The Stable Multiple Activities Problem*

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Abstract. This paper deals with the generalization of the classical stable roommates problem, called the Stable Multiple Activities problem, SMA for short. In an SMA instance a multigraph $G = (V, E)$, capacities $b(v)$ and a linear order $\prec_v$ on the set of edges incident to a vertex v for each $v \in V$ are given. A stable b-matching is sought, i.e. a set of edges M such that each vertex v is incident with at most $b(v)$ edges and for each edge $e \notin M$ a vertex v incident with e and $b(v)$ different edges $f_1, \ldots, f_{b(v)}$ incident to v exist, all of them $\prec_v$-smaller than e.

We show how to decrease the computational complexity of the SMA algorithm to run in $O(|E|)$ time and derive some properties of stable b-matchings.

Keywords. The stable roommates problem, polynomial algorithm, stable b-matching.

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1 Introduction

The theory of stable matchings began with the seminal paper of Gale and Shapley [3], where the classical problems were introduced: the Stable Mar-

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riage Problem (SM), the Stable Roommates Problem (SR) and the College Admissions Problem.

In the literature, several generalizations of the Stable Roommates Problem have been considered under the names the Stable Crews Problem [1] or the Stable Fixtures Problem (SF) [8]. We consider a generalization named in [2] the Stable Multiple Activities Problem (SMA for short).

In an instance of SMA a multigraph $G = (V, E)$, capacities $b(v)$ and a linear order on the set of edges incident to a vertex v, for each $v \in V$, are given. One seeks a stable b-matching, i.e. a set $M \subseteq E$ such that each vertex v is incident with at most $b(v)$ edges of M and a stability condition (to be formulated later) is fulfilled.

In [2], Cechlárová and Fleiner designed an $O(|E|^2)$ algorithm deciding whether an SMA instance is solvable and providing a solution if one exits. In this work, we study the properties of the SMA algorithm in greater depth and show how to speed it up to achieve complexity of $O(|E|)$. Further, we show that for a given SMA instance, each vertex is assigned the same number of edges in all stable b-matchings. Finally we present a correspondence between the set of all stable b-matchings and sets of rotation.

The organization of the paper is as follows: in Section 2 we give formal definitions of the necessary notions. Section 3 is devoted to the analysis of the SMA algorithm and Section 4 deals with its efficient implementation. In Section 5 we explore the structure of rotations and their relation to the set of all stable b-matchings. We conclude with some proposals for further research.

2 Definitions

Let $G = (V, E)$ be a finite multigraph. For each vertex $v \in V$, let $E(v, G)$ be the set of edges incident with v in G, $\prec_v$ be a linear order on the set $E(v, G)$ and $\mathcal{O} = \{\prec_v, v \in V\}$. Moreover, a function $b : V \to \mathbb{N}$ is given, called the capacity function. The triple $I = (G, \mathcal{O}, b)$ is an instance of the Stable Multiple Activities Problem (SMA for short).

We say, that an instance $I = (G, \mathcal{O}, b)$ is a *subinstance* of an instance $I' = (G', \mathcal{O}', b)$, written $I \subseteq I'$, if G is a subgraph of G', $\prec_v$ is the restriction of $\prec'_v$ to $E(v, G)$ and $b(v) = b'(v)$ for each $v \in V(G)$. A subinstance $I = (G, \mathcal{O}, b)$ is a *proper subinstance* of $I' = (G', \mathcal{O}', b)$, written $I \subset I'$, if G is a proper subgraph of G'.

We say that subset F of E *b-dominates* edge $e \in E$ if there exists a vertex v such that $e \in E(v, G)$ and different elements $f_1, f_2, \ldots, f_{b(v)}$ of $F \cap E(v, G)$ such that $f_i \prec_v e$ for $i = 1, 2, \ldots, b(v)$.

A subset M of E is a *b-matching*, if each vertex $v \in V(G)$ is incident with at most $b(v)$ edges of M. A b-matching M is *stable*, if each edge $e \notin M$ is b-dominated by M. The set of all stable b-matchings for an SMA instance I will be denoted by $\mathcal{M}(I)$.

In what follows, we denote by $s_G(v), l_G(v)$ the edges that are $(b(v)+1)st$ and last in $\prec_v$ in G, respectively. Also, if a vertex v is incident with fewer than $b(v)$ edges in a matching M, it is said to be *undersubscribed* in M.

3 The SMA Algorithm

The SMA algorithm proposed in [2] determines for a given instance $I = (G, \mathcal{O}, b)$ of the SMA whether a stable b-matching exists and if so, it finds one. This algorithm is derived from Irving's classical algorithm for SR and it also consists of two phases. The algorithm creates a sequence of instances

$$I = I_0, I_1, \ldots, I_i, I_{i+1}, \ldots, I_k$$

in such a way that for each $i = 1, 2, \ldots, k-1$

$$I_{i+1} \textit{ is a proper subinstance of } I_i \tag{1}$$

$$\textit{if } I_i \textit{ has a stable } b - \textit{matching then } I_{i+1} \textit{ has one,} \tag{2}$$

$$\textit{any stable } b - \textit{matching of } I_{i+1} \textit{ is a stable } b - \textit{matching of } I_i \tag{3}$$

The algorithm ends when $(G_k, \mathcal{O}_k, b)$ either represents a stable b-matching or its form implies that there is no stable b-matching.

3.1 Phase 1 of the SMA Algorithm

In each step of Phase 1, an edge is deleted that will never belong to any stable b-matching. Formal definitions were introduced in [2].

Let $(G, \mathcal{O}, b)$ be an SMA instance and let $u \in V(G)$. Define

$$B(u, G) := \{f \in E(u, G) : |\{g \in E(u, G) : g \prec_u f\}| < b(u)\}$$

An edge $f \in B(u, G)$ is called *a B-edge* at vertex u. Clearly, an edge $f = ux \in B(u, G)$ can be b-dominated only at its other end, thus at vertex x. We further define

$$D(u, G) := \{f = ux \in E(u, G) : f \in B(x, G)\}$$

An edge $f \in D(u, G)$ is called *a D-edge* at vertex u.

The following definition generalizes the property, summarized in Lemma 4.2.2 of [4] and Lemma 2.2 of [8], when no further reductions according to Phase 1 of the algorithm are possible.

Definition 1 *We say, that an instance $(G, \mathcal{O}, b)$ of the SMA has the first-last-property (the FLP for short), if for each vertex $u \in V(G)$ and for each edge $e \in E(u, G)$*

$$|\{f \in D(u, G) : f \prec_u e\}| < b(u). \tag{4}$$

An instance satisfying the FLP will be called an FL instance for brevity. The set of edges $e \in E(u, G)$ satisfying relation (4) at vertex u will be denoted by $FL(u, G)$, the set of edges violating (4) by $NFL(u, G)$. Edges from sets $FL(u, G)$ and $NFL(u, G)$ will be called FL-edges and non-FL edges at u, respectively.

Example 1 *Figure 1 displays an example SMA instance. Vertices are labeled v_i, $1 \le i \le 7$ and edges are e_i, $1 \le i \le 39$. The multigraph G is given by its incidence lists, written in the orders corresponding to $\mathcal{O}$. Capacities of vertices are displayed in brackets.*

Here, e.g. $B(v_2, G) = \{e_{11}, e_{12}, e_3, e_{13}\}$, $B(v_7, G) = \{e_8, e_{26}, e_{14}\}$. All D-edges are underlined.

The goal of Phase 1 of the SMA algorithm is to reach a subinstance fulfilling FLP. As proposed in [2], as long as the working SMA instance $I_i = (G_i, \mathcal{O}_i, b)$ does not satisfy the FLP, a non-FL edge $e = uv$ is found and deleted from I_i to get I_{i+1}. The correctness of Phase 1 for SMA and its basic properties were proved in [2], we repeat them here for completeness.

Lemma 1 (Lemma 4.2 [2]) *If an instance $I_{i+1} = (G_{i+1}, \mathcal{O}_{i+1}, b)$ is constructed from $I_i = (G_i, \mathcal{O}_i, b)$ by deleting a non-FL edge in a Phase-1 step then properties (1 - 3) hold.*

$v_1(3)$	e_1	e_2	$\underline{e_3}$	e_4	e_5	e_6	e_{37}	e_{38}	$\underline{e_7}$	$\underline{e_8}$	$\underline{e_9}$	e_{10}		
$v_2(4)$	e_{11}	e_{12}	$\underline{e_3}$	e_{13}	$\underline{e_{14}}$	e_{15}	e_{16}	e_{17}	e_{18}	e_{35}	e_{36}	$\underline{e_{19}}$	$\underline{e_{20}}$	$\underline{e_1}$
$v_3(2)$	e_9	e_{21}	e_{18}	e_{16}	e_{17}	e_{38}	$\underline{e_2}$	e_{22}	e_{23}	e_{24}				
$v_4(1)$	e_{19}	e_{25}	e_{10}	e_{36}	e_{24}	e_{39}	e_4	$\underline{e_{26}}$	e_{27}	e_6	$\underline{e_{11}}$	$\underline{e_{28}}$		
$v_5(3)$	e_{29}	e_{30}	e_7	e_{22}	e_{31}	e_{27}	$\underline{e_{32}}$	$\underline{e_{12}}$	e_{15}	e_{33}				
$v_6(3)$	e_{28}	e_{20}	e_{32}	e_{31}	e_5	e_{25}	e_{39}	$\underline{e_{13}}$	$\underline{e_{29}}$	e_{37}	e_{35}	$\underline{e_{21}}$	e_{34}	
$v_7(3)$	e_8	e_{26}	e_{14}	$\underline{e_{30}}$	e_{33}	e_{23}	e_{34}							

Figure 1: The preference lists for an example SMA instance

Lemma 2 (Lemma 4.1 [2]) *If an SMA instance $(G_i, \mathcal{O}_i, b)$ satisfies the FLP then $|B(u, G_i)| = |D(u, G_i)|$ for each vertex $u \in V(G_i)$.*

Although it is impossible to detect all the edges to be deleted during Phase 1 in the beginning of the algorithm, as each deletion may create some new non-FL edges, a few observations about the set of deleted edges are easy to derive.

