NOTE
Rainbowness of cubic polyhedral graphs

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Abstract. The rainbowness, \(\text{rb}(G)\), of a connected plane graph \(G\) is the minimum number \(k\) such that any colouring of vertices of the graph \(G\) using at least \(k\) colours involves a face all vertices of which have different colours. For a cubic polyhedral (i.e. 3-connected plane) graph \(G\) we prove that
\[
\frac{n}{2} + \alpha_1^* - 1 \leq \text{rb}(G) \leq n - \alpha_0^* + 1,
\]
where \(\alpha_0^*\) and \(\alpha_1^*\) denote the independence number and the edge independence number, respectively, of the dual graph \(G^*\) of \(G\). Moreover, we show that the lower bound is tight and that the upper bound cannot be less than \(n - \alpha_0^*\) in general. We also prove that if the dual graph \(G^*\) of an \(n\)-vertex cubic polyhedral graph \(G\) has a perfect matching then
\[
\text{rb}(G) = \frac{3}{4} n.
\]

AMS Subject Classification 2000: 05C15, 52B10
Keywords: vertex colouring, rainbowness, plane graph, cubic polyhedral graphs

1 Introduction

Colouring vertices of plane graphs under restrictions given by faces has recently attracted much attention, see e.g. [JKS], [K], [N], [RW] and references there. One natural problem of this kind is the following Ramsey type problem: Let us define the rainbowness of a connected plane graph \(G\), \(\text{rb}(G)\), as the minimum number \(k\) such that any surjective colour assignment \(\varphi : V(G) \rightarrow\)
\{1, 2, \ldots, k\} involves a face all vertices of which have different colours. Problem is to determine the rainbowness of the graph \(G\).

We use the standard terminology according to [BM] except for few notation defined throughout. However we recall some frequently used terms.

For a plane graph \(G\) let \(\alpha_0(G)\) be the independence number of \(G\) and \(\alpha_1(G)\) be the edge independence number of \(G\). Let \(G^*\) be the dual graph to the plane graph \(G\). Then we let \(\alpha_0^*(G) = \alpha_0(G^*)\) and \(\alpha_1^*(G) = \alpha_1(G^*)\).

The rainbowness, \(rb(T)\), of plane triangulations \(T\) has been recently studied (under the name looseness) by Negami [N]. He proved that for any triangulation \(T\)

\[
\alpha_0(T) + 2 \leq rb(T) \leq 2\alpha_0(T) + 1,
\]

where \(\alpha_0(T)\) is the independence number of \(T\). Ramamurthi and West [RW] observed that the following inequality relating \(rb(G)\) to the independence number \(\alpha_0(G)\) and the chromatic number \(\chi_0(G)\) holds

\[
rb(G) \geq \alpha_0(G) + 2 \geq \left\lceil \frac{n}{\chi_0(G)} \right\rceil + 2,
\]

where \(n = |V(G)|\), the number of vertices of a plane graph \(G\).

For an \(n\)-vertex plane graph \(G\), the Four Colour Theorem yields \(rb(G) \geq \left\lceil \frac{n}{4} \right\rceil + 2\). If \(G\) is triangles-free, then Grötzsch’s theorem (see [G], [T]) gives \(rb(G) \geq \left\lceil \frac{n}{2} \right\rceil + 2\). In [RW] Ramamurthi and West showed that the above lower bound is tight for a fixed \(n\) when \(\chi(G) = 2, 3\) and it is within one of being tight for \(\chi(G) = 4\). They conjectured the following bound for triangle-free plane graphs.

**Conjecture 1.1.** If \(G\) is \(n\)-vertex triangles-free plane graph, \(n \geq 4\), then \(rb(G) \geq \left\lceil \frac{n}{2} \right\rceil + 2\).

Ramamurthi and West proved their conjecture for plane graphs with girth at least six. Jungić, Kráľ and Škrekovski [JKS] answered the conjecture in affirmative. Moreover, they proved for plane graphs \(G\) with girth \(g \geq 5\) that the rainbowness \(rb(G)\) is at least \(\left\lceil \frac{g-3}{g-2}n - \frac{g-7}{2(g-2)} \right\rceil + 1\) if \(g\) is odd and \(\left\lceil \frac{g-3}{g-2}n - \frac{g-6}{2(g-2)} \right\rceil + 1\) if \(g\) is even. The bounds are tight for all pairs \(n\) and \(g\) with \(g \geq 4\) and \(n \geq \frac{3g}{2} - 3\).

