Non-intersecting detours in strong oriented graphs

Susan van AARDT*

Department of Mathematical Sciences, University of South Africa,
P.O. Box 392, Pretoria, 0003 South Africa
vaardsa@unisa.ac.za

Gabriel SEMANIŠIN**

Institute of Mathematics, P.J. Šafárik University,
Jesenná 5, 041 54 Košice, Slovak Republic
semanisin@science.upjs.sk

Mathematics Subject Classifications: 05C20, 05C38

Abstract

One of the classical results in graph theory states that every two longest path in a connected graph (also called detours) have a vertex in common. The corresponding problem for three longest paths in a graph is still unsolved. One can easily construct an oriented graph having q non-intersecting detours for any integer $q \geq 2$. But the situation is more complicated if we require the oriented graph to be strong.

We prove that for $k \leq 7$ there is no strong oriented graph with non-intersecting detours of order k. For $k \geq 8$ we provide a construction of an infinite class of strong oriented graphs with approximately $\sqrt{k}$ non-intersecting detours.

Keywords: digraph, oriented graph, strong connectivity, longest path

*This material is based upon work supported by the National Research Foundation under Grant number 2053848.

**This material is based upon work supported by Slovak VEGA grant 1/0424/03 and by the National Research Foundation under Grant number 2053752.

1 Introduction and preliminaries

The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively, and the order of a digraph is denoted by $n(D)$. A *directed* path (cycle) in a digraph will simply be called a path (cycle). If the order of a cycle is equal to m, we call it an m-cycle. A digraph D is *strong* if every vertex of D is reachable from every other vertex of D. An *oriented* graph is a digraph with no cycle of order 2.

We call a longest path in a digraph a *detour* of D. The number of vertices in such a path is called the *detour order* of D and is denoted by $\lambda(D)$. The out-neighbourhood and in-neighbourhood of a vertex v are denoted by $N^+(v)$ and $N^-(v)$ respectively. The vertices in $N^+(v)$ and $N^-(v)$ are called the out-neighbours and in-neighbours of v.

In [1], page 16, Bondy mentioned two basic questions regarding graphs and digraphs:

1. How are the properties of a graph reflected in the properties of its orientations?
2. Which properties of graphs extend to digraphs?

It is well-known that important information about a graph can be gained from studying the detours of its orientations. For example, the Gallai-Roy-Vitaver theorem [5] states that the chromatic number of a graph equals the minimum detour order among all its orientations.

For (undirected) graphs, T. Zamfirescu posed the following question which is formulated in Voss [4]: what is the largest integer r such that in every connected graph, every set of r longest paths have a common vertex? (See also [6].) It is well-known that the intersection of any two longest paths in a connected graph is non-empty. (See Ore [2, Theorem 2.5.1] for a proof.) Skupień [3] obtained, for every integer $q \geq 7$, a connected graph in which q longest paths have an empty intersection, but any $q-1$ longest paths have a non-empty intersection.

It therefore seems natural to investigate the intersection of detours in digraphs.

It is easy to construct, for any integer $q \geq 2$, a connected oriented graph in which q detours have an empty intersection. Simply take q disjoint paths of order k each, for some $k \geq 2$ and direct an arc from the initial vertex of one of the paths to the end vertices of each of the others. However, for *strong* digraphs, the situation is more complicated.

In this paper we construct an infinite class of strong oriented graphs of detour order k, $k \geq 8$, with $q(k)$ non-intersecting detours where $q(k) \approx \sqrt{k}$ and we show that there does not exist a strong oriented graph of detour order less than eight with two non-intersecting detours.

2 Strong oriented graphs with detour order at most 9

If W is a set of vertices of a digraph D, then $\langle W \rangle_D$ will denote the subgraph of D induced by W. If confusion is unlikely we simply write $\langle W \rangle$ instead of $\langle W \rangle_D$. If D^* is a subdigraph of D and v a vertex of D^*, then $b_{D^*}(v)$ and $e_{D^*}(v)$ denote the maximum order of a path in D^* that starts at v and ends at v, respectively.

The following easy observations give some indication of the structure of strong oriented graphs containing non-intersecting detours.

Lemma 2.1 *Let D be a strong oriented graph with detour order $k < n(D)$ and let P denote a detour $u_1 u_2 \ldots u_k$ in D. Then D has the following properties:*
(i) $u_k u_1 \notin A(D)$*;*

(ii) $N^-(u_1) \subset V(P)$ *and* $N^+(u_k) \subset V(P)$*;*

(iii) *If* $k \geq 3$*, then* $b_P(u_i) \geq 3$ *and* $e_P(u_i) \geq 3$*;*

(iv) *If there exist arcs* $u_r u_1$ *and* $u_k u_s$*, with* $s \leq r+1$*, then* $b_P(u_i) > k/2$ *and* $e_P(u_i) > k/2$*, for every* $i \in \{1, \ldots, k\}$*.*

Proof. (i) If $u_k u_1 \in A(D)$ then $b_P(u_i) = e_P(u_i) = k$ for all $u_i \in V(P)$ and hence $b_D(v) > k$ for all $v \in V(D) \setminus V(P)$.

(ii) Suppose there is a $w \in N^-(u_1)$ such that $w \notin V(P)$, then the path $wu_1 \ldots u_k$ in D is of order $k+1$. The proof for $N^+(u_k) \subset V(P)$ is similar.