Proposition 1 *If $e \in FL(u, G)$ then $f \in FL(u, G)$ for each $f \in E(u, G)$ such that $f \prec_u e$.*

Proposition 2 *If $e \in NFL(u, G)$ then $f \in NFL(u, G)$ for each $f \in E(u, G)$ such that $e \prec_u f$.*

Proposition 3 *If $e \in D(u, G_j)$, then either $e \in D(u, G_k)$ for all $k > j$ or $e \in NFL(u, G_k)$ for some $k > j$.*

Proof. $e = uv \in D(u, G_j)$ if $e \in B(v, G_j)$. Hence if e is not deleted, it remains in $B(v, G_k)$ for $k > j$ and can never enter $NFL(v, G_k)$. ■

Proposition 4 *If in $(G_j, \mathcal{O}_j, b)$ some edge $e \in NFL(u, G_j)$ is deleted, then $|D(u, G_k)| \geq b(u)$ for each $k \geq j$. In particular, $|D(u, G^*)| = b(u)$ in any instance $(G^*, \mathcal{O}^*, b)$ obtained by Phase 1.*

Proof. If $e \in NFL(u, G_j)$, then $|D(u, G_j)| \geq b(u)$. If some $f \in D(u, G_j)$ is deleted in a subsequent step k, then due to Proposition 3, it is deleted because $f \in NFL(u, G_k)$, hence $|D(u, G_k)| \geq |\{g \in D(u, G_k) : g \prec_u f\}| \geq b(u)$. In the end of Phase 1 $|D(u, G^*)| = b(u)$ holds because $(G^*, \mathcal{O}^*, b)$ fulfills the FLP. ■

Proposition 5 *If in instance* $(G_j, \mathcal{O}_j, b)$*, edge* e *enters* $NFL(u, G_j)$*, then* e *will stay in* $NFL(u, G_k)$ *for* $k \geq j$ *until it is deleted.*

Proof. An edge $e \in NFL(u, G_j)$ if $|\{f \in D(u, G_j) : f \prec_u e\}| \geq b(u)$. Again by Proposition 3, if some edge f from the above set is deleted from $G_k, k \geq j$ then $|\{g \in D(u, G_k) : g \prec_u f\}| \geq b(u)$ and the assertion follows from transitivity of $\prec_u$. ■

Proposition 6 *If an SMA instance* $(G_j, \mathcal{O}_j, b)$ *satisfies the FLP then* $l_{G_j}(u) \in D(u, G_j)$ *for all vertices* $u \in V(G_j)$*.*

Proof. As $(G_j, \mathcal{O}_j, b)$ satisfies the FLP, $|B(u, G_j)| = |D(u, G_j)|$ by Lemma 2 . If $|E(u, G_j)| \leq b(u)$, then $E(u, G_j) = B(u, G_j) = D(u, G_j)$ and so clearly $l_{G_j}(u) \in D(u, G_j)$.

Suppose now that $|E(u, G_j)| > b(u)$, and $l_{G_j}(u) \notin D(u, G_j)$. Thus all D-edges at u are better than $l_{G_j}(u)$. Again by Lemma 2, we have $|\{f \in D(u, G_j) : f \prec_u l_{G_j}(u)\}| = b(u)$, hence $l_{G_j}(u) \in NFL(u, G_j)$, a contradiction. ■

Proposition 7 *If* $|D(u, G_j)| = b(u)$ *for a vertex* u*, then the deletion of an edge* $e \in NFL(u, G_j)$ *does not create any new non-FL edge.*

Proof. Let $e = uv$. As $|D(u, G_j)| = b(u)$, $NFL(u, G_j) \cap D(u, G_j) = \emptyset$, hence $e \notin B(v, G_j)$, hence no additional edge enters $B(v, G_j)$, hence no new edge becomes a D-edge and hence no FL edge becomes non-FL edge. ■

The following result is a generalization of Lemma 4.2.1 [4] for SR as well as of Lemma 2.1 [8] for SF.

Theorem 1 *For a given SMA instance* $(G, \mathcal{O}, b)$*, all possible executions of Phase 1 of the SMA algorithm yield the same subinstance.*

Proof. Suppose that $(G^*, \mathcal{O}^*, b)$ and $(G', \mathcal{O}', b)$ are the instances produced by two different executions $\mathcal{F}$ and $\mathcal{F}'$ of Phase 1 of the SMA algorithm when applied to $(G, \mathcal{O}, b)$. Suppose that the two instances are different, so let edge $e = uv \in E(G^*)$ but $e \notin E(G')$, and that, during $\mathcal{F}'$, e was the first such edge to be deleted.

e was an FL edge in $(G, \mathcal{O}, b)$, otherwise $e \notin E(G^*)$ by Proposition 5. In step i of $\mathcal{F}'$ when e entered say $NFL(u, G'_i)$, some $f \in E(u, G'_i)$, $f \prec_u e$

entered $D(u, G'_i)$. However, since $e \in E(G^*)$, $|\{g \in D(u, G^*) : g \prec_u e\}| \leq b(u) - 1$. Hence, at least one edge from the set $\{g \in D(u, G'_i) : g \prec_u e\}$, say $f = uz$, does not belong to $D(u, G^*)$. Now distinguish two cases:

1. $f \in E(G^*)$ but $f \notin D(u, G^*)$. Take the set $S = \{g \in E(z, G^*) : g \prec_z f\}$. Then $|S| \geq b(z)$ and at least one edge from S had to be deleted during $\mathcal{F}'$ as $|\{g \in E(z, G'_i) : g \prec_z f\}| < b(z)$, even earlier than e – a contradiction with the assumption that e was the first such edge.

2. $f \notin E(G^*)$, then f was deleted during $\mathcal{F}^*$ because of vertex z (otherwise, according to Proposition 2, e would also have been deleted during $\mathcal{F}^*$). But then $|D(z, G^*)| = b(z)$, due to Proposition 4 and $g \prec_z f$ for each $g \in D(z, G^*)$. Consequently, f had to enter $B(z, G'_i)$ during $\mathcal{F}'$ and hence at least one edge from $D(z, G^*)$ had to be deleted during $\mathcal{F}'$ before f entered $D(u, G'_i)$, hence before e was deleted – again a contradiction.

■

Lemma 3 *Let $I^* = (G^*, \mathcal{O}^*, b)$ be the Phase-1 subinstance of $I = (G, \mathcal{O}, b)$.*

(i) If an edge e is absent from I^* then e is b-dominated by the set $E(G^*)$. In particular, e does not belong to any stable b-matching of I.

(ii) If each vertex u is incident with at most $b(u)$ edges in $(G^*, \mathcal{O}^*, b)$, then $E(G^*)$ determines a stable b-matching.

Proof. If edge $e = uv$ is deleted during Phase 1 then, say $e \in NFL(u, G_i)$ for some i. By Proposition 4, $|D(u, G^*)| = b(u)$ and each edge of $D(u, G^*)$ is $\prec_u$-better that e, so $E(G^*)$ b-dominates e.

Suppose that M is the stable b-matching of I and $e \in M$. Hence there exists at least one edge $f \in D(u, G^*) \setminus M$. As $f \in B(v, G^*)$, $M \cap E(v, G) \subseteq \{E(v, G^*) \setminus \{f\}\}$ and $f \prec_u e$, so f is not b-dominated by M contradicting stability of M.

(ii) is a direct consequence of the proof of assertion (i). ■

Phase 1 of the SMA algorithm can terminate in two possible ways. Either the obtained subinstance $(G^*, \mathcal{O}^*, b)$, called the Phase-1 subinstance, already represents a stable b-matching, or there exists at least one vertex u with $|E(u, G^*)| > b(u)$. In the latter case, Phase 2 of the algorithm follows. A vertex u with $|E(u, G^*)| \leq b(u)$ will be called a *Phase*-1 *vertex* and a vertex u such that $|E(u, G^*)| > b(u)$ a *Phase*-2 *vertex*.

As the result of Phase 1 is independent from the order of deletions, some results concerning the structure of $\mathcal{M}(I)$ can be derived from the form of the obtained Phase-1 subinstance I^*. The following Theorem is a generalization of the "rural hospitals" theorem for the College Admissions Problem (see Theorem 1.6.3 in [4]) and Lemma 2.4 and Corollary 3.1 of [8].

Theorem 2 *Let $(G, \mathcal{O}, b)$ be a solvable SMA instance, then*

(i) each vertex u is assigned the same number of edges in all stable b-matchings,

(ii) if a vertex u is undersubscribed in one stable b-matching then it is assigned to precisely the same set of edges in all stable b-matchings. Moreover, the set of assigned edges for such a vertex u is obtained already by Phase 1 of the SMA algorithm.

Proof. Let $I^* = (G^*, \mathcal{O}^*, b)$ be Phase-1 subinstance of $I = (G, \mathcal{O}, b)$ and suppose that $M \in \mathcal{M}(I)$ is arbitrary.

(i) If u is a Phase-1 vertex then Lemma 3 implies $E(u, G) \cap M \subseteq E(u, G^*)$. Now let $e \notin M$ for some $e = uv \in E(u, G^*)$. By Lemma 2, $e \in D(u, G^*)$, hence $e \in B(v, G^*)$ and this means that e is not b-dominated by M, so M cannot be stable.

Suppose that u is a Phase-2 vertex, i. e. $b(u) = |B(u, G^*)| = |D(u, G^*)|$. Let $D(u, G^*) = \{g_1, g_2, \ldots, g_{b(u)}\}$ and let $g_i = uv_i$ for $i = 1, \ldots, b(u)$. Then $g_i \in B(v_i, G^*)$ for each i. If $|M \cap E(u, G^*)| < b(u)$, then at least one of $g_1, g_2, \ldots, g_{b(u)}$, say g_j, does not belong to M. This immediately indicates that M does not b-dominate g_j, so M cannot be stable.

(ii) Suppose that vertex u is undersubscribed in M. Hence u is a Phase-1 vertex with $|E(u, G^*)| < b(u)$. Now the proof of (i) implies (ii). ∎

As the order of deletions of non-FL edges is immaterial, we can raise the efficiency of Phase 1 of the SMA algorithm by using Propositions 2 and 5. That is, when the algorithm finds an edge $e \in NFL(u, G_j)$ then it deletes whole set $NFL(u, G_j)$.

Example 2 *Let us use Example 1 on page 4. As the capacity of vertex v_4 is 1, each edge in $E(v_4, G)$ worse than e_{26} belongs to $NFL(v_4, G)$. Instead of deleting them one by one, we delete them all, i.e. edges $e_{27}, e_6, e_{11}, e_{28}$*

in one step. After this deletion, as edges e_{11} and e_{28} were D-edges at their other endvertices, some new B-edges arise and this leads to some more new D-edges. Figure 2 displays the situation after this step (again with D-edges underlined) and Figure 3 displays the instance I^* obtained by Phase 1 of the SMA algorithm. As vertex v_7 is in I^* incident with only two edges and its capacity is 3, it is undersubscribed in each stable b-matching (if one exists), and it is always assigned edges e_{14} and e_{30}.

$v_1(3)$	e_1	e_2	$\underline{e_3}$	e_4	e_5	e_{37}	e_{38}	$\underline{e_7}$	$\underline{e_8}$	$\underline{e_9}$	e_{10}		
$v_2(4)$	e_{12}	$\underline{e_3}$	e_{13}	$\underline{e_{14}}$	e_{15}	e_{16}	e_{17}	e_{18}	e_{35}	e_{36}	$\underline{e_{19}}$	$\underline{e_{20}}$	$\underline{e_1}$
$v_3(2)$	e_9	e_{21}	e_{18}	e_{16}	e_{17}	e_{38}	$\underline{e_2}$	e_{22}	e_{23}	e_{24}			
$v_4(1)$	e_{19}	e_{25}	e_{10}	e_{36}	e_{24}	e_{39}	e_4	$\underline{e_{26}}$					
$v_5(3)$	e_{29}	e_{30}	e_7	e_{22}	$\underline{e_{31}}$	$\underline{e_{32}}$	$\underline{e_{12}}$	e_{15}	e_{33}				
$v_6(3)$	e_{20}	e_{32}	e_{31}	e_5	e_{25}	e_{39}	$\underline{e_{13}}$	$\underline{e_{29}}$	e_{37}	e_{35}	$\underline{e_{21}}$	e_{34}	
$v_7(3)$	e_8	e_{26}	$\underline{e_{14}}$	$\underline{e_{30}}$	e_{33}	e_{23}	e_{34}						

Figure 2: The preference lists after deletion of non-FL edges at v_4

$v_1(3)$	e_2	$\underline{e_3}$	e_4	$\underline{e_5}$	e_{37}	e_{38}	$\underline{e_7}$			
$v_2(4)$	$\underline{e_3}$	e_{13}	$\underline{e_{14}}$	e_{16}	e_{17}	$\underline{e_{18}}$	e_{35}	e_{36}	$\underline{e_{19}}$	
$v_3(2)$	e_{21}	e_{18}	$\underline{e_{16}}$	e_{17}	e_{38}	$\underline{e_2}$				
$v_4(1)$	e_{19}	e_{25}	e_{36}	e_{39}	$\underline{e_4}$					
$v_5(3)$	e_{29}	$\underline{e_{30}}$	e_7	$\underline{e_{31}}$	$\underline{e_{32}}$					
$v_6(3)$	e_{32}	e_{31}	e_5	e_{25}	e_{39}	$\underline{e_{13}}$	$\underline{e_{29}}$	e_{37}	e_{35}	$\underline{e_{21}}$
$v_7(3)$	$\underline{e_{14}}$	$\underline{e_{30}}$								

Figure 3: The preference lists of the Phase-1 subinstance I^*

3.2 Phase 2 of the SMA Algorithm

In Phase 2, the algorithm further reduces preference lists of vertices by eliminating the so-called rotations, until each Phase-2 vertex u is incident with exactly $b(u)$ edges or until the algorithm determines that no stable b-matching of the given instance exists.

