General neighbour-distinguishing index of a graph

Mirko HORŇÁK*

P. J. Šafárik University
Institute of Mathematics
Jesenná 5, 040 01 Košice, Slovakia
e-mail: hornak@science.upjs.sk

Mariusz WOŹNIAK

AGH University of Science and Technology
Department of Applied Mathematics
Al. Mickiewicza 30, 30-059 Kraków, Poland
e-mail: mwozniak@uci.agh.edu.pl

Abstract

It is proved that edges of a graph G can be coloured using $\chi(G) + 2$ colours so that any two adjacent vertices have distinct sets of colours of their incident edges. In the case of a bipartite graph three colours are sufficient.

Keywords: colour set, neighbour-distinguishing edge colouring, general neighbour-distinguishing index

1 Introduction

All graphs we deal with in this paper are simple and finite. Let G be a graph and k a non-negative integer. A (*general*) *k-edge-colouring* of G is a mapping $\varphi : E(G) \to \bigcup_{i=1}^{k}\{i\}$. The *colour set* (with respect to φ) of a vertex $x \in V(G)$ is the set $S_\varphi(x)$ of colours of edges incident to x. The colouring φ is *neighbour-distinguishing* if $S_\varphi(x) \neq S_\varphi(y)$ whenever vertices x, y are adjacent. A neighbour-distinguishing colouring will be frequently shortened to an *nd-colouring*. The *general neighbour-distinguishing index* of G is the minimum k in a general k-edge-colouring of G that is neighbour-distinguishing, and will be denoted as $\mathrm{gndi}(G)$. If G has an isolated edge, then G does not have any nd-colouring, hence for

*This work was supported by Science and Technology Assistance Agency under the contract No. APVT-20-004104 and by the Slovak grant VEGA 1/0424/03.

the sake of the completeness of the definition in such a case we set $\mathrm{gndi}(G) := \infty$. For a disconnected graph G with connected components we have evidently $\mathrm{gndi}(G) = \max\ (\mathrm{gndi}(G_i) : i = 1, \dots, n)$, hence our analysis of the general neighbour-distinguishing index can be restricted to connected graphs.

The general neighbour-distinguishing index is a relaxation of two known graph invariants. If $S_\varphi(x) \neq S_\varphi(y)$ is required for any two distinct vertices x, y, the corresponding parameter $\chi_0(G)$, called the *point-distinguishing chromatic index* of G, has been introduced by Harary and Plantholt in [3]. The authors proved, among other things, that $\chi_0(K_n) = \lceil \log_2 n \rceil + 1$ for any $n \geq 3$. In spite of the fact that the structure of complete bipartite graphs is simple, it seems that the problem of determining $\chi_0(K_{m,n})$ is not easy, especially in the case $m = n$, as documented by papers of Zagaglia Salvi [8], [9], Horňák and Soták [5], [6] and Horňák and Zagaglia Salvi [7].

On the other hand, if only *proper* nd-colourings are considered, the *neighbour-distinguishing index* of G, symbolically $\mathrm{ndi}(G)$, is obtained. This invariant has been introduced only recently by Zhang et al. in [10]. It is easy to see that $\mathrm{ndi}(C_5) = 5$ and in [10] it is conjectured that $\mathrm{ndi}(G) \leq \Delta(G)+2$ for any connected graph $G \notin \{K_2, C_5\}$. The conjecture has been confirmed by Balister et al. in [1] for bipartite graphs and for graphs G with $\Delta(G) = 3$. Edwards et al. in [2] have shown even that $\mathrm{ndi}(G) \leq \Delta(G) + 1$ if G is bipartite, planar, and of maximum degree $\Delta(G) \geq 12$. In the general case a weaker statement $\mathrm{ndi}(G) \leq \Delta(G) + 300$ has been proved by Hatami in [4] for all graphs G with $\Delta(G) > 10^{20}$.