(iii) Suppose there is a vertex u_i in $V(P)$ which does not lie on a cycle of order at least 3 in $\langle V(P) \rangle$. Then from statement (ii) it follows that $u_i \notin \{u_1, u_2, u_3\}$ and $u_i \notin \{u_{k-2}, u_{k-1}, u_k\}$. But then $k \geq 7$ and any path in $\langle V(P) \rangle$ that ends or starts at such a vertex u_i is of order at least 4.

(iv) First we consider the case where $s \leq r$. For any vertex $u_i \in V(P)$ it is easy to see that

$$b_P(u_i) \geq \begin{cases} \max\{r, k-i+1\} \text{ for } 1 \leq i \leq s-1, \\ \max\{r, k-s+1\} \text{ for } s \leq i \leq r, \\ \max\{k-s+1, k-i+1+r\} \text{ for } r+1 \leq i \leq k. \end{cases}$$

Also,

$$e_P(u_i) \geq \begin{cases} \max\{r, k-s+1+i\} \text{ for } 1 \leq i \leq s-1, \\ \max\{r, k-s+1\} \text{ for } s \leq i \leq r, \\ \max\{k-s+1, i\} \text{ for } r+1 \leq i \leq k, \end{cases}$$

so that clearly $b_P(u_i) > k/2$ and $e_P(u_i) > k/2$.

Next we consider the case where $s = r+1$. There exist an $i \in \{r+1, \ldots, k\}$ and $j \in \{1, 2, \ldots, r\}$ such that there is a $u_i - u_j$-path Q with no internal vertices on P. If $n(Q) > 2$, then the path $u_{i+1}u_{i+2}\ldots u_k u_{r+1}u_{r+2}\ldots u_{i-1}Q$ $u_{j+1}u_{j+2}\ldots u_r u_1 \ldots u_{j-1}$ is of order greater than k. Hence $V(Q) = \{u_i, u_j\}$ and thus $u_iu_j \in A(D)$. For any vertex $u_\ell \in V(P)$ it is now easy to see that

$$b_P(u_\ell) \geq \begin{cases} \max\{k-\ell+1, r\} \text{ for } 1 \leq \ell \leq r, \\ \max\{k-r, r+1\} \text{ for } r+1 \leq \ell \leq k, \end{cases}$$

and

$$e_P(u_\ell) \geq \begin{cases} \max\{r, k-r+1\} \text{ for } 1 \leq \ell \leq r, \\ \max\{\ell, k-r\} \text{ for } r+1 \leq \ell \leq k, \end{cases}$$

so that $b_P(u_\ell) > k/2$ and $e_P(u_\ell) > k/2$. ■

Lemma 2.2 *Suppose D is a strong oriented graph with two non-intersecting detours $u_1u_2\ldots u_k$ and $v_1v_2\ldots v_k$ denoted by P and Q, respectively. If $u_iv_j \in A(D)$, then $i < j$ and $e_P(u_i) + b_Q(v_j) \leq k$.*

Proof. The path $u_1 \ldots u_iv_j \ldots v_k$ has order $i+k-j+1$ and a path in $\langle V(P)\rangle$ of order $e_P(u_i)$ which ends at u_i followed by a path in $\langle V(Q)\rangle$ of order $b_Q(v_j)$ which starts at v_j is clearly of order $e_P(u_i) + b_Q(v_j)$. Hence $i+k-j+1 \leq k$ and simultaneously $e_P(u_i) + b_Q(v_j) \leq k$. ■

Using the results above we now show that there does not exist a strong oriented graph with two non-intersecting detours of order less than eight.

Theorem 2.3 *Let D be a strong oriented graph with detour order $k \leq 7$ and let P and Q be two detours in D. Then $V(P) \cap V(Q) \neq \emptyset$.*

Proof. Let P and Q denote two detours $u_1u_2 \dots u_k$ and $v_1v_2 \dots v_k$ in D respectively and assume that $V(P) \cap V(Q) = \emptyset$.

The case where $k = 2$ is obvious. If $3 \leq k \leq 7$, we shall show that each of the values $b_P(u_i)$, $e_P(u_i)$, $b_Q(v_i)$, $e_Q(v_i)$ is greater than $k/2$. The result then follows from Lemma 2.2. If $k \leq 5$, this follows immediately from Lemma 2.1 (iii) and if $k = 6$, it follows from Lemma 2.1 (iv).

Now let $k = 7$: From Lemma 2.1 (iv) we only need to consider the case where $N^+(u_7) = \{u_5\}$ and $N^-(u_1) = \{u_3\}$. Since D is strong there exists some $u_i - u_j$-path in D of the form u_iWu_j where $i \in \{5, 6, 7\}$ and $j \in \{1, 2, 3\}$. It is easy to see that $n(W) \leq 1$.

If $n(W) = 0$ or $V(W) = \{u_4\}$, such a path consists only of vertices in $V(P)$ and it can easily be shown that $b_P(u_i) \geq 4$ and $e_P(u_i) \geq 4$ for all $u_i \in V(P)$.

Now suppose $V(W) = \{w\}$ where $w \neq u_4$. Then $\{w\} \cap V(Q) = \emptyset$, since $e_P(u_i) \geq 5$ for each $i \in \{5, 6, 7\}$ and $b_Q(v) \geq 3$ for each $v \in V(Q)$ by Lemma 2.1 (iii).

