THE KIDNEY EXCHANGE PROBLEM: HOW HARD IS IT TO FIND A DONOR?*

Katarína Cechlárová[1] and Vladimír Lacko[2]

[1]Institute of Mathematics, P.J. Šafárik University, Košice, Slovakia

email: katarina.cechlarova@upjs.sk

[2] Institute of Computer Science, P.J. Šafárik University, Košice, Slovakia

email: vladimir.lacko@upjs.sk

July 11, 2006

Abstract. The most effective treatment for kidney failure that is currently known is transplantation. However, the supply of kidneys from cadaveric donors does not meet the fast growing demand and the kidney from a willing living donor (genetically or emotionally relative of the patient) is often not suitable for immunological reasons. Therefore in several countries attempts have started to organize exchanges of kidneys between incompatible patient-donor pairs. Game-theoretical models have been proposed to analyze various optimality criteria for such exchanges and various search schemes have been tested. One possibility to model patients' preferences is to take into account in the first step the suitability of the donated kidney and in the second step the length of the obtained cycle of exchanges. Although the core of such a cooperative game is always nonempty and one solution can be found by the famous Top trading cycles algorithm, in this paper we show that many questions concerning the structure of the core are difficult to answer.

Keywords. Kidney transplantation, cooperative game, core, NP-completeness.

1 Introduction

Renal failure is a very serious illness for which the most effective treatment that is currently known is kidney transplantation. Ideally, a kidney from a deceased

*This work was supported by VEGA grants 1/3001/06, 1/3128/06 and Science and Technology Assistance Agency contract No. APVT-20-004104

donor could be used, but the supply of those in spite of joint efforts of national and even international organisations (for example Eurotransplant Foundation [25] and the United Network for Organ Sharing in the USA [26]) is not sufficient for the growing demand. Moreover, the waiting time for a cadaveric kidney is unpredictable. As the operation techniques improved and the risk for a living donor of a kidney (a genetic or an emotional relative of the patient) was minimized, the number of live-donor transplantations increased. Moreover, some studies [24] show that grafts from living donors have a higher survival rate.

For a transplantation to be successful, some immunological requirements must be fulfilled. Basically, ABO incompatibility and a positive cross-match are an absolute contraindication, moreover, the greater the number of HLA mismatches between the donor and the recipient, the greater the chance of rejection [12]. Hence, it often happens that a willing donor cannot donate his/her kidney to the intended recipient. Therefore in several countries systematic kidney exchange programmes have been established: in Romania [15], the Netherlands [14], USA [18, 19, 20, 21], United Kingdom [11]; in other cases there are isolated examples, e.g. in the Middle East [8].

Kidney exchange is still encompassed by difficult ethical and legal problems. However, in spite of some pessimistic expectations (the British Transplantation Society [2] estimated potential benefits from living donors' exchanges to be around 3%) more and more countries introduce paired kidney exchanges into their medical praxis. (At the time of this writing, the the most recent event was the Press releases of the Human Tissue Authority in Great Britain of April 26, 2006 [11].) Practical experiences demonstrate the benefits of kidney exchanges. To name at least two: in the Clinic in Cluj-Napoca, Romania, the monthly mean number of transplantations increased from 4.2 to 6.1 since the kidney exchange program started [15] and after just 3 runs of the allocation algorithm developed by the Dutch Transplantation Foundation a match was found for 22 of 53 participating pairs [13]. To assess a possible impact of kidney exchanges, several simulation studies modeling real statistical data of patients and donors have been performed [22, 20, 21] and they all clearly showed substantial gains from pooling greater numbers of patient-donor pairs and searching for possible exchanges by implementing some optimization techniques.