For $p, q \in \mathbb{Z}$ we denote by $[p, q]$ the *integer interval* lower bounded by p and upper bounded by q, i.e., $[p, q] := \bigcup_{i=p}^{q} \{i\}$. Let n and $l_1, \dots, l_n$ be non-negative integers. The *concatenation* of finite sequences $A_i = (a_i^1, \dots, a_i^{l_i})$, $i = 1, \dots, n$, is defined as the sequence $\prod_{i=1}^{n} A_i := (a_1^1, \dots, a_1^{l_1}, \dots, a_n^1, \dots, a_n^{l_n})$. If $A_i = A$ for each $i \in [1, n]$, we write A^n instead of $\prod_{i=1}^{n} A$. If $n = 0$, A^n is the empty sequence $(\ )$.

Let G be a graph let $x, y \in V(G)$. By $\deg_G(x)$ we denote the degree of x in G and by $d_G(x, y)$ the distance between x and y in G. An *arm* of a tree T is a maximal (non-extendable) subpath A of T such that $\deg_A(x) = \deg_T(x) = 2$ for any internal vertex $x \in V(A)$. Let $a(T)$ denote the number of arms of T. If T is (isomorphic to) an n-vertex path P_n, then $a(T) = 1$ and T itself is the only arm of T. On the other hand, if $\Delta(T) \geq 3$, any arm A of T has one endvertex of degree one, the other of degree at least three and $a(T)$ is equal to the number of pendant vertices of T.

The main aim of the present paper is to show that if $\chi(G) \geq 3$, then $\mathrm{gndi}(G) \leq \chi(G) + 2$. As an easy consequence of this bound we obtain the inequality $\mathrm{gndi}(G) \leq \Delta(G) + 2$.

2 Paths, cycles, trees and bipartite graphs

Proposition 1. *For any graph G the following statements are equivalent:*

(1) $\mathrm{gndi}(G) = 2$.

(2) *G is bipartite and there is a bipartition $\{X_1 \cup X_2, Y\}$ of $V(G)$ such that $X_1 \cap X_2 = \emptyset$ and any vertex of Y has at least one neighbour in both X_1 and X_2.*

Proof. (1) $\Rightarrow$ (2): Consider an nd-colouring $\varphi : E(G) \to [1,2]$. The only three available non-empty colour sets are $\{1\}$, $\{2\}$ and $\{1,2\}$. Since $\{1\} \cap \{2\} = \emptyset$, for any $xy \in E(G)$ exactly one of $S_\varphi(x)$ and $S_\varphi(y)$ is equal to $\{1,2\}$. Let $Y := \{y \in V(G) : S_\varphi(y) = \{1,2\}\}$ and let $X_i := \{x \in V(G) : S_\varphi(x) = \{i\}\}$, $i = 1,2$. Clearly, $X_1 \cap X_2 = \emptyset$, $(X_1 \cup X_2) \cap Y = \emptyset$, any edge of G joins a vertex of $X_1 \cup X_2$ to a vertex of Y, and any vertex of Y has at least one neighbour in both X_1 and X_2.

(2) $\Rightarrow$ (1): Let the colouring $\varphi : E(G) \to [1,2]$ be defined so that $\varphi(xy) = i$ if and only if $x \in X_i$ and $y \in Y$, $i = 1,2$. Then $S_\varphi(x) = \{i\}$ for any $x \in X_i$, $i = 1,2$, $S_\varphi(y) = \{1,2\}$ for any $y \in Y$, and so φ is neighbour-distinguishing. $\square$

An nd-colouring $\varphi : E(G) \to [1,3]$ of a bipartite graph G is said to be *canonical* if there is a *canonical ordered bipartition* (X,Y) of $V(G)$, one that satisfies $S_\varphi(x) \in \mathcal{S}_1 := \{\{1\},\{2\},\{1,3\},\{2,3\}\}$ for every $x \in X$ and $S_\varphi(y) \in \mathcal{S}_2 := \{\{3\},\{1,2\}\}$ for every $y \in Y$. The set $\mathcal{S}_1$ has the following important property: whenever $S \in \mathcal{S}_1$, then also $S \cup \{3\} \in \mathcal{S}_1$. A canonical nd-colouring φ of a tree T is 3-*canonical* if $S_\varphi(v) \neq \{3\}$ for any vertex $v \in V(T)$ with $\deg_T(v) \geq 2$. A 3-canonical nd-colouring φ of a path P_n is $(3,i)$-*canonical*, $i \in [1,2]$, if there is a pendant edge $e \in E(P_n)$ such that $\varphi(e) = i$.