Now let $H = \langle V(P) \cup \{w\} \rangle$. Then it is easy to check that $e_H(u) \geq 5$ for every $u \in H$. Since $b_Q(v) \geq 3$ for every $v \in V(Q)$, there is no path from H to Q, contradicting our assumption that D is strong. ■

We now construct a strong oriented graph D_8 (see Fig.1) with $n(D) = 16$ which has two non-intersecting detours of order 8: Let P and Q be two disjoint paths $u_1 \dots u_8$ and $v_1 \dots v_8$ of order 8 each. Now add the arcs u_3u_1, u_6u_3 and u_8u_6 in $\langle V(P) \rangle$ and the corresponding arcs v_iv_j in $\langle V(Q) \rangle$. Note that $\langle V(Q) \rangle$ is just a copy of $\langle V(P) \rangle$. Finally connect the two oriented graphs $\langle V(P) \rangle$ and $\langle V(Q) \rangle$ by adding the two arcs u_3v_6 and v_3u_6.

Theorem 2.4 *D_8 is a strong oriented graph with detour order 8 containing two non-intersecting detours.*

Proof. From the construction it is clear that D_8 is a strong oriented graph with two non-intersecting paths $u_1 \dots u_8$ and $v_1 \dots v_8$. It remains to check the detour order of D_8. Let L be a detour of D_8. Without loss of generality we assume L begins in $\langle V(P) \rangle$. If L ends in $\langle V(Q) \rangle$, then, since $e_P(u_3) = 4$ and $b_Q(v_6) = 4$, it follows that $n(L) = 8$. If L begins and ends in $\langle V(P) \rangle$, L is P or the path $u_1u_2u_3v_6v_3u_6u_7u_8$. In either case $n(L) = 8$. ■

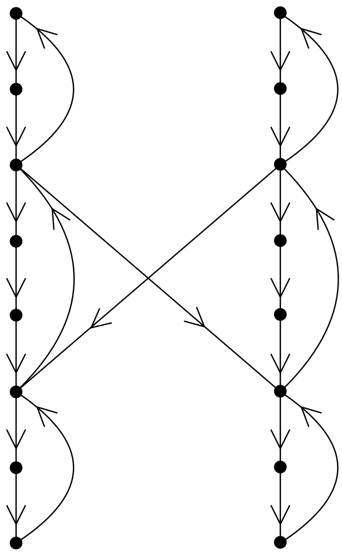

Figure 1: D_8

We would now like to show that D_8 is contained in every strong oriented graph D, $n(D) = 16$, which has two non-intersecting detours of order 8. In order to do this we need the following lemmas:

Lemma 2.5 *Let P be a path of order k in an oriented graph D and suppose $\langle V(P)\rangle$ is strong. If $\langle V(P)\rangle$ contains a cycle C of order $m < k$, then:*
(i) $b_P(u) \geq m$ and $e_P(u) \geq m$ for every $u \in V(P)$;
(ii) $b_P(u) \geq m+1$ and $e_P(u) \geq m+1$ for every $u \in V(P) - V(C)$.

Proof. Suppose $u \in V(C)$. Then clearly $b_P(u) \geq m$ and $e_P(u) \geq m$. Now suppose $u \in V(P) \setminus V(C)$. Since $\langle V(P)\rangle$ is strong, there is a path from u to C. Let $w \in V(C)$ be the first vertex on the cycle in such a path. Then the $u - w$-path followed by the remaining vertices in C is of order at least $m+1$ and therefore $b_P(u) \geq m+1$. Similarly, since there is a path from C to u, we can show that $e_P(u) \geq m+1$. ■

Lemma 2.6 *Let P denote a path $u_1u_2 \ldots u_8$ in an oriented graph D and suppose $\langle V(P)\rangle$ is strong. Then $b_P(u_i) \geq 4$ and $e_P(u_i) \geq 4$. Moreover, $e_P(u_i) \geq 5$ for $i = 1, 2$, and $b_P(u_i) \geq 5$ for $i = 7, 8$.*

Proof. If $\langle V(P)\rangle$ contains a 4-cycle, the first part of the statement holds from Lemma 2.5. We can therefore assume that $\langle V(P)\rangle$ contains only 3-cycles. But then, for every $u_i \in V(P)$ there is a 3-cycle C in $\langle V(P)\rangle$ such that $u_i \notin V(C)$ and the first part of the statement follows once again from Lemma 2.5.

Next we show that $e_P(u_1) \geq 5$. Again, from Lemma 2.5, we only need to consider the case where u_1 is contained in a 3- or a 4-cycle and all the remaining cycles are 3-cycles. Note that $u_8u_6 \in A\langle V(P)\rangle$.

First we consider the case where $u_4u_1 \in A\langle V(P)\rangle$. If $u_6u_4 \in A\langle V(P)\rangle$ then $u_7u_8u_6u_4u_1$ is a path of order 5 which ends at u_1. If $u_6u_4 \notin A\langle V(P)\rangle$, then $\{u_7u_5, u_5u_3\} \subseteq A\langle V(P)\rangle$ and $u_8u_6u_7u_5u_3u_4u_1$ is a path of order 7 which ends at u_1.