We follow the approach started in [18] and [19], in that we represent kidney exchange as a cooperative game, in which patient-donor pairs seek cyclic exchanges of kidneys. Since all operations on a cycle should be performed simultaneously (to avoid the risk that one of the donors will withdraw his or her commitment after the others have undergone nephrectomy [8]), cycles should be as short as possible for logistical reasons. Unlike in [20, 21, 22], where the cycle lengths were restricted to 2 or at most 3 directly in the constraints of the mathematical problem solved, we incorporate cycle lengths into preference models. This approach was suggested in [7]. Notice that in [18] and [20] preferences of patients mirror only the suitability of the donated kidney. In [19] the obtained cycle lengths are just observed after simulations of the algorithm used and in [20] the numbers of matched patients are derived as a function of the maximum allowed cycle length. Most authors intentionally search only for paired kidney exchanges. In [22] an optimized algorithm based on Edmonds' maximum cardinality matching algorithm was used to find the maximum number of matched pairs and the algorithm in [13] tries to ensure that highly sensitized recipients (for whom it is otherwise very difficult to find a suitable donor) have the best chance to receive a kidney. All the studies show that increasing the pool of participating pairs leads to an increase of the number of obtained exchanges, so this encourages several transplantations centers to cooperate on this issue and to organize kidney exchanges on the national level [13].

Kidney exchange may potentially present difficult conflicts of interest. Imagine a situation when a patient has to give up the most suitable kidney to ensure that a greater number of exchanges will be achieved by a different allocation. Pareto optimal solutions are often used in such situations, but if say one of the participating transplantation centers discovers that their patients could improve by another allocation between them, then such a discovery could destroy the national cooperation or at least decrease the trust of participating patients in the principle used.

Therefore in this paper we concentrate on the core of the considered kidney exchange game. In the search for the description of its structure we encountered several NP-complete problems, other problems remain unsolved. In the view of these results, we proposed two heuristics and tested their strength on several sets of randomly generated data. They are described in Section 5.

2 The Kidney Exchange game

An instance of the kidney exchange game is represented by a directed graph $G = (V, A)$ without loops where each vertex $i \in V$ corresponds to a patient and his intended incompatible donor (or donors). A pair $(i, j) \in A$ if the patient corresponding to vertex i can accept a kidney from a donor corresponding to vertex j. If $(i, j) \in A$, player j is acceptable for i, otherwise he is unacceptable. Moreover, for each vertex i there is a linear ordering $\preceq_i$ on the set of endvertices of arcs incident from i, where $j \preceq_i k$ means that vertex j is preferred by vertex i to vertex k. This ordering is represented by a preference list $P(i)$ of i. To keep things simple, we shall usually not give the graph G explicitly, as its structure is implied by the contents of the preference lists.

In this paper we suppose that preference lists do not contain ties, i.e. if $j \preceq_i k$ then $k \preceq_i j$ does not hold; we also write $j \prec_i k$ and say that i prefers j to k strictly in this case. Otherwise we say that i is indifferent between j and k and write $k \sim_i j$.

For brevity, the first entry in $P(i)$ will usually be called the favourite of i and denoted by $f(i)$ and $f^{-1}(j)$ denotes the player (players) such that $f(i) = j$.

Definition 1 *A* kidney exchange *game (KE game for short) is a triple* $\Gamma = (V, G, \mathcal{O})$*, where* V *is the set of players,* G *is a digraph with a vertex set* V *and* $\mathcal{O} = \{\preceq_i; i \in V\}$.

Definition 2 *A* solution *of a KE game* $\Gamma = (V, G, \mathcal{O})$ *is a permutation* π *of* V *such that* $i \neq \pi(i)$ *implies* $(i, \pi(i)) \in A$ *for each* $i \in V$*. If* $(i, \pi(i)) \in A$*, we say that* i *is* covered *by* π*, otherwise* i *is* uncovered*. A player* i *is assigned in a solution* π *the pair* $(\pi(i), C^\pi(i))$*, where* $C^\pi(i)$ *denotes the cycle of* π *containing* i.

A player evaluates a permutation not only according to the player (kidney) he is assigned to, but he also takes into account the length of the cycle he is contained in. The least preferred possibility for each player is $\pi(i) = i$, since this means that the corresponding patient will not receive a kidney. For technical reasons, since this notation will simplify some formulations later in this paper, we shall put $|C^\pi(i)| = \infty$ if $\pi(i) = i$.