Proposition 2. *Let n be an integer, $n \geq 3$, and let $i \in [1,2]$.*

1. *If n is odd, then* $\mathrm{gndi}(P_n) = 2$ *and there is a $(3,i)$-canonical* nd-*colouring* $\varphi : E(P_n) \to [1,2]$.
2. *If n is even, then* $\mathrm{gndi}(P_n) = 3$ *and there is a $(3,i)$-canonical* nd-*colouring* $\varphi : E(P_n) \to [1,3]$.

Proof. Let us first show that $\mathrm{gndi}(P_n) = 2$ implies $n \equiv 1 \pmod 2$. Suppose that $\mathrm{gndi}(P_n) = 2$ and let $\{X_1 \cup X_2, Y\}$ be the bipartition of $V(P_n)$ yielded by Proposition 1. The natural sequence of vertices of P_n (from one endvertex to the other) is an alternating sequence of vertices from $X_1 \cup X_2$ and Y that starts and ends with a vertex of $X_1 \cup X_2$. Therefore $|X_1 \cup X_2| = |Y| + 1$ and n is odd.

Further, if φ is a $(3,1)$-canonical nd-colouring of P_n, then the colouring $\tilde{\varphi}$, defined by $\varphi(e) = 3 \Rightarrow \tilde{\varphi}(e) = 3$ and $\varphi(e) = k \in [1,2] \Rightarrow \tilde{\varphi}(e) = 3 - k$, is a $(3,2)$-canonical nd-colouring of P_n, and uses the same number of colours as φ does.

Now, it is sufficient to present a $(3,1)$-canonical nd-colouring of P_n using the appropriate number of colours. Such a colouring is in a natural way determined by the sequence of colours of consecutive edges of P_n. Since $n \geq 3$, there is a positive integer j and $k \in [-1, 2]$ such that $n = 4j + k$. For $k = -1, 0, 1, 2$ we can use successively the sequences $(1, 2)(2, 1, 1, 2)^{j-1}$, $(3, 2, 1)(1, 2, 2, 1)^{j-1}$, $(1, 2, 2, 1)^j$ and $(3)(1, 2, 2, 1)^j$. □

Proposition 3. *Let n be an integer, $n \geq 3$.*
1. *If $n \equiv 0 \pmod 4$, then* $\text{gndi}(C_n) = 2$.
2. *If $n \not\equiv 0 \pmod 4$, then* $\text{gndi}(C_n) = 3$.

Proof. Similarly as in the proof of Proposition 2 we start by showing that $\text{gndi}(C_n) = 2$ implies $n \equiv 0 \pmod 4$. Suppose that $\text{gndi}(C_n) = 2$ and let $\{X_1 \cup X_2, Y\}$ be the bipartition of $V(C_n)$ from Proposition 1. Pick a vertex $y \in Y$, take his unique neighbour $x_1 \in X_1$ and consider the natural sequence of vertices of C_n given by the ordered pair (y, x_1) that ends with the other neigbour $x_2 \in X_2$ of y. This sequence is built up by concatenating ordered 4-tuples of vertices belonging successively to Y, X_1, Y and X_2, hence $n \equiv 0 \pmod 4$.

Now, the following (cyclic) sequences represent an nd-colouring of C_n with the minimum possible number of colours successively for $n = 4j-1$, $4j$, $4j+1$, $4j+2$: $(1, 2, 3)(1, 2, 2, 1)^{j-1}$, $(1, 2, 2, 1)^j$, $(1, 2, 2, 3, 1)(1, 2, 2, 1)^{j-1}$, $(1, 2, 3)^2(1, 2, 2, 1)^{j-1}$. □

Theorem 4. *If T is a tree with $|E(T)| \geq 2$, then* $\text{gndi}(T) \leq 3$.

Proof. We prove by induction on $a(T)$ a stronger statement, namely that there is a 3-canonical nd-colouring of T. If $a(T) = 1$, there is $n \geq 3$ such that $T \simeq P_n$ and we are done by Proposition 2.