Next, suppose $u_3u_1 \in A\langle V(P)\rangle$. If $u_6u_4 \in A\langle V(P)\rangle$, then at least one of $\{u_5u_3,\ u_4u_2\}$ is in $A\langle V(P)\rangle$ so that $u_7u_8u_6u_4u_5u_3u_1$ or $u_7u_8u_6u_4u_2u_3u_1$ are paths in $\langle V(P)\rangle$. If $u_6u_4 \notin A\langle V(P)\rangle$, then $\{u_7u_5, u_5u_3\} \subset A\langle V(P)\rangle$ which yields the path $u_8u_6u_7u_5u_3u_1$. All these paths end at u_1 and are of order at least 6.

We follow similar arguments to show that the values of $e_P(u_2)$, $b_P(u_7)$ and $b_P(u_8)$ are all at least 5. ■

Lemma 2.7 *Let D be a strong oriented graph of detour order 8 and let $n(D) = 16$. Let P and Q denote two detours $u_1 \ldots u_8$ and $v_1 \ldots v_8$ in D respectively. If $V(P) \cap V(Q) = \emptyset$ then both $\langle V(P)\rangle$ and $\langle V(Q)\rangle$ are strong.*

Proof. Suppose $\langle V(P)\rangle$ is not strong. Then there are two vertices, u_k and u_ℓ, in $V(P)$ with $k < \ell$ such that, for any $u_iu_j \in A(D)$ with $i > j$, either $j < i \leq k$ or $i > j \geq \ell$. Since u_1 has an in-neighbour and u_8 an out-neighbour on P, it follows from Lemma 2.1 (ii) that $3 \leq k \leq 5$ and $4 \leq \ell \leq 6$.

Now suppose $\langle V(Q)\rangle$ is also not strong, and let v_r and v_s be the vertices in $V(Q)$ which have the same properties as u_k and u_ℓ in $V(P)$ respectively.

Since D is strong, a $v_s - v_r$-path in D contains vertices of $V(P)$ and similarly a $u_\ell - u_k$-path in D contains vertices of $V(Q)$. Hence either $3 \leq k < \ell < r < s \leq 6$ or $3 \leq r < s < k < \ell \leq 6$. Without loss of generality we can therefore assume that $u_k = u_3$, $u_\ell = u_4$, $v_r = v_5$ and $v_s = v_6$.

Also, for the indices $b_P(u_i)$ and $e_P(u_i)$ we have the minimum values:

u_i	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
$\min(b_P(u_i))$	8	7	6	5	4	3	3	3
$\min(e_P(u_i))$	3	3	3	4	5	6	7	8

and similarly for $b_Q(v_i)$ and $e_Q(v_i)$.

Calculating the values $e_Q(v_i) + b_P(u_j)$ for $i \geq 6$ from the table above, and from Lemma 2.2, it is clear that $v_i u_j \notin A(D)$ if $i \geq 6$ so that there is no $v_6 - v_5$-path in D. This contradicts our assumption that D is strong.

Now suppose $\langle V(P) \rangle$ is strong and $\langle V(Q) \rangle$ is not strong and let v_r and v_s be the vertices as described above. From Lemma 2.6 we now have the following minimum values for $b_P(u_i)$ and $e_P(u_i)$:

u_i	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
$\min(b_P(u_i))$	8	7	6	5	4	4	5	5
$\min(e_P(u_i))$	5	5	4	4	5	6	7	8

The minimum values for $b_Q(v_i)$ and $e_Q(v_i)$ are the same as in the first part of the proof.

Since $\langle V(Q) \rangle$ is not strong, a $v_s - v_r$-path in D contains vertices of $V(P)$. But, calculating the values $e_Q(v_i) + b_P(u_j)$ and $e_P(u_i) + b_Q(v_j)$ from the tables above, it is clear that $v_i u_j \notin A(D)$ if $i \geq 5$ and similarly, $u_i v_j \notin A(D)$ if $j \leq 4$ so that $s = 4$ and $r = 5$. But this contradicts our assumption that $r < s$.

This implies that both $\langle V(P) \rangle$ and $\langle V(Q) \rangle$ are strong. ■

Theorem 2.8 *Suppose D is a strong oriented graph with $\lambda(D) = 8$ and $n(D) = 16$. If D has two non-intersecting detours, then D_8 is contained in D.*

Proof. Let P and Q denote two non-intersecting detours $u_1 \dots u_8$ and $v_1 \dots v_8$ in D respectively. Since D is strong there is at least one $u_i v_j \in A(D)$ and one $v_k u_\ell \in A(D)$ where, from Lemma 2.2, $i < j$ and $k < \ell$. Furthermore, from Lemma 2.7, both $\langle V(P) \rangle$ and $\langle V(Q) \rangle$ are strong. Hence, from the values of $e_P(u_i) + b_Q(v_j)$ listed in the last table, it is clear that if $u_i v_j \in A(D)$, then $u_i v_j \in \{u_3v_5, u_3v_6, u_4v_5, u_4v_6\}$ and similarly, if $v_k u_\ell \in A(D)$, then $v_k u_\ell \in \{v_3u_5, v_3u_6, v_4u_5, v_4u_6\}$. Also, if $j - k \leq 2$, the path $v_1 \dots v_k u_\ell \dots u_i v_j \dots v_8$ is of order at least $11 + k - j \geq 9$. Hence $j - k > 2$ and similarly $\ell - i > 2$ so that u_3v_6 and v_3u_6 are the only arcs in $A(D)$ between $V(P)$ and $V(Q)$.