The extension of preferences from $\mathcal{O}$ to preferences over (player,cycle) pairs is formally introduced in the following definiton. (The same symbol is used for preferences over players as well as over (player,cycle) pairs and permutations.)

Definition 3 *A player* i *prefers pair* (j, M) *to pair* (k, N) *if*
(i) $j \prec_i k$ *or*
(ii) $j \sim_i k$ *and* $|M| < |N|$.

Definition 4 *A coalition* $S \subseteq V$ blocks *a solution* π *if there exists a permutation* σ *of* S *such that*
$(\sigma(i), C^\sigma(i)) \prec_i (\pi(i), C^\pi(i))$ *for each* $i \in S$.

Now we define the studied solution concept.

Definition 5 *A permutation* π *is in the* core $Core(\Gamma)$ *of game* Γ *if no coalition blocks* π.

Roth et al. [18] proposed to concentrate on Pareto optimality in the context of kidney exchange. In [7] the notions of Pareto optimal, strongly Pareto optimal, core and strong core solutions of the KE game were also studied. It was shown that a strong core solution is also a core solution and this is in turn Pareto optimal, that a strong core solution is strongly Pareto optimal and also Pareto optimal, but a core solution need not be strongly Pareto optimal even in the case without indifferences.

3 The Top Trading Cycles algorithm and its shortcommings

With no indifferences, the famous Top Trading Cycles (TTC for short) algorithm can be used to find a permutation in the strong core (and hence also in the core) of the KE game. For completeness, we state it in Figure 1 and in what follows, we denote also by TTC the permutation obtained by this algorithm for a given KE game instance, so the notation of the form $TTC(i)$ or $C^{TTC}(i)$ will sometimes appear on condition that the KE game instance is clear from the context.

Input. A KE game $\Gamma = (V, G, \mathcal{O})$.
Output. A permutation $\pi = C_1C_2 \dots C_r$ of V.
Step 0. $N := V$, round $r := 0$.
Step 1. Choose an arbitrary player $i_0 \in N$.
Step 2. Player i_0 points to his favourite i_1 in N. i_1 points to his favourite i_2 in N etc. A cycle arises or for some k, player i_k cannot point.
Step 3. $r := r+1$. If a cycle C was obtained, then $C_r := C$, otherwise $C_r = (i_k)$. $N := N - C_r$.
Step 4. If $N \neq \emptyset$, go to Step 1, otherwise end.

Figure 1: The Top Trading Cycles (TTC) Algorithm

The TTC algorithm was originally proposed by Gale in [23] for housing markets, where cycle lengths were not taken into account. It was shown that the TTC algorithm outputs a permutation in the core of the housing market also in the case with indifferences (ties are broken arbitrarily). In [17] Roth and Postlewaite proved that if there are no indifferences, the strong core of the housing market is nonempty and contains a unique permutation. Further, Roth [16] proved that the TTC algorithm is strategy-proof. However, a detailed consideration of algorithmic questions connected with the TTC algorithm is quite recent; in [1] its implementation with $O(m)$ time complexity was proposed, where m is the number of arcs in G.

In [6], the TTC algorithm is called Algorithm B-stable and its output is shown to be in the strong core and hence also in the core of the KE game (called in that paper the stable partition problem with B-preferences) in the case with no indifferences.

On the other hand, in [5] it was proved that in the case with indifferences it is NP-complete to decide whether $Core(\Gamma) \neq \emptyset$. Moreover, in the case with complete indifferences (i.e. all vertices adjacent from a given vertex are indifferent = all suitable kidneys are equally good for a patient) we know that the problem of deciding nonemptyness of the strong core of the obtained simple KE game is NP-complete and although a strongly Pareto optimal permutation exists for all simple KE games, it is NP-hard to find one [7]. Hence, the case with indifferences seems to be less tractable.

However, the case with strict preferences is also not so simple. $Core(\Gamma)$ may contain many pemutations, some of them much more favourable than the TTC permutation.

