiv m \mod (l_1 + 3)\), \(l_1 \in [4, 5]\) and \(l_{r+1} = l_{r+2} = l_1 + 3\).

(32312214111) \(l_1 = 4\)

(323122141111) If \(f_4 \geq 3\) or \(f_6 \geq 2\), there is \(I \subseteq [1, r]\) such that \(\sum_{i \in I} l_i = 12\) and \(\sum_{i \in [1, r+2]-I} l_i = 2e_n \to 3.1\).

(32312214112) \(f_4 \leq 2\) and \(f_6 \leq 1\)

(323122141121) If \(l_p = 7\), then \(\text{nd}(L) = (4, 6)(7)^{2r-2}, 14r - 4 = 4e_n, \text{hence } r \text{ is even, } r = 2s \text{ and } 7(s - 1) + 6 = e_n\). By \(S(G_n)\) (see the row 2 of Table 1) and Proposition 6 there are good \(G_n\)-realisations \((T^1_1, T^1_2, T^1_3, T^1_4)\) of \((3)^3(4)^7(7)^2(7)^3(7)^4(7)^5(7)^6(7)^7\) such that \(v_{i,2} \in V(T^1_i)\), \(i = 1, 2, 3, \) and \(v_{1,1} \in V(T^1_4) \cap V(T^1_5)\). As \(V(\varphi_n(T^1_4)) \cap V(\varphi_n(T^1_5)) \supseteq \{v_{1,1}\}\), there is \(T \in \varphi_n(4T^1_4) + \varphi_n(7T^1_5)\) and \(\prod_{i=1}^4 (\varphi_n(T^1_i)) (\varphi_n(T^1_2), \varphi_n(T^1_3), T, \varphi_n(T^1_4))\varphi_n(7T^1_5)\) is a good \(H_n\)-realisation of \((3)^3(4)^7(7)^2(7)^3(7)^4(7)^5(7)^6(7)^7\) with \(v_{i,2} \in V(\varphi_n(T^1_i))\), \(i = 1, 2, 3, \) and \(S(H_n)\) (the row 7 of Table 2) there is a good \(H_n\)-realisation \((T^3_1, T^3_2, T^3_3, T^3_4)\) of \((4)^3(7)^{2r-2}\) such that \(v_{i,2} \in V(T^3_i)\), \(i = 1, 2, 3\). Since \(V(\psi_n(T^3_1)) \cap \)
there is a trail $T_i \in \psi_n^i((\varphi_n^i(T^1_i))) + \psi_n^i(T^3_i)$, \(i = 1, 2, 3, \) and \(\psi_n^i(\varphi_n^i(T^2_i))\), \(\psi_n^i(\varphi_n^i(T^3_i))\), \(T_1, T_2, T_3\), \(\psi_n^i(T)\) \(\psi_n^i((\varphi_n^i(T^1) + \varphi_n^i(T^2)) \psi_n^i(T^3)\) is a good $G_{n+1}$-realisation of $nd(L)$. 

(323122141122) If $l_p = 8$, then $nd(L) = (4)(7)^{2r-2}(8)$, \(14r-2 = 4e_n\), hence $r$ is odd, $r = 2s+1$, 3 =: $l_1^1 \leq l_p - 3$, $3 =: l_2^1 \leq l_3 - 3$, \((3)(3), (3)(3), (5)(4), (4)(4) \notin \mathcal{P}\) and $3 + 3 + 2s \cdot 7 = 2e_n \rightarrow 4.2$. 

(323122141122) If $l_1 = 5$, then $n = 0$, $l_p = 8$, $2e_0 - 2 = 148 \equiv 5f_5 + 7f_7 \pmod{2}$ and $f_5 + f_7 \geq 2$. 

(323122141121) If $f_5 \geq 2$, then $\sum_{i=3}^{r+1} l_i = 2e_0 - 4 = l_1 + l_2 + \sum_{i=r+2}^{p-1} l_i \rightarrow 4.1$. 

(323122141122) If $f_7 \geq 2$, and $l_1 = l_{j+1} = 7$ for some $j \in [2, r - 2]$, then $\sum_{i=1}^{r-2} l_i + \sum_{i=j+2}^{r+1} l_i = 2e_0 \rightarrow 3.1$. 