Suppose that $a(T) > 1$ and there is a 3-canonical nd-colouring of an arbitrary tree T' with $a(T') < a(T)$. Consider a pendant vertex $x \in V(T)$ and such a vertex $y \in V(T)$ with $\deg_T(y) \geq 3$ that $d_T(x, y)$ is minimal. The subpath A of T with endvertices x and y is an arm of T and $T' := T - (V(A) - \{y\})$ is a subtree of T with $a(T') \leq a(T) - 1$ and $|E(T')| \geq 2$. By the induction hypothesis there is a 3-canonical nd-colouring $\varphi' : E(T') \to [1, 3]$. Let (X', Y') be a canonical ordered bipartition of $V(T')$ (there is one corresponding to φ'). A 3-canonical nd-colouring $\psi : E(T) \to [1, 3]$ will be found as a continuation of φ'.

(1) $V(A) = \{x, y\}$

(11) If $S_{\varphi'}(y) \neq \{1, 2\}$, then $S_{\varphi'}(y) \in \mathcal{S}_1$. Defining $\psi(xy) := 3$ yields $S_\psi(y) = S_{\varphi'}(y) \cup \{3\} \in \mathcal{S}_1$, $S_\psi(x) = \{3\} \in \mathcal{S}_2$ and $(X', Y' \cup \{x\})$ is the canonical ordered bipartition of $V(T)$.

(12) If $S_{\varphi'}(y) = \{1, 2\}$, set $\psi(xy) := 1$. Then $S_\psi(x) = \{1\} \in \mathcal{S}_1$, $S_\psi(y) = \{1, 2\} \in \mathcal{S}_2$ and $(X' \cup \{x\}, Y')$ is the canonical ordered bipartition of $V(T)$.

(2) Provided that $|V(A)| \geq 3$, let z be the unique neighbour of y in A. Since $\deg_{T'}(y) = \deg_T(y) - 1 \geq 2$ and the colouring φ' is 3-canonical, there is $i \in S_{\varphi'}(y) \cap [1, 2]$. By Proposition 2 there exists a $(3, i)$-canonical nd-colouring $\varphi : E(A) \to [1, 3]$ with $\varphi(yz) = i$. Clearly, if (X, Y) is the canonical ordered bipartition of $V(A)$, then $y \in X$, $z \in Y$ and $S_\varphi(z) = \{1, 2\}$.

(21) If $S_{\varphi'}(y) \neq \{1,2\}$, let ψ be the common continuation of both φ' and φ. In such a case $S_\psi(v) = S_{\varphi'}(v)$ for any $v \in V(T')$, $S_\psi(v) = S_\varphi(v)$ for any $v \in V(A) - \{y\}$ and the canonical ordered bipartition of $V(T)$ is $(X' \cup X, Y' \cup Y)$.

(22) If $S_{\varphi'}(y) = \{1,2\}$, then $y \in Y'$.

(221) If $V(A) = \{x,y,z\}$, set $\psi(yz) := 2$ and $\psi(zx) := 3$ to obtain $S_\varphi(y) = \{1,2\} \in \mathcal{S}_2$, $S_\psi(z) = \{2,3\} \in \mathcal{S}_1$ and $S_\psi(x) = \{3\} \in \mathcal{S}_2$; the canonical ordered bipartition of $V(T)$ is $(X' \cup \{z\}, Y' \cup \{x\})$.

(222) If $|V(A)| \geq 4$, then $A^- := A - y$ is a path on $|V(A)| - 1 \geq 3$ vertices. By Proposition 2 there is a $(3,1)$-canonical nd-colouring $\varphi^- : E(A^-) \to [1,3]$ such that $S_{\varphi^-}(z) = \{1\}$; if (X^-, Y^-) is the canonical ordered bipartition of $V(A^-)$, then $z \in X^-$. The continuation ψ of both φ' and φ^- with $\psi(yz) := 1$ satisfies $S_\psi(v) = S_{\varphi'}(v)$ for any $v \in V(T')$, $S_\psi(v) = S_{\varphi^-}(v)$ for any $v \in V(A^-)$ and $(X' \cup X^-, Y' \cup Y^-)$ is the canonical ordered bipartition of $V(T)$. □

Theorem 5. *If G is a connected bipartite graph with $|E(G)| \geq 2$, then* $\mathrm{gndi}(G) \leq 3$.