Now consider the path $u_1u_2u_3v_6 \dots v_3u_6u_7u_8$ of order at least 8 in D. Since the detour order of D is 8, v_6v_3 is the only $v_6 - v_3$-path in $\langle V(Q) \rangle$ and $v_6v_3 \in A(D)$. Similarly u_6u_3 is the only $u_6 - u_3$-path in $\langle V(P) \rangle$ and $u_6u_3 \in A(D)$.

Also, $u_iu_1 \notin A(D)$ for $i = 4, 5$ since the path $u_iu_1u_2u_3v_6v_3u_6u_7u_8$ is of order 9. Similarly $v_iv_1 \notin A(D)$ for $i = 4, 5$. Since $\langle V(P) \rangle$ and $\langle V(Q) \rangle$ are strong it follows that $\{u_3u_1, v_3v_1\} \in A(D)$. Finally, $u_8u_i \notin A(D)$, $i = 4, 5$

since the path $u_7u_8u_i \dots u_6u_3v_6v_3v_4v_5$ is of order at least 9. Similarly $v_8v_i \notin A(D)$, $i = 4, 5$. Since D is strong, it follows that $\{u_8u_6, v_8v_6\} \in A(D)$. Hence $A(D_8) \subset A(D)$. ■

Remark Let D, with $n(D) = 16$, be a strong oriented graph with detour order 8 and let P and Q denote two non-intersecting detours $u_1 \dots u_8$ and $v_1 \dots v_8$ in D respectively. We would like to determine all possible arcs in $A(D)$ other than $A(D_8)$. In the proof of Theorem 2.8 it is shown that u_3v_6 and v_3u_6 are the only arcs between the vertices of P and Q. We will therefore only determine all possible u_iu_j arcs in $A(D)$ as the v_iv_j arcs in $A(D)$ are determined similarly. Let A_8 denote all the u_iu_j arcs in $\langle V(D_8)\rangle$. We can only add arcs to A_8 so that the values of $b_P(u_6)$ and $e_P(u_3)$ remain equal to 4. Also, keeping in mind that a path in D can start in $\langle V(P)\rangle$, use vertices of $\langle V(Q)\rangle$ and return to $\langle V(P)\rangle$, the values of $e_{\langle V(P)-\{u_6\}\rangle}(u_3)$ and $b_{\langle V(P)-\{u_3\}\rangle}(u_6)$ must remain equal to 3. Finally, u_6u_3 can be the only $u_6 - u_3$-path in D.

The only u_iu_j arcs with $i > j$ which can be added to A_8 can therefore be chosen from one of the sets $\{u_4u_2, u_5u_3\}$, $\{u_4u_2, u_7u_5\}$, $\{u_6u_4, u_7u_5\}$. Similarly the only u_iu_j arcs with $i < j$ which can be added to A_8 can be chosen from the set $B = \{u_1u_5, u_1u_6, u_1u_8, u_2u_6, u_3u_5, u_3u_7, u_3u_8, u_4u_6, u_4u_8\}$. Combining these possibilities and calculating all the index values, it follows that u_iu_j arcs which can be added to A_8 can be chosen from one of the following sets: $\{B - \{u_3u_5\}, u_4u_2, u_5u_3\}$, $\{B - \{u_4u_6\}, u_6u_4, u_7u_5\}$, $\{B, u_4u_2, u_7u_5\}$.

The construction of a strong oriented graph D_9 with detour order 9 which has two non-intersecting detours is very similar to that of D_8: Let P and Q denote two disjoint paths $u_1 \dots u_9$ and $v_1 \dots v_9$ of order 9 respectively. Now add the arcs u_4u_1, u_7u_4 and u_9u_7 in $\langle V(P)\rangle$ and the corresponding arcs v_iv_j in $\langle V(Q)\rangle$. Once again $\langle V(Q)\rangle$ is just a copy of $\langle V(P)\rangle$. Now add the two arcs u_4v_7 and v_4u_7.

Theorem 2.9 *The digraph D_9 described above is a strong oriented graph with two non-intersecting detours.*

Proof. The proof is similar to that of Theorem 2.4. ■

3 Strong digraphs with detour order at least ten

In this section we construct, for every $k \geq 10$, a strong oriented graph with approximately $\sqrt{k}$ non-intersecting detours of order k.

Let m be a nonnegative integer and let us denote by Δ_m the m-th *triangular number*, i.e. $\Delta_m = \sum_{j=1}^{m} j = \frac{1}{2}m(m+1) = \binom{m+1}{2}$. It is not difficult to see that $m = \lfloor \sqrt{2\Delta_m} \rfloor$. Then for any positive integer k there exists a positive integer m such that $\Delta_{m-1} \leq \frac{k-1}{2} < \Delta_m$. By an easy calculation we obtain that $-m \leq k-1-m^2 < m$ and we put

$$l = \begin{cases} k-1-m^2 & \text{if } k-1-m^2 \geq 0, \\ k-1-m(m-1) & \text{otherwise.} \end{cases}$$

Hence any positive integer k can be expressed either as $k = m^2+\ell+1$ or as $k = m(m-1)+\ell+1$ where $0 \leq \ell \leq m-1$. The uniqueness of this expression can be verified in a routine manner.