In [JS] the authors determined the precise values of the rainbowness for all, up to three, graphs of semiregular polyhedra.

In the present note we investigate cubic polyhedral (i.e. trivalent 3-connected plane) graphs. For this family of graphs we give better bounds than those mentioned above. The main result of this note is

**Theorem 1.2.** Let \(G\) be an \(n\)-vertex cubic polyhedral graph. Let \(\alpha_0^*\) and \(\alpha_1^*\) be an independence number and edge independence number, respectively, of the dual \(G^*\) of the graph \(G\). Then

\[
\frac{n}{2} + \alpha_1^* - 1 \leq rb(G) \leq n - \alpha_0^* + 1.
\]
2 Lower bound

Let $G$ be a cubic polyhedral graph. Let $M^* = \{e_1^*, \ldots, e_n^*\}$ be a maximum matching in $G^*$. Clearly $\alpha^*_2(G) = d = \alpha^*_1$. Every edge $e_i^* = xy$ of $M^*$ is associated in $G$ with a pair of two adjacent faces $f(x)$ and $f(y)$ which share an edge $e_i$ in common.

Let $M = \{e_1, e_2, \ldots, e_d\}$ be the set of such defined edges of $G$. Clearly $M$ is a matching. Let $V(M)$ be the set of vertices incident with the edges of $M$ (i.e. it is a set of end vertices of edges from $M$). Evidently $|V(M)| = 2d = 2\alpha^*_1$. Similarly, let $F(M)$ be the set of faces containing an edge from the set $M$. Observe that if $e_i \neq e_j$ then the pair of faces incident with the edge $e_i$ is disjoint with the pair of faces incident with $e_j$. Hence $|F(M)| = 2\alpha^*_1$. The following observation is easy to see.

**Observation 1.** Each face of $G$ has at most one edge in the set $M$. \hfill \square

**Observation 2.** If $f_1$ and $f_2$ are two distinct faces from $F(G) - F(M)$ then $f_1$ and $f_2$ do not share any common edge (and any vertex).

**Proof.** If $f_1$ and $f_2$ would share an edge $h$ then it could be added to the set $M$ and the edge $h^* = f_1^* f_2^*$ of $G^*$ corresponding to $h$ could extend maximum matching $M^*$ of $G^*$, a contradiction. \hfill \square

**Observation 3.** Every face $f \in F(G) - F(M)$ contains at least two vertices that are not in $V(M)$.

**Proof.** Let $f$ be a face that is not in $F(M)$. It is easy to see that no two consecutive vertices of $f$ belong to $V(M)$. Otherwise we have a contradiction with Observation 1. \hfill \square

The following colouring does not involve any rainbow face. First we find the set $M = \{e_1, e_2, \ldots, e_d\}$, $d = \alpha^*_1$. Next we find the set of faces $F(G) - F(M) = \{f_1, f_2, \ldots, f_m\}$ where $m = |F(G)| - 2\alpha^*_1$. We color vertices of the edge $e_i$ with colour $i$ for any $i \in \{1, \ldots, d\}$. For every face $f_j$, $j \in \{1, \ldots, m\}$, choose two vertices that are not in $V(M)$ and colour them both with the colour $d+j$. The remaining not yet coloured vertices are colored with different colours from the set $\{d+m+1, \ldots, n-d-m\}$.