Proof. We prove by induction on $\mathrm{diff}(G) := |E(G)| - |V(G)|$ that there is a canonical nd-colouring of G. If $\mathrm{diff}(G) = -1$, then G is a tree and we can use Theorem 4.

Assume that $\mathrm{diff}(G) \geq 0$ and there is a canonical nd-colouring of any connected bipartite graph H satisfying $|E(H)| \geq 2$ and $\mathrm{diff}(H) < \mathrm{diff}(G)$. From $\mathrm{diff}(G) \geq 0$ if follows that there is a cycle C in G (of an even length). If $xy \in E(C)$, then by the induction hypothesis for the connected graph $H := G - xy$ with $|E(H)| = |E(G)| - 1 \geq 3$ and $\mathrm{diff}(H) = \mathrm{diff}(G) - 1$ there exists a canonical nd-colouring $\varphi : E(H) \to [1,3]$ with a canonical ordered bipartition (X,Y) of $V(H)$. Without loss of generality we may suppose that $x \in X$ and $y \in Y$. Then there is a canonical nd-colouring $\psi : E(G) \to [1,3]$ that is a continuation of φ and has the canonical ordered bipartion (X,Y) of $V(G) = V(H)$.

Namely, if $S_\varphi(x) \cap S_\varphi(y) \neq \emptyset$, using $\psi(xy) \in S_\varphi(x) \cap S_\varphi(y)$ leads to $S_\psi(x) = S_\varphi(x)$ and $S_\psi(y) = S_\varphi(y)$.

If $S_\varphi(x) \cap S_\varphi(y) = \emptyset$, there is $i \in [1,2]$ such that $S_\varphi(x) = \{i\}$ and $S_\varphi(y) = \{3\}$; in such a case setting $\psi(xy) := 3$ yields $S_\psi(x) = \{i,3\} \in \mathcal{S}_1$ and $S_\psi(y) = \{3\} \in \mathcal{S}_2$. □

3 Main result

Let G be a connected k-chromatic graph, $k \geq 3$. Any proper vertex k-colouring of G can be seen as a sequence $(V_1, \dots, V_k)$ such that $\{V_i : i \in [1,k]\}$ is a decomposition of $V(G)$ with the following property: whenever $xy \in E(G)$, $x \in V_i$ and $y \in V_j$, then $i \neq j$. We denote by $\mathrm{Col}_k(G)$ the set of all sequences $(V_1, \dots, V_k)$ described above. For $\mathcal{V} = (V_1, \dots, V_k) \in \mathrm{Col}_k(G)$ and $i, j \in [1,k]$, $i \neq j$, let $E_{i,j}(\mathcal{V})$ be the set of all edges of G joining a vertex of V_i to a vertex of V_j, define $e_{i,j}(\mathcal{V}) := |E_{i,j}(\mathcal{V})|$, $e_i(\mathcal{V}) := \sum_{j=1}^{i-1} e_{i,j}(\mathcal{V}) + \sum_{j=i+1}^{k} e_{i,j}(\mathcal{V})$ and $e(\mathcal{V}) := (e_1(\mathcal{V}), \dots, e_k(\mathcal{V}))$.

Lemma 6. *Let G be a connected graph with $k = \chi(G) \geq 3$ and let $\hat{\mathcal{V}} = (\hat{V}_1, \dots, \hat{V}_k) \in \mathrm{Col}_k(G)$ be a sequence lexicographically maximal in the set $\mathrm{Col}_k(G)$. Then the following hold:*

1. *For any $i \in [2, k]$, $x \in \hat{V}_i$ and $j \in [1, i-1]$ there is $y \in \hat{V}_j$ such that $xy \in E(G)$.*
2. *Pendant vertices of G belong to $\hat{V}_1 \cup \hat{V}_2$.*
3. *If a pendant edge $xy \in E_{1,2}(\hat{\mathcal{V}})$ is not adjacent to any edge of $E_{1,2}(\hat{\mathcal{V}})$, then its pendant vertex x is in $\hat{V}_2$.*