We say two cycles are *linked* if they have exactly one vertex in common. This common vertex is called a *link*.

Let P denote a path $u_1 \dots u_k$ of distinct vertices. We now add s arcs of the form $u_i u_j$ where $i > j$ so that $\langle V(P) \rangle$ consists of s linearly linked cycles $C_1, C_2, \dots, C_s$ with $u_1 \in V(C_1)$ and $u_k \in V(C_s)$. We will denote the link between two cycles C_i and C_{i+1} by x_i, $i = 1, \dots, s-1$. Note that $x_{i+1}x_i \in A(\langle V(P) \rangle)$, $i = 1, \dots, s-2$. C_1 is the cycle $u_1 \dots x_1 u_1$ and C_s is the cycle $x_{s-1} \dots u_k x_{s-1}$. $\langle V(P) \rangle$ is clearly a strong oriented graph. The number of cycles in $\langle V(P) \rangle$ as well as their order are given in the following table:

k	No. of cycles s	No. of cycles with $n(C) = m+1$	No. of cycles with $n(C) = m+2$
$m^2+\ell+1$	m	$m-\ell$	ℓ
$m(m-1)+\ell+1$	$m-1$	$m-1-\ell$	ℓ

Table 1: Number and order of cycles C_i

We follow the convention that the cycles C_s or C_s and C_1 will be of the smallest order so that, for $k = m^2 + \ell + 1$,

$$n(C_1) = \begin{cases} m+1, & 0 \le \ell < m-1, \\ m+2, & \ell = m-1, \end{cases} \quad n(C_s) = m+1, \ 0 \le \ell \le m-1, \tag{1}$$

and for $k = m(m-1) + \ell + 1$,

$$n(C_1) = \begin{cases} m+1, & 0 \le \ell < m-2, \\ m+2, & \ell = m-2, m-1, \end{cases} \quad n(C_s) = \begin{cases} m+1, & 0 \le \ell < m-1, \\ m+2, & \ell = m-1. \end{cases} \tag{2}$$

For example, if $k = 10 = 3.3 + 0 + 1$, $\langle V(P) \rangle$ consists of three 4-cycles. We therefore add the arcs u_4u_1, u_7u_4 and $u_{10}u_7$ to the path $u_1 \dots u_{10}$;
If $k = 11 = 3.3 + 1 + 1$, $\langle V(P) \rangle$ consists of three cycles; two 4-cycles and one 5-cycle. Using our convention we add the arcs u_4u_1, u_8u_4 and $u_{11}u_8$ to the path $u_1 \dots u_{11}$;
If $k = 12 = 3.3 + 2 + 1$, $\langle V(P) \rangle$ has one 4-cycle and two 5-cycles. We therefore add the arcs u_5u_1, u_9u_5 and $u_{12}u_9$ to the path $u_1 \dots u_{12}$;
If $k = 13 = 4.3 + 0 + 1$, $\langle V(P) \rangle$ has three 5-cycles. We therefore add the arcs u_5u_1, u_9u_5 and $u_{13}u_9$ to the path $u_1 \dots u_{13}$, and so on.

Lemma 3.1 *Let $\langle V(P) \rangle$ be the strong oriented graph of detour order k which consists of s linearly linked cycles $C_1, C_2, \dots, C_s$, $s \ge 3$, as described above. If $k = m^2 + \ell + 1$, $0 \le \ell \le m-1$, then*

$$e_P(x_1) = 2m-1, \ 0 \le \ell \le m-1, \quad b_P(x_{s-1}) = \begin{cases} 2m-1, & 0 \le \ell < m-1 \\ 2m, & \ell = m-1, \end{cases}$$

else if $k = m(m-1) + \ell + 1$, $0 \le \ell \le m-1$,

$$e_P(x_1) = \begin{cases} 2m-2, & 0 \le \ell < m-1 \\ 2m-1, & \ell = m-1, \end{cases} \quad b_P(x_{s-1}) = \begin{cases} 2m-2, & 0 \le \ell < m-2 \\ 2m-1, & \ell \in \{m-2, m-1\}. \end{cases}$$

Proof. Let x_i^- and x_i^+ respectively denote the predecessor and successor of x_i on P.

Any path in $\langle V(P) \rangle$ that starts in some cycle C_i and ends at x_1 uses at most the $n(C_i)$ vertices of C_i together with the $i-2$ links $x_{i-2}, x_{i-3}, \dots, x_1$. Since $s \ge 3$ and $|n(C_i) - n(C_j)|$ is either equal to 0 or 1, it is easy to see that the $x_{s-1}^+ - x_1$-path is always a longest path in $\langle V(P) \rangle$ which ends at x_1.

Hence $e_P(x_1) = n(C_s) + s - 2$. Similarly the $x_{s-1} - x_1^-$-path is always a longest path in $\langle V(P) \rangle$ which starts at x_{s-1} so that $b_P(x_{s-1}) = n(C_1) + s - 2$.