(323122141123) If $f_5 = f_7 = 1$, then $nd(L) = (5, 7)(8)^{36}$. By $S(G_0)$ and Propositions 6 and 16.1 there are good $G_0$-realisations $(T^1_1, T^2_1, T^3_1)$ of (3)(4)(8)(8) and $(T^2_1, T^2_2, T^2_3)T^2$ of (4)(3)(8)(8) such that $\psi_{i+1} \in V(T^1_i) \cap V(T^2_i)$, \(i = 1, 2, 3\). As $V(\varphi_n^i(T^1)) \cap V(\varphi_n^i(T^2)) \subseteq \{v_{i+1}^1\}$, there is a trail $T_i \in \varphi_n^i(T^1) + \varphi_n^i(T^2)$, \(i = 1, 2, 3\), \(\varphi_n^i(T^1), \varphi_n^i(T^2), T_1, T_2, \varphi_n^i(T^1) \varphi_n^i(T^2)\) is a good $(3, \mathcal{P})$-global $H_0$-realisation of $(3, 4)(7)(8)^{17}$. Since $3 =: l_3^1 \leq l_3 - 3$ and $3 \leq l_4^1 := 4 \leq l_4 - 3$, we are done by Claim 3.3.

(323122141122) If $l_p - 1 = l_4 + 4$, then $l_1 \leq 4$ and $f_{i+2} = 0$. 

(323122141122) If $l_1 = 3$, there is $I \subseteq [1, r]$ such that $\sum_{i \in I} l_i = 12$ and $\sum_{i \in [1, r] - I} l_i + l_p - 1 + l_p = 2e_n \rightarrow 3.1$.

(323122141122) If $l_1 = 4$, then $\sum_{i=2}^{r-1} l_i + l_p = 2e_n - 5$ and $l_1 + \sum_{i=r+2}^{p-2} l_i = 2e_n - 3 \rightarrow 4.1$.

(323122141122) If $l_r = l_1 + 4$, then $2e_n + 2 \equiv 0 \pmod{l_1 + 4}$, $2e_n - 2 \equiv l_1 \pmod{l_1 + 4}$, $f_{i+2} = 0$ and $l_1 \leq 4$.

(323122151) $l_1 = 3$ 

(3231221511) If $f_3 \geq 4$, $f_4 \geq 3$ or $f_6 \geq 2$, there is $I \subseteq [1, r]$ such that $\sum_{i \in I} l_i = 12$ and $\sum_{i \in [1, r+2] - I} l_i = 2e_n \rightarrow 3.1$.

(3231221512) $f_3 \leq 3 \land f_4 \leq 2 \land f_6 \leq 1$ 

(3231221512) If $l_2 = 2$ and $l_4 = l_{j+1} = 4$ for some $j \in [2, r - 2]$, then $\sum_{i=1}^{r-1} l_i + \sum_{i=j+2}^{r+1} l_i = 2e_n - 3$ and $l_j + l_{j+1} + \sum_{i=r+2}^{p-1} l_i = 2e_n - 4 \rightarrow 4.1$.

(3231221512) $f_3 = 1$

(32312215122) If $nd(L) = (3)(2)(4)(7)^{4e_n-10}$, by $S(H_n)$ (the rows 1 and 7 of Table 2) there are good $H_n$-realisations $(T^1_1, T^2_1, T^3_1)T^1$ of $(3)(4)(7)^{2e_n-12}$ and $(T^3_2, T^3_2, T^3_3)T^2$ of $(4)(3)(7)^{2e_n-12}$ such that $\psi_n^i(T^1) \cap \varphi_n^i(T^2) \cap \psi_n^i(T^3) \subseteq \{v_{i+1}\}$, \(i = 1, 2, 3\) \rightarrow 3.2 (with $i_1 := 4$, $i_2 := 5$, $I^1 := [1, 2] \cup [6, r+2]$, $I^2 := \{3\} \cup [r + 3, 2r - 1]$ and $l_{1k}^i := 3$, $l_{2k}^i := 4$, $k = 1, 2$).