The idea of a rotation was invented for SR by Irving in [5] originally under the name *all-or-nothing cycle*. Later, Irving and Leather [6] used rotations for enumeration of all stable matchings in an instance of SM and in SF problem [8] again rotations were used to find a stable b-matching. However, for stable matchings in multigraphs the classical definition of rotations does not exhibit all the necessary properties, e.g. it does not ensure the presence of a rotation in each Phase-1 instance [1]. A generalized definition of a rotation for a multigraph was given in [2].

Definition 2 *A rotation exposed in* $(G_i, \mathcal{O}_i, b)$ *is a pair of edge sets*

$$\varrho = (\{e_0^\varrho, e_1^\varrho, \ldots, e_{r-1}^\varrho\}\{f_0^\varrho, f_1^\varrho, \ldots, f_{r-1}^\varrho\})$$

such that

$$e_j^\varrho = u_j^\varrho\ v_j^\varrho,\ f_j^\varrho = u_j^\varrho\ v_{j+1}^\varrho$$

(subscripts are taken modulo r*),* e_j^ϱ *is worst in* $\prec_{v_j^\varrho}$ *and* f_j^ϱ *is the* $(b(u_j^\varrho)+1)$*st best element of* $\prec_{u_j^\varrho}$*.*

The superscript ϱ may be omitted if the rotation is understood from the context. We denote by ϱ_E the set $\{e_0^\varrho, e_1^\varrho, \ldots, e_{r-1}^\varrho\}$ and by ϱ_F the set $\{f_0^\varrho, f_1^\varrho, \ldots, f_{r-1}^\varrho\}$. A vertex w that is incident with some edge of $\varrho_E \cup \varrho_F$ is said to be *covered* by rotation ϱ.

Lemma 4 (Lemma 4.4, [2]) *If* $(G, \mathcal{O}, b)$ *is an FL instance and* $E(G)$ *is not a* b*-matching then there is a rotation* ϱ *exposed in* $(G, \mathcal{O}, b)$ *such that* $|E(v, G)| > b(v)$ *for each vertex* v *covered by* ϱ*.*

The previous Lemma was proved constructively: Given an FL instance $(G, \mathcal{O}, b)$ of SMA, define the auxiliary digraph $H(G) = (V(G), A)$ by $\vec{a} = \vec{vw} \in A$ if $e = vu$ is the worst edge in $\prec_v$ and $f = uw$ is the $(b(u)+1)$st best edge in $\prec_u$. Each Phase-2 vertex has one outgoing arc in H, moreover, such an arc leads to another Phase-2 vertex. In such a digraph always a cycle exists, say $v_0, v_1, \ldots, v_{r-1}$, and this cycle determines a rotation ϱ by taking as $e_i^\varrho = u_i v_i$ the $\prec_{v_i}$-worst edge and $f_i^\varrho = u_i v_{i+1}$ the $(b(u_i)+1)$st best edge for u_i. The vertices of H on a directed path leading to a cycle corresponding to a rotation ϱ are said to *lead* to the rotation ϱ.

Example 3 *The auxiliary digraph $H(G^*)$ for the Phase-1 subinstance I^* from Figure 3 is depicted in Figure 4. $H(G^*)$ contains just one cycle (v_2, v_6) which defines a unique rotation exposed in I^*, namely $\varrho = (\{e_1^\varrho, e_2^\varrho\}\{f_1^\varrho, f_2^\varrho\})$ with $e_1^\varrho = e_{19}, e_2^\varrho = e_{21}, f_1^\varrho = e_{25}, f_2^\varrho = e_{16}$. Rotation ϱ is illustrated in this Figure by edges from ϱ_E depicted in circles and edges from ϱ_F in double circles.*

v_1	e_2	$\underline{e_3}$	e_4	$\underline{e_5}$	e_{37}	e_{38}	$\underline{e_7}$		
$v_1^{\varrho^1} = v_2$	$\underline{e_3}$	e_{13}	$\underline{e_{14}}$	e_{16}	e_{17}	$\underline{e_{18}}$	e_{35}	e_{36}	$\underline{e_{19}}$
$u_2^{\varrho^1} = v_3$	e_{21}	e_{18}	$\underline{e_{16}}$	e_{17}	e_{38}	$\underline{e_2}$			
$u_1^{\varrho^1} = v_4$	e_{19}	e_{25}	e_{36}	e_{39}	$\underline{e_4}$				
v_5	e_{29}	$\underline{e_{30}}$	e_7	$\underline{e_{31}}$	$\underline{e_{32}}$				
$v_2^{\varrho^1} = v_6$	e_{32}	e_{31}	e_5	e_{25}	e_{39}	$\underline{e_{13}}$	$\underline{e_{29}}$	e_{37}	e_{35}	$\underline{e_{21}}$
v_7	$\underline{e_{14}}$	$\underline{e_{30}}$							

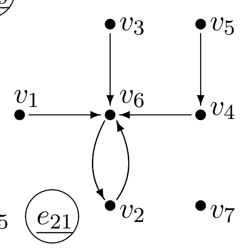

Figure 4: Phase-1 subinstance $(G^*, \mathcal{O}^*, b)$ with the auxiliary digraph $H(G^*)$ and the corresponding rotation

To eliminate the rotation $\varrho = (\{e_0, e_1, \ldots, e_{r-1}\}\{f_0, f_1, \ldots, f_{r-1}\})$ exposed in $(G_i, \mathcal{O}_i, b)$ means according to [2] to delete the edge set ϱ_E from the graph G_i. The correctness of this step for SMA was proved in [2].

Lemma 5 (Lemma 4.5 [2]) *Let $(G_i, \mathcal{O}_i, b)$ be an FL instance and let ϱ be a rotation exposed in $(G_i, \mathcal{O}_i, b)$.*

(i) Sets $\{e_0^\varrho, e_1^\varrho, \ldots, e_{r-1}^\varrho\}$ and $\{f_0^\varrho, f_1^\varrho, \ldots, f_{r-1}^\varrho\}$ are disjoint or identical. In the latter case, $(G_i, \mathcal{O}_i, b)$ has no stable b-matching.

(ii) If $(G_{i+1}, \mathcal{O}_{i+1}, b)$ is the SMA subinstance obtained by the elimination of rotation ϱ then properties (1) - (3) hold.

In the SMA, the subinstance $(G_{i+1}, \mathcal{O}_{i+1}, b)$ obtained by rotation elimination may not satisfy the FLP, hence in [2] it was suggested that the algorithm returns to Phase 1 again. Because all possible executions of Phase 1 yield the same reduced instance (see Theorem 1), the instance obtained by the elimination of ϱ and the subsequent application of Phase 1 is uniquely determined. The following Lemma characterizes which edges exactly are to be deleted. We denote by $k_{G_i}(u)$ the $\prec_u$ −worst edge of $D(u, G_i) \setminus l_{G_i}(u)$.

Lemma 6 *Let $\varrho = (\{e_0, e_1, \ldots, e_{r-1}\}\{f_0, f_1, \ldots, f_{r-1}\})$ be a rotation exposed in $I_i = (G_i, \mathcal{O}_i, b)$ with $e_j = u_j v_j = l_{G_i}(v_j)$, $f_j = u_j v_{j+1} = s_{G_i}(u_j)$. If $\varrho_E \cap \varrho_F = \emptyset$, then all the edges to be deleted during the application of Phase 1 after the elimination of ϱ are incident with some v_j, moreover, they are exactly those that are $\prec_{v_j}$ −worse than f_{j-1} as well as $k_{G_i}(v_j)$ in $I_{i+1} = (G_{i+1}, \mathcal{O}_{i+1}, b)$.*

Proof. The instance I_i satisfies the FLP, so $|D(w, G_i)| \leq b(w)$ for each vertex $w \in V(G_i)$ and $l_{G_i}(w) = wz \in D(w, G_i)$ by Proposition 6, so $l_{G_i}(w) \in B(z, G_i)$.

During the elimination of rotation ϱ we delete edges $e_j = u_j v_j = l_{G_i}(v_j)$ for $j = 0, \ldots, r-1$. As $\varrho_E \cap \varrho_F = \emptyset$, edge $f_j = u_j v_{j+1} = s_{G_i}(u_j)$ enters $B(u_j, G_{i+1})$ and hence also $D(v_{j+1}, G_{i+1})$, $0 \leq j \leq r-1$. More precisely, for $j = 0, 1, \ldots, r-1$:

$$D(v_j, G_{i+1}) = D(v_j, G_i) \setminus \{e_j\} \cup \{f_{j-1}\}. \tag{5}$$

For vertices w that are not covered by ϱ as some v_k, $k = 0, \ldots, r-1$, we have $D(w, G_{i+1}) = D(w, G_i)$ as each edge $e = wz \in E(w, G_i)$ neither enters $B(z, G_{i+1})$ nor is deleted from $B(z, G_i)$ during the elimination of ϱ.

As $|E(v_j, G_i)| > b(v_j)$ for all v_j $(0 \leq j \leq r-1)$, by Lemma 2 $|D(v_j, G_i)| = b(v_j)$ and from (5), it is clear that also $|D(v_j, G_{i+1})| = b(v_j)$.

I_i satisfies the FLP and sets of D-edges change only for vertices covered as v_j, $j = 0, \ldots, r-1$, thus non-FL edges in I_{i+1} are only those edges incident with v_j that are for v_j worse than the worst of D-edges at v_j. Denote by d_j the $\prec_{v_j}$-worst edge from $D(v_j, G_{i+1})$. From (5) it follows, that d_j is either $k_{G_i}(v_j)$ or f_{j-1}.

As $|D(v_j, G_{i+1})| = b(v_j)$ for all v_j, by Proposition 7 the FLP property will be restored after deletion of all edges $\prec_{v_j}$-worse than d_j. ■

Hence, from now on we shall understand by rotation elimination the deletion of the set ϱ_E plus all the edges in the subsequent Phase 1. This is summarized in the following definition. Now we have the rotation elimination analogous to that defined for the SF problem in [8].