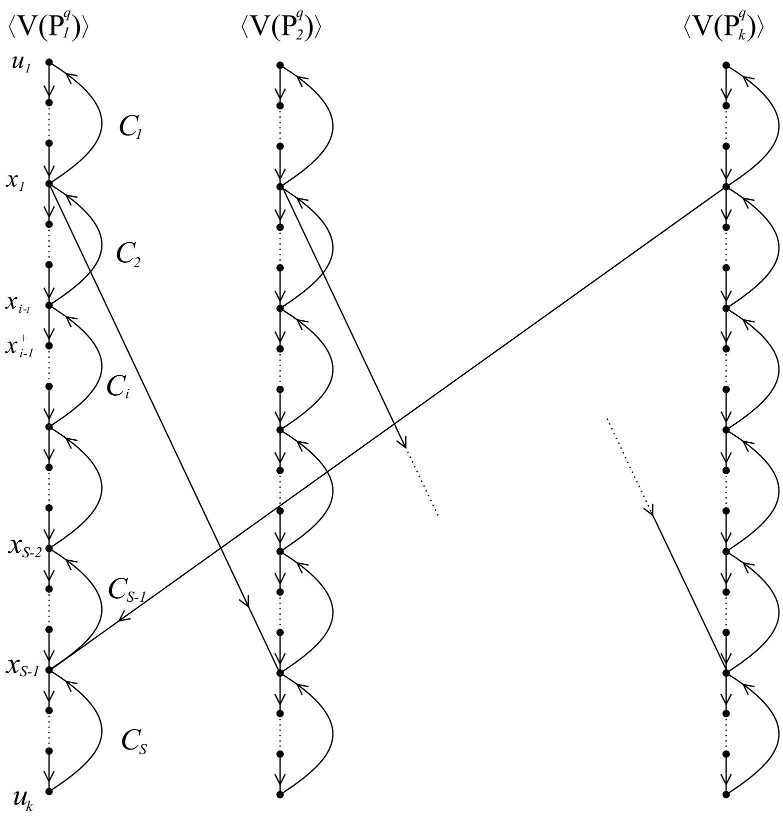

Figure 2: D_k^q

The result now follows from Table 1 and the equalities (1) and (2). ■

We would now like to connect a number of copies, say q, of $\langle V(P)\rangle$ in such a way that the resulting digraph is a strong oriented graph with q non-intersecting detours of order k. In order to do this we denote these copies by $\langle V(P_r)\rangle$, $r = 1, \ldots, q$ and the links in each copy by x_i^r, $i = 1, \ldots, s-1$. The strong oriented graph D_k^q (see Fig.2) is formed by adding the arcs $x_1^r x_{s-1}^{r+1}$, $r = 1, \ldots, q-1$ and $x_1^q x_k^1$. Next we calculate a lower bound, $L(q)$, on the detour order of D_k^q and we solve for q by using $L(q) \leq \lambda(D_k^q) = k$. We will omit sub- and superscripts where they are deemed unnecessary.

Lemma 3.2 *If W is a path in D_k^q which starts in $\langle V(P_i)\rangle$ and ends in $\langle V(P_j)\rangle$ where $i \neq j$, then for $k = m^2 + \ell + 1$,*

$$n(W) \leq \begin{cases} 2m + (m-1)q, & 0 \leq \ell < m-1 \\ 2m + 1 + (m-1)q, & \ell = m-1, \end{cases}$$

and for $k = m(m-1) + \ell + 1$,

$$n(W) \leq \begin{cases} 2m + (m-2)q, & 0 \leq \ell < m-2 \\ 2m + 1 + (m-2)q, & \ell = m-2 \\ 2m + 2 + (m-2)q, & \ell = m-1. \end{cases}$$

Proof. W consists of at most $e_P(x_i)$ vertices of $\langle V(P_i)\rangle$. If W enters some $\langle V(P_j)\rangle$ at x_{s-1} and exists it at x_1, W uses all $s-1$ of the links of $\langle V(P_j)\rangle$. W can do so for $q-2$ of the copies of $\langle V(P)\rangle$ before entering $\langle V(P_{i-1})\rangle$ at x_{s-1} where it can use at most $b_P(x_{s-1})$ of the vertices of $\langle V(P_{i-1})\rangle$. Hence $n(W) \leq e_P(x_1) + b_P(x_{s-1}) + (s-1)(q-2)$ and the result follows immediately from Table 1 and Lemma 3.1. ■

Lemma 3.3 *If* T *is a path in* D_k^q *which starts and ends* $\langle V(P_i)\rangle$ *and* $T \neq P_i$, *then for* $k = m^2 + \ell + 1$,

$$n(T) \leq \begin{cases} 2m + (m-1)q, & 0 \leq \ell < m-1, \\ 2m + 1 + (m-1)q, & \ell = m-1, \end{cases}$$

and for $k = m(m-1) + \ell + 1$,

$$n(T) \leq \begin{cases} 2m + (m-2)q, & 0 \leq \ell < m-2, \\ 2m + 1 + (m-2)q, & \ell = m-2, m-1. \end{cases}$$

Proof. If T exits $\langle V(P_i)\rangle$ at x_1 and re-enters it at x_{s-1}, T uses all the $(s-1)(q-1)$ links in $\langle V(D_k^q) - V(P_i)\rangle$. In order to determine the maximum number of vertices of T in $V(P_i)$, we define the path Y_i, $i = 1, \ldots, s-2$ as the longest path in $\langle V(P)\rangle$ that starts in C_i and ends at x_1 and does not use x_{s-1}. It is easy to see that

$$n(Y_i) = \begin{cases} n(C_1), & i = 1, \\ n(C_i) + i - 2, & i = 2, \ldots, s-2, \ s \geq 4. \end{cases}$$