(32312215122) If $nd(L) = (4, 6)(7)^{4e_n-10}$, by $S(H_n)$ (the rows 4 and 7 of Table 2) there are good $H_n$-realisations $(T^1_1, T^2_1)T^1$ of $(3)(2)(6)(7)^{2e_n-12}$
and \((T_1^2, T_2^2)T^2\) of \((4)^3(7)^{\frac{2n-12}{7}}\) such that \(V(T_1^4) \cap V(T_2^4) \cap V_{n,2} \supseteq \{v_{i,2}\}\), \(i = 1, 2 \rightarrow 3.2\).

(32312215123) If \(f_4 = 0\), then \(nd(L) = (3)^3(7)^{\frac{2n-3}{7}}\) and we can proceed analogously as in (323122151221) (with \(i_1 = 2\), \(i_2 = 3\), \(i_3 = 4\), \(I_1 = \{1\} \cup [5, r + 2]\), \(I_2 = [r + 3, 2r]\) and \(l_{i,k}^1 = 3\), \(l_{i,k}^2 = 4\), \(k = 1, 2, 3\).

(3231222) If \(\varepsilon = -1\), then \(l_p \leq l_1 + 3\).

(32312221) If \(l_r = l_1\), then \(2e_n - 1 \equiv 0 \pmod{2n_1}\) in contradiction with (H1)–(H6).

(32312222) If \(l_r = l_1 + 1\), then \(l_j = l_1 + 3\) for any \(j \in \{r + 1, 2\}\), \(2e_n + 1 \equiv 0 \pmod{2n_1 + 3}\), \(l_1 \leq 5\) and, because of (H1)–(H6), \(l_1 = 4\). Since \(4f_4 + 5f_5 = 2e_n - 1 \geq 149\), there is \(I \subseteq \{1, r\}\) such that \(\sum_{i \in I} l_i = 20\) and \(\sum_{i \in [1,r+3]} l_i = 2e_n \rightarrow 3.1\).

(32312223) If \(l_r = l_1 + 2\), then \(l_j = l_1 + 2\) for any \(j \in \{r + 1, 2\}\), hence \(2e_n + 1 \equiv 0 \pmod{2n_1 + 2}\), \(l_1 \leq 6\) and, because of (H1)–(H6), \(l_1 = 5\) and \(2e_n - 1 \equiv 5 \pmod{7}\). Moreover, \(l_1 \in \{5, 7\}\) for any \(i \in \{1, r\}\).

(32312224) If \(f_5 \geq 4\), then \(\sum_{i=1}^{r+3} l_i = 2e_n \rightarrow 3.1\).

(323122232) If \(f_5 \leq 3\), then \(nd(L) = (5)^3(7)^{\frac{2n-5}{7}}\), \(l_{i,k}^1 \equiv 3 \leq l_i - 3, i = 2, 3\), by \(S(G_1)\) there is a good \(G_n\)-realisation \(T^4\) of \((3)^3(7)^{\frac{2n-1}{7}}\) and \(\varphi_n^1(T^4)\) is a good \((3, 3)\)-global \(H_n\)-realisation of \((3)^3(7)^{\frac{2n-6}{7}} \rightarrow 3.3\).

(32312224) If \(l_r = l_1 + 3\), then \(2e_n + 1 \equiv 0 \pmod{2n_1 + 3}\), \(l_1 \leq 5\), from (H1)–(H6) it follows that \(l_1 = 4\), hence \(f_6 = 0\) and \(2e_n - 1 \equiv 5 \pmod{7}\).

(323122241) \(\exists j \in \{2, r - 1\} l_j = 5\)

(3231222411) If \(f_4 \geq 2\), then \(\sum_{i=3}^{r-1} l_i + \sum_{i=j+1}^{r+2} l_i = 2e_n \rightarrow 3.1\).

(3231222412) \(f_4 = 1\)

(32312224121) If \(f_5 \geq 4\), then \(l_1 + \sum_{i=6}^{r+3} l_i = 2e_n \rightarrow 3.1\).