Definition 3 *Let $\varrho = (\{e_0, e_1, \dots, e_{r-1}\}\{f_0, f_1, \dots, f_{r-1}\})$ be a rotation exposed in an FL instance $I = (G_i, \mathcal{O}_i, b)$ with $e_j = u_j v_j = l_{G_i}(v_j)$, $f_j = u_j v_{j+1} = s_{G_i}(u_j)$. The elimination of rotation ϱ is the deletion of all edges of the form $g = v_j w$, where $f_{j-1} \prec_{v_j} g$ as well as $k_{G_i}(v_j) \prec_{v_j} g$, for all $j = 0, 1, \dots, r-1$. The obtained subinstance will be denoted by $(G_i \setminus \varrho, \mathcal{O}_i \setminus \varrho, b)$ or $I_i \setminus \varrho$.*

The correctness of rotation elimination defined in this way is justified by the following lemma, which is an immediate consequence of Lemma 6, Lemma 1 and Theorem 1.

Lemma 7 *Let $(G_i, \mathcal{O}_i, b)$ be an FL instance and let ϱ be a rotation exposed in $(G_i, \mathcal{O}_i, b)$ such that $\varrho_E \cap \varrho_F = \emptyset$. If $(G_i \setminus \varrho, \mathcal{O}_i \setminus \varrho, b)$ is the SMA instance obtained by the elimination of rotation ϱ then properties (1) - (3) hold.*

Lemma 8 is a generalization of Lemma 4.2.7 [4] and Lemma 3.4 [8].

Lemma 8 *Let $\varrho = (\{e_0, e_1, \dots, e_{r-1}\}\{f_0, f_1, \dots, f_{r-1}\})$ be a rotation exposed in $(G_i, \mathcal{O}_i, b)$. Let $(G_i \setminus \varrho, \mathcal{O}_i \setminus \varrho, b)$ be its subinstance obtained by elimination of ϱ. If $\varrho_E \cap \varrho_F = \emptyset$ then*

(i) $B(u, G_i \setminus \varrho) = B(u, G_i) \setminus \{e_j\} \cup \{f_j\}$ *for each u covered by ϱ as u_j*

(ii) $D(v, G_i \setminus \varrho) = D(v, G_i) \setminus \{e_j\} \cup \{f_{j-1}\}$ *for each v covered by ϱ as v_j*

(iii) $B(u, G_i \setminus \varrho) = B(u, G_i)$ *for each vertex u not covered by ϱ as u_j and* $D(v, G_i \setminus \varrho) = D(v, G_i)$ *for each vertex v not covered by ϱ as v_j, $j = 0, \dots, r-1$.*

Proof. During the elimination of rotation ϱ we delete for each j, $(0 \le j \le r-1)$ edges $g = v_j w$ fulfilling $f_{j-1} \prec_{v_j} g$ and $k_{G_i}(v_j) \prec_{v_j} g$. Similarly as in the proof of Lemma 6 we get, that except for the edge set ϱ_E, none of these edges is a B-edge nor a D-edge at any v_j. Therefore assertion (iii) of Lemma 8 follows. As $\varrho_E \cap \varrho_F = \emptyset$, we do not delete edge f_j for any j, $0 \le j \le r-1$, so $f_j \in B(u_j, G_i \setminus \varrho)$, or equivalently $f_j \in D(v_j, G_i \setminus \varrho)$ for $0 \le j \le r-1$ (assertions (i) and (ii)). ■

Lemma 9 *If $I_i = (G_i, \mathcal{O}_i, b)$ is an FL subinstance of $I = (G, \mathcal{O}, b)$, then the edge $e = uv \in V(G)$ is absent from $E(G_i)$ if and only if e is b-dominated by $D(u, G_i)$ or $D(v, G_i)$.*

Proof. If e is b-dominated either by $D(u, G_i)$ or $D(v, G_i)$, then either $e \in NFL(u, G_i)$ or $e \in NFL(v, G_i)$, so FLP of I_i implies that $e \notin E(G_i)$.

For the converse implication suppose that $e \notin E(G_i)$. If I_i is the Phase-1 subinstance then by the proof of Lemma 3 e is b-dominated by $D(u, G_i)$. Now suppose that the assertion of this Lemma holds for a subinstance $I_k = (G_k, \mathcal{O}_k, b)$ and $I_i = I_k \backslash \varrho$ for some rotation ϱ. If $e = uv$ is absent already from I_k, then the assumption implies that either $D(u, G_i)$ or $D(v, G_i)$ b-dominates e. If $e \in E(G_k) \backslash E(G_i)$, then e was deleted during the elimination of rotation ϱ, i. e. for some j $(0 \leq j \leq r-1)$, $e = v_j^{\varrho} w$ and $f_{j-1}^{\varrho} \prec_{v_j^{\varrho}} e$ and $k_{G_k}(v_j^{\varrho}) \prec_{v_j^{\varrho}} e$. Now the assertion follows from Lemma 8(ii). ∎

Lemma 10 *Let $I_i = (G_i, \mathcal{O}_i, b)$ be an FL subinstance of $I = (G, \mathcal{O}, b)$. Then*

(i) if $M \in \mathcal{M}(I_i)$, then M b-dominates each edge absent from $E(G_i)$.

(ii) if each Phase-2 vertex u is incident with exactly $b(u)$ edges in $(G_i, \mathcal{O}_i, b)$, than $E(G_i)$ determines a stable b-matching,

(iii) if $I_l = (G_l, \mathcal{O}_l, b)$ is also an FL subinstance of $I = (G, \mathcal{O}, b)$ and $B(u, G_l) = B(u, G_i)$ for each vertex u, or equivalently $D(u, G_l) = D(u, G_i)$ for each u, then $I_i = I_l$.

Proof. (i) This is implied by Lemma 9.

(ii) This is a consequence of assertion (i).

(iii) By the definition of B-edges and D-edges, $B(u, G_l) = B(u, G_i)$ holds for all u if and only if $D(v, G_l) = D(v, G_i)$ holds for all v. So the sets of D-edges b-dominate the same edges in both instances I_i and I_k, hence Lemma 9 implies that $I_i = I_l$. ∎

Lemma 11 *Suppose $I_k = (G_k, \mathcal{O}_k, b)$ and $I_l = (G_l, \mathcal{O}_l, b)$ are FL instances and $I_k \subseteq I_l$. If $\varrho = (\{e_0^{\varrho}, e_1^{\varrho}, \ldots, e_{r-1}^{\varrho}\}\{f_0^{\varrho}, f_1^{\varrho}, \ldots, f_{r-1}^{\varrho}\})$ is a rotation exposed in I_l and if $B(w, G_k) \neq B(w, G_l)$ for at least one vertex w that leads to ϱ, then $I_k \subseteq I_l \setminus \varrho$.*

Proof. Let w be such that $B(w, G_k) \neq B(w, G_l)$. If w leads to ϱ in I_l then there is a sequence of edge pairs $(g_0, h_0), (g_1, h_1), \ldots, (g_{t-1}, h_{t-1})$ such that $g_i = w_i z_i = l_{G_l}(z_i)$, $h_i = w_i z_{i+1} = s_{G_l}(w_i)$, $0 \leq i \leq t-1$, $w = w_0$ and w_t is covered by ϱ as some u_s^{ϱ} where $0 \leq s \leq r-1$.

As $B(w_0, G_k) \neq B(w_0, G_l)$ and $I_k \subseteq I_l$, there exists at least one edge $f \in B(w_0, G_k) \setminus B(w_0, G_l)$. As w_0 is Phase-2 vertex with $|B(w_0, G_k)| > b(w_0)$, we distinguish two cases: $h_0 = f$ and $h_0 \prec_{w_0} f$.

If $f = h_0$ then $h_0 \in D(z_1, G_k) \setminus D(z_1, G_l)$. By Lemma 9, h_0 was not b-dominated by $D(z_1, G_l)$, so $h_0 \prec_{z_1} l_{G_l}(z_1)$ and hence $l_{G_l}(z_1) \in D(z_1, G_l) \setminus D(z_1, G_k)$ what implies that $l_{G_k}(z_1)$ must be equal to, or better than the worse of h_0 and $k_{G_l}(z_1)$ (notice, that $k_{G_l}(z_1)$ does not need to be in $E(G_k)$).

If $h_0 \prec_{w_0} f$, then $h_0 \notin B(w_0, G_k)$ and it had to be deleted as a non-FL edge at z_1. So it is b-dominated by $D(z_1, G_k)$ but not $D(z_1, G_l)$. Hence $l_{G_k}(z_1)$ must be better than the worse of h_0 and $k_{G_l}(z_1)$.

But clearly $l_{G_l}(z_1) = g_1 = w_1 z_1$ is worse than h_0 and $k_{G_l}(z_1)$, so $l_{G_k}(z_1) \prec_{z_1} l_{G_l}(z_1)$. It follows that $B(w_1, G_l) \neq B(w_1, G_k)$. If we repeat this argument for vertices $w_1, \ldots, w_t = u_s^\varrho, u_{s+1}^\varrho, \ldots$ we get that for each $p = 0, 1, \ldots, r-1$ we have $B(u_p^\varrho, G_l) \neq B(u_p^\varrho, G_k)$ and $l_{G_k}(v_{p+1}^\varrho)$ is for v_{p+1}^ϱ better than or equal to the worse of $f_p^\varrho = u_p^\varrho v_{p+1}^\varrho$ and $k_{G_l}(v_{p+1}^\varrho)$.

During the elimination of ϱ from I_l we delete only edges incident with v_p^ϱ $(0 \leq p \leq r-1)$, worse than both f_{p-1}^ϱ and $k_{G_l}(v_p^\varrho)$. It follows, that none of these edges is present in I_k and so I_k is a subinstance of $I_l \setminus \varrho$. ■

Lemma 12 *Let $I_k = (G_k, \mathcal{O}_k, b)$ and $I_l = (G_l, \mathcal{O}_l, b)$ be FL instances. If $I_k \subseteq I_l$, then I_k can be obtained from I_l by elimination of a rotation sequence. In particular, each FL subinstance can be obtained from the Phase-1 subinstance by elimination of an appropriate sequence of rotations.*

Proof. Suppose that $I_k \neq I_l$ (if they are equal, the first part of the lemma is trivial). Lemma 10(iii) implies that $B(w, G_l) \neq B(w, G_k)$ for some w. Moreover, w has to be a Phase-2 vertex, so in $H(G_l)$ it leads to some rotation ϱ^1. By Lemma 11, $I_k \subseteq I_1 = (G_l \setminus \varrho^1, \mathcal{O}_l \setminus \varrho^1, b)$. Repeating this argument, we can produce a sequence $I_1, I_2, \ldots, I_s$ of instances such that $I_k \subseteq I_t = (G_l \setminus \varrho^1 \setminus \cdots \setminus \varrho^t, \mathcal{O} \setminus \varrho^1 \setminus \cdots \setminus \varrho^t, b)$, for each $t = 1, 2, \ldots, s$. Moreover $B(w, G_k) = B(w, G_l \setminus \varrho^1 \setminus \cdots \setminus \varrho^s)$ for all w. Then by Lemma 10(iii) $I_s = I_k$.

Each FL instance is a subinstance of the Phase-1 subinstance, so the second part of the lemma is straightforward. ■

As each stable b-matching is an FL subinstance of the starting instance $(G, \mathcal{O}, b)$, we get the following generalization of Corollary 4.2.2 of [4].