Proof. 1. If there is $i \in [2, k]$, $x \in \hat{V}_i$ and $j \in [1, i-1]$ such that $xy \notin E(G)$ for each $y \in \hat{V}_j$, then

$$\mathcal{V} := \prod_{l=1}^{j-1}(\hat{V}_l)(\hat{V}_j \cup \{x\}) \prod_{l=j+1}^{i-1} (\hat{V}_l)(\hat{V}_i - \{x\}) \prod_{l=i+1}^{k} (\hat{V}_l) \in \mathrm{Col}_k(G)$$

and the sequence

$$e(\mathcal{V}) = \prod_{l=1}^{j-1}(e_l(\hat{\mathcal{V}}))(e_j(\hat{\mathcal{V}}) + \deg_G(x)) \prod_{l=j+1}^{i-1} (e_l(\hat{\mathcal{V}}))(e_i(\hat{\mathcal{V}}) - \deg_G(x)) \prod_{l=i+1}^{k} (e_l(\hat{\mathcal{V}}))$$

is lexicographically greater than $e(\hat{\mathcal{V}})$, a contradiction.

2. A consequence of Lemma 6.1.

3. If $x \in \hat{V}_1$, then $\deg_G(y) \geq 2$ (note that $|E(G)| \geq 3$),

$$\mathcal{V} := (\hat{V}_1 - \{x\} \cup \{y\}, \hat{V}_2 - \{y\} \cup \{x\}) \prod_{l=3}^{k}(\hat{V}_l) \in \mathrm{Col}_k(G)$$

and the sequence

$$e(\mathcal{V}) = (e_1(\hat{\mathcal{V}}) + \deg_G(y) - 1, e_2(\hat{\mathcal{V}}) + 1 - \deg_G(y)) \prod_{l=3}^{k}(e_l(\hat{\mathcal{V}}))$$

is lexicographically greater than $e(\hat{\mathcal{V}})$, which is not possible. □

Theorem 7. *If G is a connected graph with $\chi(G) \geq 3$, then $\mathrm{gndi}(G) \leq \chi(G) + 2$.*

Proof. Set $k := \chi(G)$ and let $(\hat{V}_1, \dots, \hat{V}_k) \in \mathrm{Col}_k(G)$ be a sequence that is lexicographically maximal in $\mathrm{Col}_k(G)$. The graph $G_{1,2}$, induced in G on the vertex set $\hat{V}_1 \cup \hat{V}_2$, is bipartite. Therefore, if $\hat{E} \subseteq E_{1,2}(\hat{\mathcal{V}})$ is the set of all isolated edges of $G_{1,2}$, by Theorem 5 we have $\mathrm{gndi}(G_{1,2} - \hat{E}) \leq 3$ and there is an nd-colouring $\varphi : E(G_{1,2} - \hat{E}) \to [1, 3]$.

We are going to find an nd-colouring $\psi : E(G) \to [1, k+2]$ as a continuation of φ. Namely, we define $\psi(e) := 1$ for any $e \in \hat{E}$, $\psi(e) := k+2$ for any $e \in E_{1,j}(\hat{\mathcal{V}})$ with $j \in [2, k]$ and $\psi(e) := j+1$ for any $e \in E_{i,j}(\hat{\mathcal{V}})$ with $i \in [2, k-1]$ and $j \in [i+1, k]$.

Let us check that ψ is an nd-colouring. For that purpose consider vertices $u \in \hat{V}_i$ with $i \in [1, k-1]$ and $v \in \hat{V}_j$ with $j \in [i+1, k]$ such that $uv \in E(G)$.

If $(i, j) = (1, 2)$ and $uv \notin \hat{E}$, then $S_\varphi(u) \subseteq S_\psi(u) \subseteq S_\varphi(u) \cup \{k+2\}$ and $S_\varphi(v) \subseteq S_\psi(v) \subseteq S_\varphi(v) \cup [4, k+1]$; as $S_\varphi(u) \neq S_\varphi(v)$ and both $S_\varphi(u), S_\varphi(v)$ are subsets of $[1, 3]$, it is clear that also $S_\psi(u) \neq S_\psi(v)$.