(32312224122) If \(f_5 \leq 3\), then \(nd(L) = (4)^3(7)^{\frac{2n-12}{7}}\), \(l_1 + l_2 + \sum_{i=5}^{r+1} l_i = 2e_n - 4\) and \(l_3 + l_4 + \sum_{i=r+2}^{n-2} l_i = 2e_n - 3 \rightarrow 4.1\).

(323122242) \(f_5 = 0\)

(3231222421) If \(f_4 \geq 5\), then \(\sum_{i=1}^{r+3} l_i = 2e_n \rightarrow 3.1\).

(3231222422) If \(f_4 \leq 4\), then \(nd(L) = (4)^3(7)^{\frac{2n-12}{7}}\). By \(S(H_n)\) (the rows 4 and 6 of Table 2) and Proposition 19.1 there are good \(H_n\)-realisations \((T_1^3, T_2^3)T^2\) of \((3)^3(7)^{\frac{2n-6}{7}}\) and \(\prod_{i=1}^{5}(T_i)\) of \((4)^3(7)^{\frac{2n-20}{7}}\) such that \(v_{i,3} \in V(T_1^3), i = 1, 2, \psi_{v_{i,3}}(T_2^3) \in V(T_1^3), i = 1, 2, 3\), and \(v_{i,3} \in V(T_3, T_5)\), \(i = 1, 2\). As \(V(\psi_{v_{i,3}}(T_1^3)) \cap V(\psi_{v_{i,3}}(T_3, T_5)) \supseteq \{v_{i,3}^1\} = \{v_{i,3}^1\}\), there is a trail \(T_1 \in \psi_{v_{i,3}}(T_1^3) + \psi_{v_{i,3}}(T_3, T_5), i = 1, 2, 3\), and \(\psi_{v_{i,3}}(T_2^3, T_6)T_2^3T_6^3(T_1, T_2)\psi_{v_{i,3}}(T_1)\) is a good \(G_{n+1}\)-realisation of \(nd(L)\) with \(v_{i,2} \in \psi_{v_{i,3}}(T_1^3), i = 1, 2, 3\). Since \(p_{i(3-2n+1)}(v_{i,2}^1) = 2\), \(i = 1, 2, 3\), by Proposition 6 there is a good \(G_{n+1}\)-realisation \((T_1, T_2, T_3)T^6\) of \(nd(L)\) such that \(v_{i,1} \in V(T_1), i = 1, 2, 3\) (satisfying the conditions of the rows 5 of Table 1).
(3232) If $\varepsilon \in [1, 2]$, then $l_r = \sum_{i=1}^{r} l_i - \sum_{i=1}^{r-1} l_i \geq 2e_n + \varepsilon - (2e_n - 3) = 3 + \varepsilon$.
With $l := \min(l_i : i \in [1, r], l_i \geq 3 + \varepsilon)$ we have $3 + \varepsilon \leq l \leq l_r$.

(32321) If $l_p \geq l + 3 - \varepsilon$, let $j \in [1, r]$ be such that $l_j = l$. Then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+1}^{r} l_i = 2e_n + \varepsilon - l \leq 2e_n + \varepsilon - (3 + \varepsilon)$ and $l_j + \sum_{i=r+1}^{p-1} l_i = l + (2e_n - \varepsilon) - l_p \leq 2e_n - 3 \rightarrow 4.1$.

(32322) $\forall i \in [1, p] l_i \in [3, 2 + \varepsilon] \cup [l, l + 2 - \varepsilon]$.

(323221) $\varepsilon = 1$

(3232211) If $l_r = l + 1$, then $2e_n - 1 \equiv 0 \pmod{l + 1}$ in contradiction with $l \leq 7$ and (H1)–(H6).

(3232212) $l_r = l$

(32322112) If $l_{p-1} = l$, then $2e_n - 1 \equiv k \pmod{l}$ for some $k \in [0, 1]$ and then from (H1)–(H6) it follows that $l = 4$, $n = 0$, $2e_0 + 1 \equiv 3 \pmod{4}$ and $f_3 \geq 1$.

(323221211) If $f_3 \geq 3$, then $\sum_{i=1}^{r+2} l_i = 2e_0 \rightarrow 3.1$.