Corollary 1 *For a solvable SMA instance, each stable b-matching can be found by the SMA algorithm.*

Example 4 *As we saw earlier, in the SMA instance of Figure 4 on page 11, a unique rotation is exposed,* $\varrho^1 = (\{e_{21}, e_{19}\}\{e_{16}, e_{25}\})$. *As* $f_2^{\varrho_1} = e_{16} \prec_{v_2} k_{G_i}(v_2) = e_{18}$, *edges* e_{35}, e_{36} *and* e_{19} *will be deleted from the list of* v_2. *Similarly,* $f_1^{\varrho_1} = e_{25} \prec_{v_6} k_{G_i}(v_6) = e_{29}$, *hence* v_6 *will delete edges* e_{37}, e_{35} *and* e_{21}. *In the new subinstance, two rotations are exposed:* $\varrho^2 = (\{e_{18}\}\{e_{17}\})$ *and* $\varrho^3 = (\{e_{29}\}\{e_{31}\})$. *Altogether, 7 rotations become exposed in all the subinstances for this example, their complete list is*

$$\begin{aligned} \varrho^1 &= (\{e_{21}, e_{19}\}\{e_{16}, e_{25}\}) \\ \varrho^2 &= (\{e_{18}\}\{e_{17}\}) & \varrho^7 = (\{e_{17}\}\{e_{18}\}) \\ \varrho^3 &= (\{e_{29}\}\{e_{31}\}) \\ \varrho^4 &= (\{e_2, e_{13}\}\{e_5, e_{17}\}) \\ \varrho^5 &= (\{e_7, e_{25}\}\{e_{32}, e_4\}) & \varrho^6 = (\{e_4, e_{32}\}\{e_7, e_{25}\}) \end{aligned}$$

and Figure 5 illustrates how they become exposed. For this instance, $\mathcal{M}(I) = \{M_1, M_2, M_3, M_4\}$ *and these b-matchings are shown in Figure 6.*

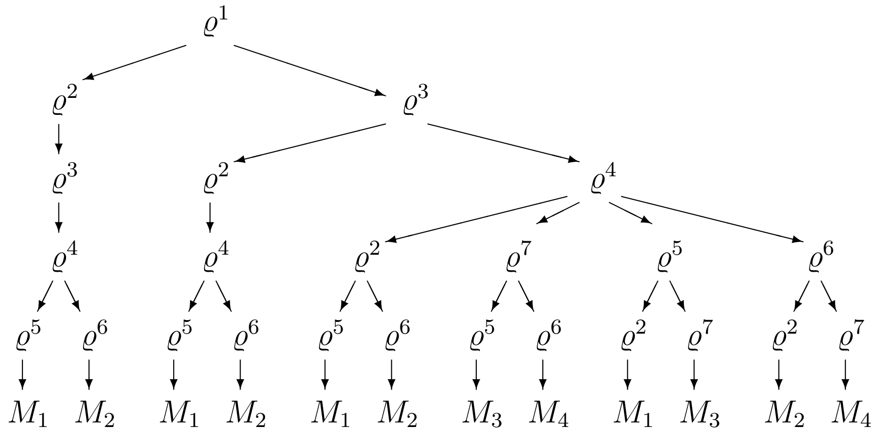

Figure 5: The diagram illustrating how rotations are exposed

4 The Complexity of the SMA Algorithm

The SMA algorithm decides whether an SMA instance $(G, \mathcal{O}, b)$ admits a stable b-matching or not, and if so, it finds one. The algorithm, as proposed in [2], runs in $O(m^2)$ time, where m is the number of edges of G.

$$M_1 \left| \begin{array}{l|llll} v_1(3) & e_3 & e_4 & e_5 & \\ v_2(4) & e_3 & e_{14} & e_{16} & e_{17} \\ v_3(2) & e_{16} & e_{17} & & \\ v_4(1) & e_4 & & & \\ v_5(3) & e_{30} & e_{31} & e_{32} & \\ v_6(3) & e_{32} & e_{31} & e_5 & \\ v_7(3) & e_{14} & e_{30} & & \end{array} \right. \qquad M_2 \left| \begin{array}{l|llll} v_1(3) & e_3 & e_5 & e_7 & \\ v_2(4) & e_3 & e_{14} & e_{16} & e_{17} \\ v_3(2) & e_{16} & e_{17} & & \\ v_4(1) & e_{25} & & & \\ v_5(3) & e_{30} & e_7 & e_{31} & \\ v_6(3) & e_{31} & e_5 & e_{25} & \\ v_7(3) & e_{14} & e_{30} & & \end{array} \right.$$

$$M_3 \left| \begin{array}{l|llll} v_1(3) & e_3 & e_4 & e_5 & \\ v_2(4) & e_3 & e_{14} & e_{16} & e_{18} \\ v_3(2) & e_{18} & e_{16} & & \\ v_4(1) & e_4 & & & \\ v_5(3) & e_{30} & e_{31} & e_{32} & \\ v_6(3) & e_{32} & e_{31} & e_5 & \\ v_7(3) & e_{14} & e_{30} & & \end{array} \right. \qquad M_4 \left| \begin{array}{l|llll} v_1(3) & e_3 & e_5 & e_7 & \\ v_2(4) & e_3 & e_{14} & e_{16} & e_{18} \\ v_3(2) & e_{18} & e_{16} & & \\ v_4(1) & e_{25} & & & \\ v_5(3) & e_{30} & e_7 & e_{31} & \\ v_6(3) & e_{31} & e_5 & e_{25} & \\ v_7(3) & e_{14} & e_{30} & & \end{array} \right.$$

Figure 6: Stable b-matchings of the example instance

The improvements in the SMA algorithm suggested in this paper are:

1. for Phase 1: elimination of several non-FL edges in one step;
2. for Phase 2: elimination of a rotation plus deletion of all the edges that would be identified in the subsequent Phase 1 in a single step.

Figure 7 displays the modified SMA algorithm in the pseudocode.

To achieve the worst-case time complexity $O(m)$ for the modified SMA algorithm, we have to use some special data structures and techniques; they are similar to those used for SR in [4] and for SF in [8].

Let us suppose that the underlying multigraph G of the SMA instance $(G, \mathcal{O}, b)$ is given by its vertex-edge incidence matrix and orders $\prec_v \in \mathcal{O}$ are represented for each vertex v by a double linked structure of edges incident with v. Moreover, suppose that for each edge we have links to positions in the preference lists of its endvertices (the links can be e.g. a part of the incidence matrix of G or of the structure representing orders; if not, they can be created from the above structures in $O(m)$ time). Such a structure enables us to delete one edge in constant time.

In Phase 1, sets $B(u, G)$ are built successively by scanning the ordered lists of vertices in the direction from the best edge. Each time another edge

```
The modified SMA algorithm
        Input: SMA instance (G, O, b);
        Output: stable b-matching of (G, O, b) if one exists;
begin i:=0; { Phase 1 }
        (G_0, O_0, b) := (G, O, b); B(u, G_0) := ∅ for all u ∈ V(G_0);
        while there exists u with |B(u, G_i)| < b(u) and |E(u, G_i)| > |B(u, G_i)| do
                begin e := uv = the ≺_u −best edge of E(u, G_i) \ B(u, G_i);
                      add e to B(u, G_i) and to D(v, G_i) ;
                      if |D(v, G_i)| ≥ b(v) then delete NFL(v, G_i);
                      increase i by 1;
                end;

        (G_i, O_i, b) =the Phase-1 subinstance; { Phase 2 }
        while E(G_i) is not a b-matching do
                begin find rotation ϱ exposed in (G_i, O_i, b);
                      if ϱ_E = ϱ_F
                      then STOP: No stable b-matching of (G, O, b) exists
                      else eliminate ϱ
                      increase i by 1;
                end;
        STOP: Output stable b-matching E(G_i) of (G, O, b)
end.
```

Figure 7: Pseudocode of the modified SMA algorithm

$e = uv$ enters $B(u, G)$, the number of D-edges at its second endvertex is increased by 1 and the edge is marked to be a D-edge. If at some vertex v the number of D-edges reaches its capacity $b(v)$, the edges from $NFL(v, G)$ are deleted starting from the end of the preference list of v until the worst of D-edges is found. The number of backwards steps and deletions is bounded by $2m$. So in the worst case, the complexity of Phase 1 is $O(m)$.

In Phase 2, efficient search for rotations in the auxiliary digraph $H(G) = (V(G), A)$ is ensured by using a stack. The stack is initialized by some Phase-2 vertex v. The next vertex to be pushed in the stack is a vertex w, such that $(v, w) \in A$. A rotation is found as soon as the algorithm reaches a vertex that is already on the stack. Then the rotation ϱ can be built up by popping the stack till the first appearance of that vertex; for each popped vertex u, the $\prec_u$-worst edge belongs to ϱ_E. At this point, the algorithm checks whether $\varrho_E = \varrho_F$. If this is the case, no stable b-matching of the original instance exists, otherwise, the rotation elimination is performed. This is done efficiently by tracing the preference lists of vertices v_i^ϱ from their ends until for the first time a D-edge, different from e_i^ϱ, or the edge f_i^ϱ is reached.

Vertices that remain in the stack after the found rotation ϱ is eliminated, are not covered by ϱ. So Lemma 8(iii) implies, that their sets of B-edges and D-edges are not affected, so these vertices are incident with more edges than their capacity. By Lemma 4, there exists a rotation and moreover, for the vertices left in the stack the arcs in the old and new auxiliary digraph are the same.

Hence each time a rotation is sought, the algorithm starts from the top vertex of the stack, or, if the stack is empty, from another Phase-2 vertex v with more than $b(v)$ incident edges. This approach guarantees that the algorithm will not traverse the same long path in the auxiliary digraph more than once.

This implies that the numbers of push and pop operations are equal. Realize, that the only vertices that can be pushed into the stack are those incident with more edges than their capacity. Therefore the stack will be empty when the algorithm terminates. Further, each time a vertex is popped from the stack, at least one edge is deleted from the graph. Consequently, the number of pop, and therefore push, operations is bounded by the total number of edges m. As mentioned above, each deletion is a constant time operation and so is each other operation. Therefore, the whole algorithm runs in $O(m)$ time.

Since the Stable marriage problem is a special case of the SMA and it was shown to be $\Omega(m)$ [9], the SMA algorithm is asymptotically optimal.

5 The structure of stable *b*-matchings

As we have seen, an SMA instance may admit several stable b-matchings. As the result of Phase 1 is unique, the particular stable b-matching obtained by the SMA algorithm is determined by the set of eliminated rotations.

The structure of the stable matchings in SR is known to be related to the structure of the set of rotations corresponding to the given SR instance [4]. In [8], an SF algorithm was presented, but the structure of stable b-matchings was not studied. In this section we explore the structure of rotations in more detail, trying to obtain some analogies to the stable roommates case. Each instance in this section is supposed to have FLP and we restrict our work only to solvable instances, so for each rotation ϱ we suppose that $\varrho_E \cap \varrho_F = \emptyset$.