Because each face of $G$ is adjacent with a monochromatic edge or with two vertices of the same colour, $G$ does not contain any rainbow face. In this colouring we have used the following number of colours

$$n - d - m = n - \alpha^*_1 - |F(G)| + 2\alpha^*_1 = n + \alpha^*_1 - |F(G)| .$$

(2)

Because $G$ is a cubic polyhedral graph we have $3n = 2|E(G)|$. Using this fact and the Euler’s polyhedral formula $n - |E(G)| + |F(G)| = 2$ we obtain

$$|F(G)| = \frac{n + 4}{2} .$$

(3)
Substituting for $|F(G)|$ from (2) in (3) we find out that the number of used colours is $\frac{n}{2} + \alpha_1 - 2$. Hence we have proved that

$$rb(G) \geq \frac{n}{2} + \alpha_1 - 1.$$ 

3 Upper bounds

**Lemma 3.1.** Let $G$ be an $n$-vertex plane graph and let $\{f_1, \ldots, f_k\}$ be a set of faces of $G$ such that no two among them have a common vertex. Then

$$rb(G) \leq n - k + 1.$$ 

*Proof.* Let $V(f_i)$ be the set of vertices incident with the face $f_i$. Suppose there is a $(n - k + 1)$-colouring of $G$ that has no rainbow face. Then there are at most $|V(f_i)| - 1$ colours at $f_i$ and at most $n - \sum_{i=1}^{k} |V(f_i)|$ colours at the vertices outside of the set $\bigcup_{i=1}^{k} V(f_i)$. This means that there are at most

$$n - \sum_{i=1}^{k} |V(f_i)| + \sum_{i=1}^{k} (|V(f_i)| - 1) = n - k$$

colours used at the vertices of $G$; a contradiction. \qed

Observe that maximum number of faces in a cubic polyhedral graph $G$ that no two among them have a vertex in common is $\alpha_0^* = \alpha_0(G^*)$, the independence number of $G^*$, the dual of $G$. This observation together with Lemma 3.1 yield

**Lemma 3.2.** For any $n$-vertex cubic polyhedral graph $G$ there is

$$rb(G) \leq n - \alpha_0^* + 1.$$ 

*□*

The upper bound in Lemma 3.2 can be improved if $\alpha_0^* < \frac{|F(G)|}{2}$.

**Theorem 3.3.** Let $G$ be an $n$-vertex cubic polyhedral graph with $m$ faces. If $\alpha_0^* < \frac{m}{2}$ then

$$rb(G) \leq n - \alpha_0^*.$$ 

Moreover, this bound is tight.
Proof. Suppose there is a \((n - \alpha_0^*)\)-colouring \(\varphi\) of a cubic polyhedral graph \(G\) that has no rainbow face. Let \(V(j)\) be the set of vertices coloured with colour \(j\) and let \(F(j)\) be the set of faces that are not rainbow because they contain at least two vertices from the set \(V(j)\). Let us estimate the number of pairs \((v, f)\) with a vertex \(v\) from \(V(j)\) and a face \(f\) from \(F(j)\). Let \(|V(j)| = a_j\) and \(|F(j)| = d_j\). Each face of \(F(j)\) contains at least two vertices from \(V(j)\), hence there are at least \(2d_j\) such pairs. On the other hand each vertex can be incident with at most three faces of \(F(j)\) therefore there are at most \(3a_j\) such pairs. Altogether we have

\[
2d_j \leq 3a_j. \quad (4)
\]

It is easy to see that if \(a_j = 1\) then \(d_j = 0\), if \(a_j = 2\) then \(d_j \leq 2\), and for \(a_j \geq 3\) we have from (4) that

\[
d_j \leq \left\lfloor \frac{3a_j}{2} \right\rfloor \leq 2(a_j - 1).
\]

The number of non-rainbow faces in \(G\) is at most

\[
\sum_{j=1}^{n-\alpha_0^*} d_j \leq \sum_{j=1}^{n-\alpha_0^*} 2(a_j - 1) = 2 \sum_{j=1}^{n-\alpha_0^*} a_j - 2(n - \alpha_0^*) = 2\alpha_0^*.
\]

Because \(2\alpha_0 < m\) there is a rainbow face in \(G\); a contradiction. For tightness of the bound see the last section of this paper. \(\square\)

**Theorem 3.4.** Let \(G\) be an \(n\)-vertex cubic polyhedral graph. Then

\[
\text{rb}(G) \leq \frac{3}{4}n.
\]

Moreover, the bound is tight.