(323221121) If $f_3 \leq 2$, then $\text{nd}(L) = (3)(4)^2(5)$. If $T$ is the sequence of closed trails from (3231221322), then $((v_{1,1}, v_{1,2}, v_{1,3}, v_{1,1}), (v_{1,3}, v_{2,1}, v_{2,2}, v_{1,3}), T((v_{1,1}, v_{2,2}, v_{1,2}, v_{2,3}, v_{1,1}))$ is a good $G_1$-realisation of $\text{nd}(L)$.

(32322122) $l_{p-1} = l + 1$

(323221221) If $f_3 = 0$, then $2e_n + 1 = rl$, hence from $l \leq 7$ and (H1)–(H4) it follows that $l = 7$, $\sum_{i=1}^{r-2} l_i + l_p = 2e_n - 5$ and $\sum_{i=r-1}^{p-2} l_i = 2e_n - 3 \rightarrow 4.1$.

(323221222) $f_3 \geq 1$

(3232212221) $l = 4$

(3232212222) If $f_3 \geq 1$, then $2e_n + 1 \equiv 3 \pmod{4}$, hence from (H3) it follows that $n = 0$ and $2e_0 - 1 \equiv 1 \pmod{4}$. Further, $2e_0 - 1 \equiv f_5 \pmod{4}$, $f_5 \geq 5$ (as $l_{p-1} = l_p = 5$) and $\sum_{i=4}^{r-1} l_i + \sum_{i=p-2}^{r} l_i = 2e_0 \rightarrow 3.1$.

(32322122212) If $f_3 \geq 2$, then $\sum_{i=1}^{r-2} l_i + l_p = 2e_n \rightarrow 3.1$.

(32322122222) $l \in [5, 7]$.

(32322122221) If $l_{r-1} = 3$, then $2e_n + 1 \equiv l \pmod{3}$, hence, by (H1), $l = 7$ and $\sum_{i=4}^{r-1} l_i + l_p = 2e_n \rightarrow 3.1$.

(32322122222) If $l_{r-1} = l$, then $\sum_{i=1}^{r-2} l_i + l_{p-1} = (2e_n + 1 - 2l) + (l + 1) \leq 2e_n - 3$ and $\sum_{i=r-1}^{p-2} l_i = 2e_n - 3 \rightarrow 4.1$.

(323222) If $\varepsilon = 2$, then $2e_n - 2 \equiv 0 \pmod{l}$ in contradiction with (H1) and (H4).

Now, it remains to be proved that there are good $G_{n+1}$-realisations of four sequences from $\text{Sct}(G_{n+1})$ according to the row $t$ of Table 1, $t \in [1, 4]$.

$t = 1$: By $S(H_n)$ (the row 4 of Table 2) and Proposition 19.1 there are good $H_n$-realisations $(T_1^1, T_2^1)T^1$ and $(T_1^2, T_2^2)T^2$ of $(3)^2(7)^{2+\varepsilon}$ such that $v_{i,2}^1 \in V(T_i^1)$, $i = 1, 2$, and $v_{i,2}^3 \in V(T_i^2)$. If $T_i := \psi_i^1(T_i^1)$, $i = 1, 2$ and $T_3 := \psi_i^2(T_i^2)$, then $(T_1, T_2, T_3)\psi_i^1(T_i^1)\psi_i^2(T_i^2)$ is a good $G_{n+1}$-realisation of $(4)^3(7)^{\frac{4+\varepsilon}{1-\varepsilon}}$ with $v_{i,2}^{1,1} \in V(T_i)$, $i = 1, 2, 3$. As $p(5, 2^{n+1})_3(v_{i,2}^{1,1}) = 2$, $i = 1, 2, 3,$
it suffices to use Proposition 6.