Definition 4 *Let* $\varrho = (\{e_0^\varrho, e_1^\varrho, \dots, e_{r-1}^\varrho\}\{f_0^\varrho, f_1^\varrho, \dots, f_{r-1}^\varrho\})$ *be a rotation with* $e_i = u_i v_i$, $f_i = u_i v_{i+1}$, $0 \le i \le r-1$. *If the pair of edge sets*

$$\bar{\varrho} = (\{f_0^\varrho, f_1^\varrho, \dots, f_{r-1}^\varrho\}\{e_1^\varrho, e_2^\varrho, \dots, e_{r-1}^\varrho, e_0^\varrho\})$$

is also a rotation, i. e. if

$$f_i^\varrho = v_{i+1}^\varrho u_i^\varrho \text{ is the worst edge in } \prec_{u_i^\varrho} \text{ and}$$

$$e_{i+1}^\varrho = v_{i+1}^\varrho u_{i+1}^\varrho \text{ is the } (b(v_{i+1}^\varrho)+1)st \text{ edge in } \prec_{v_{i+1}^\varrho}$$

for $0 \le i \le r-1$, *than* ϱ *is called nonsingular and the rotation* $\bar{\varrho}$ *is called the dual rotation to rotation* ϱ. *If* $\bar{\varrho}$ *is not a rotation, then* ϱ *is called singular.*

Note, that ϱ and $\bar{\varrho}$ cover the same set of vertices. Moreover if a rotation ϱ is dual to a rotation σ, then also σ is dual to rotation ϱ.

Lemma 13 *Let* ϱ *and* σ *be two rotations exposed in* $I = (G, \mathcal{O}, b)$. *Then*

(i) $\varrho_E \cap \sigma_E \neq \emptyset$ *if and only if* $\varrho = \sigma$

(ii) $\varrho_E \cap \sigma_F \neq \emptyset$ *if and only if* $\varrho = \bar{\sigma}$

Proof. Let $\varrho = (\{e_0^\varrho, e_1^\varrho, \dots, e_{r-1}^\varrho\}\{f_0^\varrho, f_1^\varrho, \dots, f_{r-1}^\varrho\})$ and $\sigma = (\{e_0^\sigma, e_1^\sigma, \dots, e_{k-1}^\sigma\}\{f_0^\sigma, f_1^\sigma, \dots, f_{k-1}^\sigma\})$. We will prove the 'only if' implications, as the 'if' direction is trivial.

(i) Suppose that $e \in \varrho_E \cap \sigma_E$, thus there exist j and l, $0 \le j \le r-1$, $0 \le l \le k-1$, such that $e = e_j^\varrho = u_j^\varrho v_j^\varrho = e_l^\sigma = u_l^\sigma v_l^\sigma$.

By the definition of a rotation, $e = l_G(v_j^\varrho)$ and by Proposition 6, $e \in D(v_j^\varrho, G)$, thus $e \in B(u_j^\varrho, G)$. As $|E(u_j^\varrho, G)| > b(u_j^\varrho)$, e cannot be the worst edge in $\prec_{u_j^\varrho}$, so $u_j^\varrho \neq v_l^\sigma$. Hence e is the worst edge for the same vertex $v_j^\varrho = v_l^\sigma$ and by induction we get $\varrho_E = \sigma_E$ and $\varrho_F = \sigma_F$. The argument also implies that $\varrho = \sigma$.

(ii) Suppose that $g \in \varrho_E \cap \sigma_F$, thus there exist j and l, $0 \le j \le r-1$, $0 \le l \le k-1$, such that $g = e_j^\varrho = u_j^\varrho v_j^\varrho = f_l^\sigma = u_l^\sigma v_{l+1}^\sigma$.

By the definition of a rotation, $g = l_G(v_j^\varrho) = s_G(u_l^\sigma)$. By Proposition 6, $g \in B(u_j^\varrho, G)$, so $u_j^\varrho \neq u_l^\sigma$ and thus $u_j^\varrho = v_{l+1}^\sigma$, $v_j^\varrho = u_l^\sigma$. Hence, $|E(v_j^\varrho, G)| = |E(u_l^\sigma, G)| = b(u_l^\sigma) + 1 = b(v_j^\varrho) + 1$. By Lemma 2, $|B(u_l^\sigma, G)| = |D(u_l^\sigma, G)| = b(u_l^\sigma)$, so $|B(u_l^\sigma, G) \cap D(u_l^\sigma, G)| = b(u_l^\sigma) - 1$. Again by Proposition 6, the worst edge in $\prec_{u_l^\sigma}$ is in $D(u_l^\sigma, G)$, so

$$g = l_G(u_l^\sigma) \in D(u_l^\sigma, G) \setminus B(u_l^\sigma, G) = D(v_j^\varrho, G) \setminus B(v_j^\varrho, G)$$

and for a unique edge h:

$$h \in B(u_l^\sigma, G) \setminus D(u_l^\sigma, G) = B(v_j^\varrho, G) \setminus D(v_j^\varrho, G).$$

Edges $f_{j-1}^\varrho = u_{j-1}^\varrho v_j^\varrho$ and $e_l^\sigma = u_l^\sigma v_l^\sigma$ have vertex $v_j^\varrho = u_l^\sigma$ in common. $f_{j-1}^\varrho = s_G(u_{j-1}^\varrho)$, so $f_{j-1}^\varrho \notin D(v_j^\varrho, G)$ and therefore $u_{j-1}^\varrho v_j^\varrho = h$. Also $e_l^\sigma = l_G(v_l^\sigma)$ and by Proposition 6, $e_l^\sigma \in D(v_l^\sigma, G)$, so $e_l^\sigma \in B(u_l^\sigma, G)$. Moreover, $|E(v_l^\sigma, G)| \geq b(v_l^\sigma) + 1$, so $e_l^\sigma \notin B(v_l^\sigma, G)$, hence $e_l^\sigma \notin D(u_l^\sigma, G)$ and therefore $e_l^\sigma = h$. Consequently $u_{j-1}^\varrho v_j^\varrho = h = u_l^\sigma v_l^\sigma$. So the assumption $g = e_j^\varrho = f_l^\sigma$ leads to $f_{j-1}^\varrho = e_l^\sigma = h$.

Now, if we take the edge $h = f_{j-1}^\varrho = e_l^\sigma$ and interchange the roles of rotations ϱ and σ in the proof above, we get $e_{j-1}^\varrho = f_{l-1}^\sigma$.

By induction we get that $\varrho_E = \sigma_F$ and $\varrho_F = \sigma_E$ in the same cyclic order and from the argument it is clear that $\varrho = \bar{\sigma}$. ■

Although $\varrho_E \cap \sigma_E \neq \emptyset$ implies $\varrho = \sigma$ for arbitrary ϱ and σ exposed in the same instance, the implication $\varrho_F \cap \sigma_F \neq \emptyset \Longrightarrow \varrho = \sigma$ is not valid in general. The following example illustrates such a situation.

Example 5 *The following table represents an SMA instance I (with $b(u) = 1$ for all vertices for simplicity). The rotations exposed in I are written to the right of the preference lists.*

u_1	e_1	e_2	e_3
u_2	e_4	e_2	e_1
u_3	e_3	e_5	
u_4	e_6	e_7	
u_5	e_5	e_8	
u_6	e_7	e_9	e_{10}
u_7	e_{10}	e_9	e_4
u_8	e_8	e_6	

$$\begin{aligned} \varrho &= (\{e_1\}\{e_2\}) \\ \sigma &= (\{e_{10}\}\{e_9\}) \\ \tau &= (\{e_4, e_3, e_8, e_7\}\{e_2, e_5, e_6, e_9\}) \end{aligned}$$

It is easy to see that:
$\varrho_F \cap \tau_F = \{e_2\} \neq \emptyset$ *but* $\varrho \neq \tau$ *and also* $\sigma_F \cap \tau_F = \{e_9\} \neq \emptyset$ *but* $\sigma \neq \tau$.

Lemma 14 *Let $\varrho^1, \varrho^2, \ldots, \varrho^k$, $k \geq 2$ be different rotations exposed in an SMA instance $I = (G, \mathcal{O}, b)$. Then each vertex w can be covered by at most two rotations. Moreover, if w is covered by ϱ^i and ϱ^j, then w is covered by ϱ^i only as some $u_l^{\varrho^i}$ and by ϱ^j only as some $v_r^{\varrho^j}$.*

Proof. Suppose that vertex w is covered by rotation ϱ^i. If $w = v_l^{\varrho^i}$, then w cannot be covered by any other rotation ϱ^j as some $v_r^{\varrho^j}$, as then $e_l^{\varrho^i} = e_r^{\varrho^j}$ and Lemma 13(i) implies $\varrho^i = \varrho^j$.

Suppose that $w = u_l^{\varrho^i}$. If w is covered by ϱ^j also as some $u_r^{\varrho^j}$, then $u_l^{\varrho^i} v_{l+1}^{\varrho^i} = s_G(u_l^{\varrho^i}) = s_G(u_r^{\varrho^j}) = u_r^{\varrho^j} v_{r+1}^{\varrho^j}$ and so $v_{l+1}^{\varrho^i} = v_{r+1}^{\varrho^j}$. Hence $e_{l+1}^{\varrho^i} = e_{r+1}^{\varrho^j} \in \varrho_E^i \cap \varrho_E^j$ which is by Lemma 13(i) a contradiction with $\varrho^i \neq \varrho^j$. ■

The following lemmas generalize Lemmas 4.3.1, 4.3.4 and 4.3.2 of [4] (in this order) and they explain the significance of the notion of dual rotation.

Lemma 15 *Let two rotations* $\varrho = (\{e_0^\varrho, e_1^\varrho, \ldots, e_{r-1}^\varrho\}\{f_0^\varrho, f_1^\varrho, \ldots, f_{r-1}^\varrho\})$ *and* $\sigma = (\{e_0^\sigma, e_1^\sigma, \ldots, e_{k-1}^\sigma\}\{f_0^\sigma, f_1^\sigma, \ldots, f_{k-1}^\sigma\})$, $\varrho \neq \sigma$ *be exposed in the same FL instance* $I = (G, \mathcal{O}, b)$. *Then either* ϱ *is exposed in* $I \setminus \sigma$ *or* $\sigma = \overline{\varrho}$.

Proof. Take any vertex w covered by ϱ as some v_i^ϱ, $0 \leq i \leq r-1$. By Lemma 14, it cannot be covered by σ as v_j^σ and thus by Lemma 8(iii) $D(v_i^\varrho, G \setminus \sigma) = D(v_i^\varrho, G)$. In particular, edge $e_i^\varrho = l_G(v_i^\varrho)$ is not deleted during the elimination of σ for any i, $0 \leq i \leq r-1$. So $\varrho_E \subseteq E(G \setminus \sigma)$.

Now distinguish two cases.

1. $\varrho_F \subseteq E(G \setminus \sigma)$. Take any vertex w covered by ϱ as some u_i^ϱ. By Lemma 14, w cannot be covered by σ as u_j^σ, so $B(u_i^\varrho, G \setminus \sigma) = B(u_i^\varrho, G)$ by Lemma 8(iii). Hence $f_i^\varrho = u_i^\varrho v_{i+1}^\varrho$ remains the $(b(u_i^\varrho)+1)$th best edge for u_i^ϱ in $I \setminus \sigma$ for each i, $i = 0, 1, \ldots, r-1$ and so ϱ is exposed in $I \setminus \sigma$.