Similarly we define the path Z_i, $i = 3, \ldots, s$ as the longest path in $\langle V(P)\rangle$ that starts at x_{s-1} and ends in C_i and does not use x_1, so that

$$n(Z_i) = \begin{cases} n(C_i) + s - i - 1, & i = 3, \ldots, s-1, \ s \geq 4, \\ n(C_s), & i = s. \end{cases}$$

T uses at most $\max\{n(Y_i) + n(Z_{i+2})\}$, $i = 1, \ldots, s-2$ vertices of $V(P_i)$. Now let $M = \max\{n(Y_i) + n(Z_{i+2})\}$, $i = 1, \ldots, s-2$. Then $n(T) \leq M +$

$(s-1)(q-1)$ and it suffices to calculate an upper bound on M. For $s=3$, $M = n(Y_1)+n(Z_3) = n(C_1)+n(C_3)$, and for $s \geq 4$,

$$M \leq \max \begin{cases} n(C_1)+n(C_3)+s-4, \\ n(C_i)+n(C_{i+2})+s-5, \ i=2,\ldots,s-3, \\ n(C_{s-2})+n(C_s)+s-4. \end{cases}$$

Using Table 1 and equations (1) and (2), we have, for $k=m^2+\ell+1$, $s \geq 3$,

$$M \leq \begin{cases} 3m-1, & 0 \leq \ell < m-1, \\ 3m, & \ell = m-1, \end{cases}$$

and for $k=m(m-1)+\ell+1$, $s \geq 4$,

$$M \leq \begin{cases} 3m-2, & 0 \leq \ell < m-2, \\ 3m-1, & \ell = m-2, m-1. \end{cases}$$

■

Corollary 3.4 *Suppose $\lambda(D_k^q)=k$. Then for $k=m^2+\ell+1$,*

$$k \geq \begin{cases} 2m+(m-1)q, & 0 \leq \ell < m-1 \\ 2m+1+(m-1)q, & \ell = m-1, \end{cases}$$

and for $k=m(m-1)+\ell+1$,

$$k \geq \begin{cases} 2m+(m-2)q, & 0 \leq \ell < m-2 \\ 2m+1+(m-2)q, & \ell = m-2 \\ 2m+2+(m-2)q, & \ell = m-1. \end{cases}$$

Proof. P, W and T clearly represent all possible detours of D_k^q so that $\lambda(D_k^q) = \max\{n(P),\, n(W),\, n(T)\}$ and the result follows from Lemma 3.2, Lemma 3.3 and $n(P)=k$. ■

Theorem 3.5 *Let k be any integer greater than or equal to 10. Then D_k^q is a strong oriented graph with q non-intersecting detours of order k, where q is determined as follows:*
If $k=m^2+\ell+1$, then

$$q = m-1, \quad 0 \leq \ell \leq m-1;$$

else if $k=m(m-1)+\ell+1$, then

$$q = \begin{cases} m-2, & \ell = 0, \\ m-1, & 0 < \ell \leq m-1. \end{cases}$$

Proof. Since $k \geq 10$, $m \geq 3$ if $k = m^2+\ell+1$ and $m \geq 4$ if $k = m(m-1)+\ell+1$. Suppose $k = m^2 + \ell + 1$. From Corollary 3.4 we have, for $0 \leq \ell < m-1$,

$$q \leq \frac{k-2m}{m-1} = m - 1 + \frac{\ell}{m-1} < m,$$

so we can choose $q = m-1$. For $\ell = m-1$,

$$q \leq \frac{k-2m-1}{m-1} = \frac{m^2-m-1}{m-1} = m - \frac{1}{m-1} < m$$

and we choose $q = m-1$. Now suppose $k = m(m-1)+\ell+1$. For $0 \leq \ell < m-2$,

$$q \leq \frac{m^2-3m+1+\ell}{m-2} = m-1+\frac{\ell-1}{m-2} < \begin{cases} m-1, & \ell = 0, \\ m, & 1 \leq \ell < m-2. \end{cases}$$

We therefore choose

$$q = \begin{cases} m-2, & \ell = 0, \\ m-1, & 1 \leq \ell < m-2. \end{cases}$$

Finally, if $\ell = m-2, m-1$,

$$q \leq \frac{m^2-2m-2}{m-2} = m - \frac{2}{m-2} < m, \ m \geq 4,$$

and we choose $q = m-1$. ∎

Acknowledgement: Part of this research was done during a visit to UNISA by the second author. Financial support by UNISA is gratefully acknowledged.

References

[1] J.A. Bondy, Basic graph theory: Paths and Circuits, in: Handbook of Combinatorics Vol I., eds. R.L. Graham, M. Grötschel, and L. Lovász, The MIT Press, Cambridge, MA, 1995, 5-110.

[2] O. Ore, Theory of graphs, AMS Colloq. Publ. 38. Porvidence, RI.

[3] Z. Skupień, Smallest sets of longest paths with empty intersection. Combin. Probab. Comput. 5 (1996), no 4, 429-436.

[4] J.H. Voss, Cycles and Bridges in Graphs, Deutscher Verlag Wiss., Germany, 1991.

[5] D.B. West, Introduction to Graph Theory, Prentice-Hall, Inc., London, second edition, 2001.

[6] T. Zamfirescu, On longest paths and circuits in graphs. Math. Scand. 38 (1976), 211-239.