Example 1 *An example, constructed already in [6] shows that although the TTC algorithm outputs one long cycle, there may exist a core permutation giving a cycle of length 2 to each player. In this example there are $2n$ players $a_1, a_2, \dots, a_{2n}$ and for $i = 1, 2, \dots, n$ the preference list of a_{2i-1} contains just a_{2i}, while the preference list of a_{2i} contains a_{2i+1} and a_{2i-1} in this order (indices are modulo $2n$ when necessary). The TTC permutation is $(a_1, a_2, \dots, a_{2n})$, while the permutation consisting of pairs (a_{2i-1}, a_{2i}) for $i = 1, 2, \dots, n$ also belongs to $Core(\Gamma)$.*

However, shorter cycles may be caused not only by second-preferences arcs, as the following construction shows. Take $2n$ players, $n \geq 5$, denoted by $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n$ with preferences

$$P(a_1) : a_2 \qquad\qquad P(b_1) : b_2$$
$$P(a_2) : a_3, a_4, \dots, a_{n-1}, b_1 \qquad\qquad P(b_2) : b_3, b_4, \dots, b_{n-1}, a_1$$
$$P(a_i) : a_{i+1} \qquad\qquad P(b_i) : b_{i+1}$$
$$P(a_n) : a_1, a_3 \qquad\qquad P(b_n) : b_1, b_3$$

where $i = 3, 4, \dots, n-1$. The TTC permutation for this game is $(a_1, a_2, \dots, a_n$ $(b_1, b_2, \dots, b_n)$ with two cycles of length n. It is easy to see that permutation $(a_1, a_2, b_1, b_2)(a_3, a_4, \dots, a_n)$ $(b_3, b_4, \dots, b_n)$ is also in the core and each player has a shorter cycle than in the TTC permutation.

Example 2 *This example shows that in some cases there may exist core permutations covering more players than the TTC permutation does. Let the set of players be $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n, c_1, c_2, \dots, c_n$ with preferences*

$$P(a_i) : b_i \qquad\qquad P(b_i) : a_{i+1}, c_i \qquad\qquad P(c_i) : a_i$$

for $i = 1, 2, \dots, n$. The TTC algorithm outputs one cycle $(a_1, b_1, a_2, b_2, \dots, a_n, b_n)$ and the players $c_1, c_2. \dots, c_n$ remain uncovered. However, $Core(\Gamma)$ contains a permutation σ covering all players, namely σ consists of n cycles of the form (a_i, b_i, c_i). Notice that even all players a_i are better off under σ than under the TTC permutation.

4 The structure of the core of the KE game

For some time we tried in vain to find a description of $Core(\Gamma)$. When the efforts turned out to be fruitless, we formulated instead some more specific decision problems concerning its structure. As it is possible to decide in polynomial time [4] whether a given permutation belongs to $Core(\Gamma)$, all the studied problems are clearly in NP. To our surprize, many of them are NP-complete even in the case with strict preferences. Now we formulate the first decision problem:

> *Name:* ALL-SHORTHER-CYCLES-KE
>
> *Instance:* Arbitrary instance $\Gamma = (V, G, \mathcal{O})$ of the KE game with strict preferences
>
> *Question:* Does $Core(\Gamma)$ contain a permutation π such that $|C^{\pi}(i)| < |C^{TTC}(i)|$ for each $i \in V$?

Theorem 1 *Problem* ALL-SHORTER-CYCLES-KE *is NP-complete.*

Proof. To prove the NP-completeness, we shall use a polynomial transformation from the problem RESTRICTED-3-SAT showed to be NP-complete in [10]:

> *Name:* RESTRICTED-3-SAT
>
> *Instance:* A boolean formula B in CNF containing n boolean variables $x_1, x_2, \dots, x_n$ and m clauses $K_1, K_2, \dots, K_m$ such that each clause contains exactly 3 literals and each variable appears at most twice nonnegated and at most twice negated in B.
>
> *Question:* Is B satisfiable?