\( t = 2 \): By \( S(H_n) \) (the row 2 of Table 2) and Proposition 19.1 there are good \( H_n \)-realisations \((T^1_1, T^2_1)T^1\) of \((3)^2(4)(7)^{\frac{2n-10}{2}}\) and \((T^2_2)T^2\) of \((3)(7)^{\frac{2n-3}{2}}\) such that \(v^{1,1}_{i,2} \in V(T^1_i), i = 1, 2,\) and \(v^{1,2}_{3,2} \in V(T^2_1)\). If \( \bar{T}_i := \psi^1_n(T^1_i), i = 1, 2,\) and \( \bar{T}_3 := \psi^2_n(T^2_1)\), then \((\bar{T}_1, \bar{T}_2, \bar{T}_3)\psi^1_n(T^1_1)\psi^2_n(T^2_1)\) is a good \( G_{n+1} \)-realisation of \((3)^3(4)(7)^{\frac{2n-12}{2}}\) with \(v^{1,1}_{i,2} \in V(\bar{T}_i), i = 1, 2, 3\). As for \( t = 1 \), employ Proposition 6.

\( t = 3 \): By \( S(H_n) \) (the rows 7 and 8 of Table 2) and Proposition 19.1 there are good \( H_n \)-realisations \((T^1_1, T^2_1, T^3_1)T^1\) of \((3)^3(4)(7)^{\frac{2n-14}{2}}\), \( j = 1, 2,\) such that \(v^{1,1}_{i,3} \in V(T^1_i), i = 1, 2, 3, v^{1,2}_{i,2} \in V(T^2_i), i = 1, 2,\) and \(v^{1,1}_{1,1} \in V(T^3_2)\). If \( \bar{T}^*_i := \psi^1_n(T^1_i), i = 1, 2, 3, j = 1, 2,\) then \(\prod_{i=1}^3(\bar{T}^*_i)\prod_{i=1}^3(\bar{T}^*_i)\psi^1_n(T^1_1)\psi^2_n(T^2)\) is a good \( G_{n+1} \)-realisation of \((4)^6(7)^{\frac{4n-24}{2}}\) with \(v^{1,1}_{i,3} \in V(\bar{T}^*_i), i = 1, 2, 3, v^{1,2}_{i,2} \in V(\bar{T}^*_i), i = 1, 2,\) and \(v^{1,1}_{1,1} \in V(\bar{T}^*_i)\). Since \(p_{(5,2n+1)}(v^{1,1}_{1,3}) = 3, i = 1, 2, 3, p_{(5,2n+3)}(v^{1,2}_{1,2}) = 2, i = 1, 2,\) and \(p_{(5,2n+1)}(v^{1,2}_{1,1}) = 1,\) we are done again due to Proposition 6.

\( t = 4 \): By \( S(H_n) \) (the row 5 of Table 2) and Proposition 19.1 there are good \( H_n \)-realisations \(\prod_{i=1}^5(T^1_i)T^1\) of \((4)^6(7)^{\frac{4n-24}{2}}\) and \((T^2_2)T^2\) of \((3)(7)^{\frac{2n-3}{2}}\) such that \(v^{1,2}_{i,2} \in V(T^1_i), v^{2,3}_{i,3} \in V(T^1_3), i = 1, 2, 3,\) and \(v^{1,1}_{3,3} \in V(T^2_2)\). As \(V(\psi^1_n(T^1_6)) \cap V(\psi^2_n(T^2_1)) \ni \{v^{2,1}_{1,3}\} = \{v^{1,2}_{3,3}\}\), there is a trail \( \bar{T} \in \psi^1_n(T^1_6) + \psi^2_n(T^2_1)\) and \(\prod_{i=1}^5(\psi^1_n(T^1_i)))(\bar{T})\psi^1_n(T^1_1)\psi^2_n(T^2_1)\) is a good \( G_{n+1} \)-realisation of \((4)^6(7)^{\frac{4n-20}{2}}\) with \(v^{1,1}_{i,2} \in V(\bar{T}_i), i = 1, 2, 3,\) and \(v^{2,1}_{i,3} \in V(\bar{T}^*_3), i = 1, 2.\) Because of \(p_{(5,2n+1)}(v^{1,1}_{1,2}) = 2, i = 1, 2, 3,\) and \(p_{(5,2n)}(v^{2,1}_{1,3}) = 3, i = 1, 2,\) we are done with help of Proposition 6. □
References


[8] D. Sotteau, Decomposition of $K_{m,n}$ ($K^*_{m,n}$) into cycles (circuits) of length $2k$, J. Combin. Theory Ser. B 30 (1981) 75–81