2. $\varrho_F \not\subseteq E(G \setminus \sigma)$. Without loss of generality, suppose that edge $f_0^\varrho = u_0^\varrho v_1^\varrho$ was deleted during the elimination of σ. Thus $f_0^\varrho = v_j^\sigma w$ for some j, $0 \leq j \leq k-1$ and some $w \in V(G)$ and $u_{j-1}^\sigma v_j^\sigma = f_{j-1}^\sigma \prec_{v_j^\sigma} f_0^\varrho$ and $k_G(v_j^\sigma) \prec_{v_j^\sigma} f_0^\varrho$. By Lemma 14, $u_0^\varrho = v_j^\sigma$, and so $f_0^\varrho = s_G(v_j^\sigma)$. But $k_G(v_j^\sigma) \prec_{v_j^\sigma} f_0^\varrho$, so $k_G(v_j^\sigma) \in B(v_j^\sigma, G)$ and hence $|B(v_j^\sigma, G) \cap D(v_j^\sigma, G)| = b(v_j^\sigma) - 1$. So there is a unique edge $h \in B(v_j^\sigma, G) \setminus D(v_j^\sigma, G)$. As $f_{j-1}^\sigma \prec_{v_j^\sigma} f_0^\varrho$, we have $f_{j-1}^\sigma \in B(v_j^\sigma, G)$. But also $f_{j-1}^\sigma = u_{j-1}^\sigma v_j^\sigma = s_G(u_{j-1}^\sigma) \notin D(v_j^\sigma, G)$, so $f_{j-1}^\sigma = h$. But as $v_j^\sigma = u_0^\varrho$ and $e_0^\varrho = u_0^\varrho v_0^\varrho$ is the last for v_0^ϱ, $f_0^\varrho \notin D(u_0^\varrho = v_j^\sigma, G)$ and so $e_0^\varrho = h = f_{j-1}^\sigma$. So $\varrho_E \cap \sigma_F \neq \emptyset$ and $\varrho = \bar{\sigma}$ by Lemma 13(ii).

■

Lemma 16 *Let rotations* $\varrho = (\{e_0, e_1, \dots, e_{r-1}\}\{f_0, f_1, \dots, f_{r-1}\})$ *and* $\bar{\varrho} = (\{f_0, f_1, \dots, f_{r-1}\}\{e_1, e_2, \dots, e_{k-1}, e_0\})$ *be both exposed in an instance* $I = (G, \mathcal{O}, b)$*. Then*

1. $|E(w, G)| = b(w) + 1$ *and* $|E(w, G \setminus \varrho)| = |E(w, G \setminus \bar{\varrho})| = b(w)$ *for each vertex* w *covered by* ϱ *(or equivalently, by* $\bar{\varrho}$*)*

2. $E(w', G \setminus \varrho) = E(w', G) = E(w', G \setminus \bar{\varrho})$ *holds for each vertex* w' *not covered by* ϱ*.*

Proof. As both ϱ and $\bar{\varrho}$ are exposed in I, we have $f_j = s_G(u_j) = l_G(u_j)$ and $e_j = l_G(v_j) = s_G(v_j)$. Hence $|E(w, G)| = b(w) + 1$ holds for each covered vertex w. Consequently, during the elimination of ϱ or $\bar{\varrho}$, only edges from ϱ_E or ϱ_F, respectively, are deleted. So the assertion follows. ■

Lemma 17 *If* ϱ *and* σ *are two different rotations exposed in an instance* $I = (G, \mathcal{O}, b)$ *and* $\varrho \neq \bar{\sigma}$*, then* $I \setminus \varrho \setminus \sigma = I \setminus \sigma \setminus \varrho$*.*

Proof. As $\varrho \neq \bar{\sigma}$, Lemma 15 implies that ϱ is exposed in $I \setminus \sigma$ and σ is exposed in $I \setminus \varrho$. Therefore, instances $I' = I \setminus \varrho \setminus \sigma$ and $I'' = I \setminus \sigma \setminus \varrho$ are defined properly. We have three types of vertices:
If vertex w is not covered by any rotation ϱ, σ then by Lemma 8(iii)

$$B(w, G \setminus \varrho \setminus \sigma) = B(w, G \setminus \varrho) = B(w, G) = B(w, G \setminus \sigma) = B(w, G \setminus \sigma \setminus \varrho). \quad (6)$$

Suppose that vertex w is covered by exactly one rotation of ϱ and σ, without loss of generality say by ϱ. If w is not covered as any u_i^{ϱ}, Lemma 8(iii) implies that (6) holds. If $w = u_i^{\varrho}$ for some i, then

$$B(w, G \setminus \varrho \setminus \sigma) = B(w, G \setminus \varrho) = B(w, G) \setminus \{e_i^{\varrho}\} \cup \{f_i^{\varrho}\} \quad (7)$$

$$B(w, G \setminus \sigma \setminus \varrho) = B(w, G \setminus \sigma) \setminus \{e_i^{\varrho}\} \cup \{f_i^{\varrho}\} = B(w, G) \setminus \{e_i^{\varrho}\} \cup \{f_i^{\varrho}\}. \quad (8)$$

If vertex w is covered by both ϱ and σ then Lemma 14 implies, without loss of generality, $w = u_i^{\varrho} = v_j^{\sigma}$. So we have the same case as (7) and (8).

Therefore for each vertex w, $B(w, G \setminus \varrho \setminus \sigma) = B(w, G \setminus \sigma \setminus \varrho)$ and as both I' and I'' satisfy the FLP, Lemma 10(iii) implies that they are equal. ■

The previous Lemma implies that if an instance $I' = (G', \mathcal{O}', b)$ was obtained by elimination of a sequence $\varrho^1, \varrho^2, \dots, \varrho^k$ of rotations from an instance

$I = (G, \mathcal{O}, b)$, then *the order of eliminated rotations is immaterial.* (However, note that a rotation cannot be eliminated until it has become exposed.) Because of that, we shall use the notation $(G', \mathcal{O}', b) = (G \setminus R, \mathcal{O} \setminus R, b)$, or $I' = I \setminus R$ where $R = \{\varrho^1, \varrho^2, \ldots, \varrho^k\}$, as the instance I' is completely determined by I and $R = \{\varrho^1, \varrho^2, \ldots, \varrho^k\}$. Lemma 19 is a generalization of Lemma 4.3.3 of [4] and it shows, that also the set of rotations R is completely determined by I and I'. At first, we state one more proposition that is a generalization of Lemma 13(i).

Lemma 18 *Let I, I' be instances satisfying the FLP and $I' \subseteq I$. Let rotation ϱ be exposed in I and rotation σ be exposed in I'. Then*

$$\varrho_E \cap \sigma_E \neq \emptyset \Longleftrightarrow \varrho = \sigma$$

Proof. We shall prove only the "$\Rightarrow$" implication for the case $I' \subset I$, as the case $I' = I$ follows from Lemma 13(i). Let $\varrho_E \cap \sigma_E \neq \emptyset$. By Lemma 12, I' can be obtained from I by elimination of a sequence $\tau^1, \tau^2, \ldots, \tau^k$ of rotations. Hence τ^1 and ϱ are both exposed in I. If $\varrho_E \cap \tau^1_E \neq \emptyset$, then $\varrho = \tau^1$ (by Lemma 13) and no edge of ϱ_E is present in I', contradicting $\varrho_E \cap \sigma_E \neq \emptyset$.

Hence $\varrho_E \cap \tau^1_E = \emptyset$. Suppose that $\tau^1 = \bar{\varrho}$. Then $|E(w, G \setminus \tau^1)| = b(w)$ for all vertices w covered by τ^1 by Lemma 16. Therefore no vertex w covered by ϱ can be covered by any other rotation after elimination of τ^1, so necessarily $\sigma_E \cap \varrho_E = \emptyset$, again a contradiction.

It follows that ϱ is exposed in $I \setminus \tau^1$. We can now repeat the arguments for $\tau^2, \tau^3, \ldots, \tau^k$ until we get a contradiction with $\sigma_E \cap \varrho_E \neq \emptyset$ or until ϱ is exposed in I'. Then by Lemma 13(i), $\varrho = \sigma$. ∎

Lemma 19 *If $I' \subset I$ and $I' = I \setminus R = I \setminus R'$ then $R = R'$.*

Proof. Suppose that $R = \{\varrho^1, \varrho^2, \ldots, \varrho^k\}$ and $R' = \{\sigma^1, \sigma^2, \ldots, \sigma^r\}$ and that these rotations were eliminated in this order to get I' from I.

Suppose that $\varrho^1 \neq \sigma^1$. If ϱ^1 is dual to σ^1, then by Lemma 16(i), no vertex covered by ϱ^1 and σ^1 can be covered by any other rotation in any following subinstance. So $E(v, G \setminus \varrho^1) = E(v, G \setminus R)$ and $E(v, G \setminus \sigma^1) = E(v, G \setminus R')$ for each vertex v covered by ϱ^1 and σ^1. Hence, together with Lemma 8 this implies that $\varrho^1_E \subseteq E(G \setminus R')$ and $\varrho^1_F \subseteq E(G \setminus R)$. As we supposed that $\varrho_E \cap \varrho_F = \emptyset$, Lemma 10(iii) implies $I \setminus R \neq I \setminus R'$ – a contradiction. So ϱ^1 is exposed in $I \setminus \sigma^1$. We can now repeat this discussion for rotations

$\sigma^2, \sigma^3, \ldots, \sigma^k$. As $I \setminus R' = I \setminus R$ and $\varrho_E^1 \cap I \setminus R = \emptyset$, it follows that ϱ^1 has to be eliminated to get I', so $\varrho^1 \in R'$ and $\varrho^1 = \sigma^j$ for some j, $0 \leq j \leq r$. As the order of rotations elimination is immaterial, we can suppose that $\varrho^1 = \sigma^1$ and by repeating the arguments above for instance $I \setminus \varrho^1$ and sets $R \setminus \{\varrho^1\}$ and $R' \setminus \{\varrho^1\}$ we get $R' = R$. ■

Lemma 20 *Let $I = (G, \mathcal{O}, b)$ and $I' = (G', \mathcal{O}', b)$, $I' \subseteq I$ be FL instances. If rotation ϱ is exposed in I and rotation σ is exposed in I', then either*

(i) $\sigma = \varrho$,

(ii) $\sigma = \bar{\varrho}$, *or*

(iii) there is an FL subinstance of $I \setminus \varrho$, in which σ is exposed.

Proof. Let I' be a maximal subinstance of I in which σ is exposed, that is, such that there is no subinstance I'' of I in which σ is exposed and $I' \subset I''$. Suppose that $I' = I \setminus R$ for a set R of rotations.

If $\varrho \in R$, then clearly I' is a subinstance of $I \setminus \varrho$ and (iii) holds.

If $\bar{\varrho} \in R$, then clearly $\varrho \notin R$. Denote by I'' the subinstance of I obtained by the elimination of a rotation subset of R such that $\bar{\varrho}$ becomes exposed. By Lemma 15, ϱ remains exposed. So in I'', both ϱ and $\bar{\varrho}$ are exposed and also $I' \subseteq I''$. By Lemma 16, elimination of $\bar{\varrho}$ does not affect the preference list for any vertex u not covered by $\bar{\varrho}$ (i. e. neither by ϱ). It follows, that σ is exposed in $I \setminus (R \setminus \{\bar{\varrho}\})$, which is a contradiction with our assumption that I' is maximal.