For each formula B we construct an instance $\Gamma = (V, G, \mathcal{O})$ of the KE game with the following properties. For each variable x_j there will be a cell of 6 *variable players* $x_j^1, x_j^2, y_j^1, y_j^2, z_j^1, z_j^2$, called a *variable cell* $\Gamma(x_j)$. Players x_j^1, x_j^2 correspond to the first and to the second occurence of literal x_j, while players y_j^1, y_j^2 correspond to the first and to the second occurence of literal $\overline{x_j}$. Players $x_j^1, x_j^2, y_j^1, y_j^2$ will be called the *proper variable* players. For each clause K_k there is a cell of 6 *clause players* $p_k^1, p_k^2, p_k^3, r_k^1, r_k^2, r_k^3$, called a *clause cell* $\Gamma(K_k)$. Players p_k^1, p_k^2, p_k^3 correspond to the first, second and third entry of K_k, respectively. Again, these players will be called the *proper clause* players.

For brevity, in this game we shall use the following notation:

- the proper clause player that corresponds to the clause entry containing the literal corresponding to a proper variable player v, will be denoted by $c(v)$;
- the proper variable player that corresponds to the literal contained in the clause entry corresponding to a proper clause player c, will be denoted by $v(c)$.

For example, if say $K_2 = x_1 + \overline{x_2} + x_3$ and for $x_1, \overline{x_2}$ this is their first occurence in B and for x_3 its second one, then $v(p_2^1) = x_1^1, v(p_2^2) = y_2^1, v(p_2^3) = x_3^2$ and $c(x_1^1) = p_2^1, c(y_2^1) = p_2^2, c(x_3^2) = p_2^3$.

Preferences of players are given in Figure 2; preferences of clause players are on the left and those of variable players on the right. If some literal is not present in the formula, then $c(v)$ does simply not appear in the preference list of player v, which will not influence our subsequent arguments.

$P(p_k^1) : p_k^2, r_k^1$	$P(x_j^1) : y_j^1, x_j^2$
$P(p_k^2) : p_k^3, r_k^2, v(p_k^1)$	$P(x_j^2) : y_j^2, x_j^1$
$P(p_k^3) : r_k^1, v(p_k^2)$	$P(y_j^1) : z_j^1, c(x_j^1), x_j^2$
$P(r_k^1) : r_k^2, v(p_k^3)$	$P(y_j^2) : z_j^2, c(x_j^2), x_j^1$
$P(r_k^2) : r_k^3$	$P(z_j^1) : x_j^2, c(y_j^1), z_j^2$
$P(r_k^3) : p_k^1, p_k^3$	$P(z_j^2) : x_j^1, c(y_j^2), z_j^1$

Figure 2: Preferences of players in Theorem 1

It is easy to see that the TTC permutation for this game consists of n 6-cycles in variable cells of the form $(x_j^1, y_j^1, z_j^1, x_j^2, y_j^2, z_j^2)$ and of m 6-cycles in clause cells of the form $(p_k^1, p_k^2, p_k^3, r_k^1, r_k^2, r_k^3)$. Notice that the construction of preferences implies that there are no acceptable players outside player's own cell except for players $v(f^{-1}(c))$ and $c(f^{-1}(v))$, if c is the favourite of a proper clause player and v is the favourite of a proper variable player, respectively. Also notice that there are no cycles involving players from two different cells with length less than 4. On the other hand, for each proper clause player c there is a cycle of length 4 of the form $(c, f(c), v(c), f(v(c)))$ and for each proper variable player v there is a cycle of length 4 of the form $(v, f(v), c(v), f(c(v)))$. These cycles will play a very important role in our construction.

Now we prove a few lemmas on the structure of core permutations in variable and clause cells.