Proof. Suppose there is a \(\left\lfloor \frac{3n}{4} \right\rfloor\)-colouring \(\varphi\) of \(G\) without any rainbow face. Analogously as in the proof of Theorem 3.3 we can show that for \(r = \left\lfloor \frac{3n}{4} \right\rfloor\)

\[
\sum_{i=1}^{r} d_i \leq 2 \sum_{i=1}^{r} (a_i - 1) = 2(n - r) = 2\left(n - \left\lfloor \frac{3n}{4} \right\rfloor\right) = 2\left\lfloor \frac{n^2}{4} \right\rfloor.
\]

If \(m\) is the number of faces of \(G\) then, because \(G\) does not contain rainbow faces, \(2\left\lfloor \frac{n}{4} \right\rfloor \geq m\). If \(e\) is the number of edges of \(G\) then \(2e = 3n\). Using these two relations and Euler’s polyhedral formula we obtain the inequality

\[
4\left\lfloor \frac{n}{4} \right\rfloor \geq n + 4
\]

which immediately yields a contradiction. For tightness of the bound \(\frac{3}{4}n\) see below. \(\square\)
Theorem 3.5. Let $G$ be an $n$-vertex polyhedral graph and let $G^*$ have a perfect matching. Then

$$
\text{rb}(G) = \frac{3n}{4}.
$$

Proof. If $G^*$ has a perfect matching then $\alpha_1^*(G^*) = |F(G)|$ and, by (3), $\alpha_1^*(G) = \frac{n^2}{4}$.

This together with the lower bound of Theorem 1.2 and Theorem 3.4 yields our equality.

4 Quality of the bounds

Consider a $d$-sided prism $D_d$. It is a 2d-vertex cubic polyhedral map which is in fact a cartesian product $P_2 \times C_d$ of a path $P_2$ on two vertices and a cycle $C_d$ on $d$-vertices. It is easy to see that $\alpha_0^*(D_d) = \left[\frac{d}{2}\right]$ and $\alpha_1^*(D_d) = \left[\frac{d+1}{2}\right]$. In [JS] there is proved that $\text{rb}(D_d) = \left[\frac{3d-1}{2}\right]$ for $d \geq 3$. Because for the prism $D_d$ there is $n + \alpha_1^* - 1 = d + \left[\frac{d+1}{2}\right] - 1 = \left[\frac{3d-1}{2}\right]$ the lower bound in Theorem 1.2 is tight.

Let the dual graph $G^*$ of an $n$-vertex cubic polyhedral graph has an almost perfect matching, i.e. let $\alpha_1^*(G) = |F(G)| - 1$. In this case $n = 4k + 2$ for some $k \geq 1$. Then $\alpha_1^*(G^*) = \frac{n^2}{4} = k + 1$. By the lower bound in Theorem 2.1 we have

$$
\text{rb}(G) \geq 2k + 1 + k + 1 - 1 = 3k + 1.
$$

On the other hand, by Theorem 3.3, there is

$$
\text{rb}(G) \leq \frac{3}{4}n = \left[\frac{3}{4}(4k + 2)\right] = 3k + \frac{6}{4}.
$$

Hence $\text{rb}(G) = 3k + 1$. So we have proved that the rainbowness of such graphs equals to the lower bound in Theorem 2.1.

We believe that the following is true.

Conjecture 4.1. For every $n$-vertex cubic polyhedral graph $G$ there is

$$
\text{rb}(G) = \frac{n}{2} + \alpha_1^*(G) - 1.
$$

We do not know any example of a cubic polyhedral graph $G$ with $\text{rb}(G) = n - \alpha_0^*(G) + 1$. For the $d$-sided prism with $d$ even there is $\text{rb}(D_d) = n - \alpha_0^*(G) = 2d - \frac{d}{2} = \frac{3d}{2}$. This means that the upper bound of Theorem 3.3 is sharp. We believe that the following is true.

Conjecture 4.2. Let $G$ be an $n$-vertex polyhedral graph. Then

$$
\text{rb}(G) \leq \frac{3n}{4}.
$$

Acknowledgement.

This work was supported by Science and Technology Assistance Agency under the contract No. APVT-20-004104. Support of Slovak VEGA Grant 1/3004/06 is acknowledged as well.
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