If neither ϱ nor $\bar{\varrho}$ belong to R, Lemma 15 implies that ϱ must be exposed in I'. So if $\sigma \neq \varrho, \bar{\varrho}$, then σ must be exposed in $I' \setminus \varrho$, which is a subinstance of $I \setminus \varrho$. ■

The first main result of this section is a generalization of Theorem 4.3.1 of [4] and it says that each stable b-matching of the given SMA instance $(G, \mathcal{O}, b)$ is associated with a unique set of rotations.

Theorem 3 *For a given solvable SMA instance let $I^* = (G^*, \mathcal{O}^*, b)$ be its Phase-1 subinstance and let $M = I^* \setminus R$ be any stable b-matching of I. Then R contains every singular rotation and exactly one rotation of each dual pair.*

Proof. Suppose that $R = \{\varrho^0, \varrho^1, \ldots, \varrho^{t-1}\}$, that ϱ^0 is exposed in I^*, and that ϱ^j is exposed in $I_j = I^* \setminus \{\varrho^0, \ldots, \varrho^{j-1}\}$ $(1 \leq j \leq t-1)$, so that $I_t = I^* \setminus R = M$. Let σ be a rotation, and I' an FL instance in which σ is exposed.

Suppose that σ is a singular rotation and $\sigma \notin R$. As I^* is the Phase-1 subinstance, I' is its subinstance. We supposed, that $\sigma \neq \varrho^0$ and as σ is singular, $\bar{\sigma} \neq \varrho^0$, so by Lemma 20 there exists a subinstance of $I_1 = I^* \setminus \varrho^0$ in which σ is exposed. Likewise, there is a subinstance of I_2, a subinstance of I_3, $\ldots$, and a subinstance of I_t in which σ is exposed. But this is a contradiction since $I_t = M$ is a stable b-matching and hence there is no exposed rotation.

Suppose that σ is nonsingular. The definition of the dual rotation and Lemma 16 imply that R cannot contain both σ and $\bar{\sigma}$ since elimination of one prevents the possibility of the elimination of the other. So suppose that neither σ nor $\bar{\sigma}$ belong to R. Again, since $I' \subseteq I^*$ and $\varrho^0 \neq \sigma, \bar{\sigma}$, by Lemma 20 there exists a subinstance of $I_1 = I^* \setminus \varrho^0$ in which σ is exposed. Likewise, for $I_2, \ldots, I_t$, giving a contradiction as above. ■

Theorem 3 describes a mapping from $\mathcal{M}(I)$ to the family of sets of rotations that contain every singular rotation and exactly one rotation of each dual pair. By Lemma 19, this mapping is one-one. However, since a rotation cannot be eliminated until it has become exposed, not each such set of rotations necessarily represents a stable b-matching. Therefore, this mapping might be not onto. However there exists a one-one correspondence between the stable b-matchings and certain sets of rotations. At first, we state one more lemma that is a generalization of Lemma 4.3.6 of [4].

Lemma 21 *A rotation ϱ exposed in $I = (G, \mathcal{O}, b)$ is singular if and only if there is a subinstance $I' = (G', \mathcal{O}', b) \subseteq I$ in which ϱ is the only exposed rotation.*

Proof. Suppose first that ϱ is singular. If in I, some other rotation is exposed, say σ, then as it cannot be dual to ϱ, Lemma 15 implies that ϱ is exposed also in $I \setminus \sigma$. This argument can be repeated and a sequence of subinstances with ϱ exposed is produced. As it has to be finite, the sequence ends with one, in which ϱ is the only exposed rotation, i. e. with I'.

Now suppose that ϱ is nonsingular, i. e. $\bar{\varrho}$ exists. Suppose that ϱ is the only exposed rotation in some subinstance I'. If $I^* = (G^*, \mathcal{O}^*, b)$ is the Phase-1 subinstance, then $I' = I^* \setminus R$ for a set of rotations R, for which $\bar{\varrho} \notin R$. Repeated applications of Lemma 20 imply that there exists a subinstance

$I'' \subseteq I'$ in which $\bar{\varrho}$ is exposed. Moreover, by Lemma 12 this subinstance can be obtained from I' by elimination of some set of rotations that cannot include ϱ. So ϱ cannot be the only exposed rotation in I'. ■

We adopt some notions from [4]. Recall, that $I^* = (G^*, \mathcal{O}^*, b)$ is the Phase-1 subinstance of a given SMA instance.

Definition 5 *Rotation σ is said to be a predecessor of rotation ϱ, $\sigma \prec \varrho$, if for each subinstance $I = I^* \setminus R \subseteq I^*$ in which ϱ is exposed, σ belongs to R.*

If σ is a predecessor of ϱ, rotation σ has to be eliminated for ϱ to become exposed. Similarly as for SR, we can easily verify, that the reflexive closure $\preceq$ of the predecessor relation is a partial order on the set of rotations of a given SMA instance. The set of rotations under relation $\preceq$ will be referred to as *the SMA rotation poset* and denoted by $\Pi(I)$. A set $R \in \Pi(I)$ will be called *closed*, if for each $\varrho \in R$ we have $\tau \in R$ whenever $\tau \prec \varrho$. A subset of the rotation poset containing all singular rotations and exactly one of each dual pair will be called *complete*.

The following lemmas generalize Lemmas 4.3.7 and 4.3.8 of [4].

Lemma 22 *If ϱ, σ are nonsingular rotations and π a singular one, then*

(i) $\varrho \not\prec \bar{\varrho}$;

(ii) $\varrho \prec \sigma \Longleftrightarrow \bar{\sigma} \prec \bar{\varrho}$;

(iii) $\tau \prec \pi \Longrightarrow \tau$ *is singular; i. e. a predecessor of a singular rotation is also singular.*

Proof. (i) Let $\varrho = (\varrho_E, \varrho_F)$. During the elimination of ϱ, set of edges ϱ_E is deleted. But as $\bar{\varrho} = (\varrho_F, \varrho_E)$, clearly $\varrho \not\prec \bar{\varrho}$.

(ii) Suppose that $\varrho \prec \sigma$ and $\bar{\sigma} \not\prec \bar{\varrho}$, so there exists a subinstance $I = I^* \setminus R$, where $\bar{\varrho} \in R$ and $\bar{\sigma} \notin R$. It follows that $\varrho \notin R$, and so $\sigma \notin R$ either. σ is a rotation, so there exists a subinstance, in which it is exposed. As neither $\sigma \in R$ nor $\bar{\sigma} \in R$, repeated applications of Lemma 20 imply that there exists a subinstance $I' \subseteq I$ in which σ is exposed. By Lemma 12, I' can be obtained from I by elimination of a set of rotations. So σ is exposed in $I' = I^* \setminus R'$, where $\varrho \notin R'$, contradicting $\varrho \prec \sigma$.

(iii) Suppose that $\tau \prec \pi$ and that τ is nonsingular. So $\bar{\tau}$ is also a rotation and it is exposed in some subinstance $I = I^* \setminus R$, where $\tau \notin R$. From Lemmas

7 and 12, and the fact that our given SMA instance is solvable it follows, that there is at least one stable b-matching embedded in a subinstance $I \setminus \bar{\tau}$. This stable b-matching can be clearly obtained also from the Phase-1 subinstance I^* by eliminating a set of rotations that does not include τ. But as $\tau \prec \pi$, this set can include neither the singular rotation π, contradicting Theorem 4. ■

Lemma 23 *If R_0 denotes the set of all singular rotations, then R_0 is closed and every stable b-matching is embedded in the subinstance $I^* \setminus R_0$.*

Proof. First we will show that R_0 is closed. Suppose that $\varrho \in R_0$ and $\sigma \prec \varrho$ for some rotation σ. As ϱ is singular, σ is also singular (Lemma 22(iii)), i. e. $\sigma \in R_0$ and R_0 is closed.

By Theorem 3, for finding any stable b-matching of a given SMA instance, each singular rotation needs to be eliminated. Notice, that the order of rotation elimination is immaterial, subject to the precedence relation, so each stable b-matching is embedded in subinstance $I^* \setminus R_0$. ■

The following Theorem gives a one-one correspondence between $\mathcal{M}(I)$ and particular closed subsets of $\Pi(I)$. It is a generalization of Theorem 4.3.2 of [4].

Theorem 4 *For a solvable SMA instance $I = (G, \mathcal{O}, b)$, there is a one-one correspondence between $\mathcal{M}(I)$ and the complete closed subsets of $\Pi(I)$.*

Proof. According to Theorem 3 and Lemma 23, every stable b-matching can be obtained from I^* by elimination of the rotations in a particular complete subset R of $\Pi(I)$. The definition of $\Pi(I)$ implies that each such subset has to be closed.

Suppose that R is a complete closed subset of $\Pi(I)$. So R can be eliminated from I^*. If there is a Phase-2 vertex v with more than $b(v)$ incident edges, then Lemma 4 implies that there is a rotation ϱ exposed in $I^* \setminus R$, such that neither $\varrho \in R$ nor $\bar{\varrho} \in R$, so R could not be complete. So by Lemma 10(ii), the set of edges $E(G^* \setminus R)$ determines a stable b-matching, where $I^* \setminus R = (G^* \setminus R, \mathcal{O}^* \setminus R, b)$. ■

Example 6 *For our example instance, the diagram in Figure 5 illustrates which set of rotations is associated with which stable b-matching. Let us*

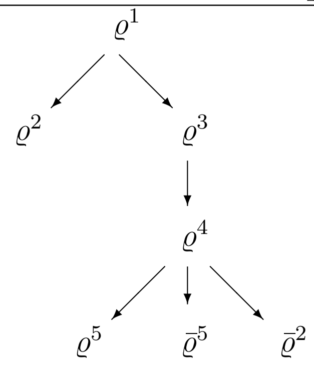

Figure 8: The rotation poset of the example SMA instance

notice that here we have two pairs of dual rotations, namely $\varrho^7 = \bar{\varrho}^2$ *and* $\varrho^6 = \bar{\varrho}^5$*. Figure 8 shows the rotation poset for our example instance. The closed complete subsets of this rotation poset are:*

$$R_1 = \{\varrho^1, \varrho^2, \varrho^3, \varrho^4, \varrho^5\} \quad R_2 = \{\varrho^1, \varrho^2, \varrho^3, \varrho^4, \bar{\varrho}^5\}$$
$$R_3 = \{\varrho^1, \varrho^3, \varrho^4, \bar{\varrho}^2, \varrho^5\} \quad R_4 = \{\varrho^1, \varrho^3, \varrho^4, \bar{\varrho}^2, \bar{\varrho}^5\}$$

and from Figure 5 it follows, that they corresponds to the four stable b-matchings of the given example instance.

6 Conclusions

In this paper, we studied the SMA algorithm proposed in [2]. We showed that the result of its Phase 1 is independent from the order of deletions and that each vertex is assigned the same number of edges in all stable b-matchings. We also proved that each stable b-matching can be found by this algorithm and studied the relation between the set of all stable b-matchings and the structure of rotations in the given SMA instance. We also showed how to modify the SMA algorithm as to run in $O(m)$ time.

For a further study we suggest the following topics:

1. As far as we know, stable b-matchings with indifferences have not been studied yet. It would be therefore interesting to explore various generalizations of the notions of stability as defined e.g. in [7].

2. Is there any analogue of the "medians" results for the stable roommates, namely that the so-called median of any three stable b-matchings is

itself a stable b-matching? This question was posed in [8] for the stable fixtures problem.

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