Lemma 1 *Let $\Gamma(K_k) = \{p_k^1, p_k^2, p_k^3, r_k^1, r_k^2, r_k^3\}$ be a clause cell. The only possibilities how a core permutation that covers all players of $\Gamma(K_k)$ and is different from the one given by TTC can look like on $\Gamma(K_k)$ are*

(i) $(p_k^1, p_k^2, v(p_k^1), f(v(p_k^1)))(p_k^3, r_k^1, r_k^2, r_k^3)$

(ii) $(p_k^2, p_k^3, v(p_k^2), f(v(p_k^2)))(r_k^1, r_k^2, r_k^3, p_k^1)$

(iii) $(p_k^3, r_k^1, v(p_k^3), f(v(p_k^3)))(r_k^2, r_k^3, p_k^1, p_k^2)$

Proof. First we notice that having a permutation π acting on $\Gamma(K_k)$ like in (i), each player a is on the shortest possible cycle containing $\pi(a)$, hence the only possibility for him to improve would be via a permutation σ such that

$\sigma(a) \prec_a \pi(a)$. If $a \in \{p_k^3, r_k^1, r_k^2, p_k^1\}$, then $\pi(a) = f(a)$, so these players will not participate in any blocking coalition. Player r_k^3 has his second choice, but he needs p_k^1 to improve and we already know that p_k^1 will not be in any blocking coalition. Player p_k^2 has his third choice, but neither p_k^3 nor r_k^2 will join him to block.

Analogous arguments will be used in cases (ii) and (iii).

To show that a core permutation cannot partition $\Gamma(K_i)$ differently, realize again that except for players p_k^2, p_k^3, r_k^1 no other player has an acceptable player outside this cell. Further, any feasible permutation π covering r_k^2 must have $\pi(r_k^2) = r_k^3$ and since both r_k^3 and p_k^1 only have two acceptable players, we must get one of the cases described in $(i), (ii)$ and (iii). ■

In the following lemmas concerning variable cells we always take the superscript t modulo 2.

Lemma 2 *Let a core permutation π contain the cycle $(x_j^t, y_j^t, c(x_j^t), f(c(x_j^t)))$ for some j and $t \in \{1,2\}$. Then players x_j^t and y_j^t cannot be in a blocking coalition.*

Proof. Notice that player x_j^t can only improve on the cycle $(x_j^t, y_j^t, x_j^{t-1})$, but then player y_j^t would be worse off. For player y_j^t to improve via a permutation σ, one must have $\sigma(y_j^t) = z_j^t$. If the cycle of σ containing y_j^t and z_j^t is within this variable cell, then it must also contain x_j^t and have length at least 4, which will not make player x_j^t better off. If, however, $\sigma(z_j^t) = c(y_j^t)$, then σ must return to y_j^t either via x_j^t (who, as we already know, will not participate in any blocking) or via a clause player c, for whom $\sigma(c) = v(c) = y_j^t$. However, should $\pi \in Core(\Gamma)$ then clause players are partitioned according to Lemma 1, which means that they are all on cycles of length 4 and since $v(c)$ is the last choice for each clause player c, this player will prevent any blocking. ■

Lemma 3 *Let $\Gamma(x_j) = \{x_j^1, x_j^2, y_j^1, y_j^2, z_j^1, z_j^2\}$ be a variable cell. Let a permutation π restricted to players of $\Gamma(x_j)$ be either*

$$(i)\ (x_j^t, y_j^t, c(x_j^t), f(c(x_j^t)))(z_j^t, x_j^{t-1}, y_j^{t-1}, z_j^{t-1})\ or$$

$$(ii)\ (y_j^t, z_j^t, c(y_j^t), f(c(y_j^t)))(x_j^{t-1}, y_j^{t-1}, z_j^{t-1}, x_j^t)$$

Then no player of $\Gamma(x_j)$ will be in a blocking coalition.

Proof. Case (i). As $\pi(a) = f(a)$ for $a \in \{z_j^t, y_j^{t-1}\}$ and $C^\pi(a)$ is shortest possible on condition $\pi(a) = f(a)$, these players cannot improve. Players x_j^t and y_j^t cannot improve due to Lemma 2. Player x_j^{t-1} could improve only by getting the cycle $(x_j^{t-1}, y_j^{t-1}, x_j^t)$, but the other players on this cycle will be worse off. Player z_j^{t-1} needs for improvement either $\sigma(z_j^{t-1}) = x_j^t$ (who will refuse) or $\sigma(z_j^{t-1}) = c(y_j^{t-1})$ (leading to the cycle $(z_j^{t-1}, c(y_j^{t-1}), f(c(y_j^{t-1})), y_j^{t-1})$, but here player y_j^{t-1} will not improve), or cycle (z_j^{t-1}, z_j^t), which is no improvement for z_j^t.