If $(i, j) = (1, 2)$ and $uv \in \hat{E}$, then, by Lemma 6.3, the vertex u is not pendant, hence has a neighbour in $\bigcup_{l=3}^{k} \hat{V}_l$ and $S_\psi(u) = \{1, k+2\} \neq S_\psi(v) \subseteq \{1\} \cup [4, k+1]$.

Suppose that $j \in [3, k]$. From Lemma 6.1 we obtain $j+1 \in S_\psi(v) \subseteq [j+1, k+2]$. If $i = 1$, then $j+1 \notin S_\psi(u) \subseteq \{1, 2, 3, k+2\}$. If $i = 2$, then $S_\psi(u) \cap [1, 3] \neq \emptyset$, while $S_\psi(v) \cap [1, 3] = \emptyset$. Finally, if $i \in [3, j-1]$, Lemma 6.1 yields $i+1 \in S_\psi(u) - S_\psi(v)$.

Thus $uv \in E(G)$ implies $S_\psi(u) \neq S_\psi(v)$ and we are done. □

Corollary 8. *If G is a connected planar graph with $|E(G)| \geq 2$, then* $\mathrm{gndi}(G) \leq 6$.

It may be a little bit surprising that $\mathrm{gndi}(I) = 3$ for the icosahedron graph I. In fact, we do not know any planar graph whose general neighbour-distinguishing index is greater than 3.

Problem 1. *Does there exist a planar graph G with* $\mathrm{gndi}(G) \in [4, 6]$ *?*

Theorem 9. *If n is an integer, $n \geq 3$, then* $\mathrm{gndi}(K_n) = \lceil \log_2 n \rceil + 1$.

Proof. In an nd-colouring of K_n any two distinct vertices must have distinct colour sets. So, $\mathrm{gndi}(K_n) = \chi_0(K_n)$ and the result follows from [3]. □

Corollary 10. *If G is a connected graph with $|E(G)| \geq 2$, then* $\mathrm{gndi}(G) \leq \Delta(G) + 2$.

Proof. If there is $n \geq 3$ such that $G \simeq C_n$ or $G \simeq K_n$, use Proposition 3 or Theorem 9. Otherwise, by Brooks' Theorem, $\chi(G) \leq \Delta(G)$, and the statement follows from Theorem 7. □

We conjecture that Theorem 7 can be strengthened to $\mathrm{gndi}(G) \leq \chi(G) + 1$.

References

[1] P.N. Balister, E. Győri, J. Lehel, R.H. Schelp, Adjacent vertex distinguishing edge-colorings, preprint, http://www.msci.memphis.edu/preprint/avdec.pdf.

[2] K. Edwards, M. Horňák, M. Woźniak, On the neighbour-distinguishing index of a graph, preprint MD/2004/005, http://www.ii.uj.edu.pl/ preMD/ index.htm.

[3] F. Harary, M. Plantholt, The point-distinguishing chromatic index, in: F. Harary, J.S. Maybee (Eds.), Graphs and Applications, Wiley-Interscience, New York, 1985, pp. 147–162.

[4] H. Hatami, $\Delta + 300$ is a bound on the adjacent vertex distinguishing edge chromatic number, preprint, http://www.cs.toronto.edu/˜hamed.

[5] M. Horňák, R. Soták, The fifth jump of the point-distinguishing chromatic index of $K_{n,n}$, Ars Combin. 42 (1996) 233–242.

[6] M. Horňák, R. Soták, Localization of jumps of the point-distinguishing chromatic index of $K_{n,n}$, Discuss. Math. Graph Theory 17 (1997) 243-251.

[7] M. Horňák, N. Zagaglia Salvi, On the point-distinguishing chromatic index of complete bipartite graphs, Ars Combin. (to appear).

[8] N. Zagaglia Salvi, On the point-distinguishing chromatic index of $K_{n,n}$, Ars Combin. 25B (1988) 93–104.

[9] N. Zagaglia Salvi, On the value of the point-distinguishing chromatic index of $K_{n,n}$, Ars Combin. 29B (1990) 235–244.

[10] Z. Zhang, L. Liu, J. Wang, Adjacent strong edge coloring of graph, Appl. Math. Lett. 15 (2002) 623-626.