Case (ii). Since $\pi(a) = f(a)$ for $a \in \{y_j^t, z_j^{t-1}, y_j^{t-1}\}$ and $C^\pi(a)$ is shortest possible on condition $\pi(a) = f(a)$, these players cannot improve. Player x_j^{t-1} could improve on the cycle $(x_j^{t-1}, y_j^{t-1}, x_j^t)$, but in this case player y_j^{t-1} would be worse off. Player x_j^t needs for improvement either his more preferred player y_j^t or a shorter cycle, requiring cooperation of x_j^{t-1} or y_j^{t-1}, but since neither of those players can improve, x_j^t also cannot be in a blocking coalition. Finally, player z_j^t could only improve by getting $\sigma(z_j^t) = x_j^{t-1}$, but player x_j^{t-1} cannot be in any blocking coalition. ■

Lemma 4 *Let* $\Gamma(x_j) = \{x_j^1, x_j^2, y_j^1, y_j^2, z_j^1, z_j^2\}$ *be a variable cell. Let a permutation* π *restricted to players of* $\Gamma(x_j)$ *be either*

$$(i)\ (x_j^1, y_j^1, c(x_j^1), f(c(x_j^1)))(x_j^2, y_j^2, c(x_j^2), f(c(x_j^2)))(z_j^1, z_j^2)$$

or

$$(ii)\ (y_j^1, z_j^1, c(y_j^1), f(c(y_j^1)))(y_j^2, z_j^2, c(y_j^2), f(c(y_j^2)))(x_j^1, x_j^2)$$

then no player of $\Gamma(x_j)$ *will be in a blocking coalition.*

Proof. For case (i) Lemma 2 implies that players $x_j^1, y_j^1, x_j^2, y_j^2$ will not be in any blocking coalition. So as to improve, player z_j^t, $t = 1, 2$ needs either player x_j^{t-1} (who cannot improve) or $c(y_j^t)$ and hence also y_j^t, who will again not join.

On the other hand, in case (ii) we have similarly as in the proof of Lemma 3 that players y_j^1, y_j^2 will not block. Players x_j^1 and x_j^2 could only improve by getting their favourite player y_j^1 or y_j^2 respectively, who, as we already know, cannot improve. Finally, players z_j^1 and z_j^2 need for improvement players x_j^1 and x_j^2 respectively, who will not cooperate as the previous sentence implies. ■

Lemma 5 *Let* $\Gamma(x_j) = \{x_j^1, x_j^2, y_j^1, y_j^2, z_j^1, z_j^2\}$ *be a variable cell. Let a core permutation* π *restricted to players of* $\Gamma(x_j)$ *be*

$$(x_j^1, y_j^1, x_j^2, y_j^2)(z_j^1, z_j^2)$$

then no player of $\Gamma(x_j)$ *will be in a blocking coalition.*

Proof. Player x_j^t, $t = 1, 2$ could only improve on cycle $(x_j^t, y_j^t, x_j^{t-1})$, but in this cycle player x_j^{t-1} would be worse off. Players y_j^t, $t = 1, 2$ could improve either by getting

- player $c(x_j^t)$ – but then the obtained cycle must also contain player x_j^t and have length at least 4, which means no improvement for player x_j^t and so no blocking coalition arises,
- or player z_j^t and cycle $(y_j^t, z_j^t, c(y_j^t), f(c(y_j^t)))$, but since $\pi \in Core(\Gamma)$, Lemma 1 implies that players $c(y_j^t), f(c(y_j^t))$ cannot improve.

Player z_j^t, $t = 1, 2$ needs either player x_j^{t-1} (who cannot improve) or $c(y_j^t)$ and hence also y_j^t, who will again not join. ∎

Now we show that there exists a permutation $\pi \in Core(\Gamma)$ such that each player in V has a shorter cycle than the one he has in the TTC permutation if and only if formula B is satisfiable.

Let B be satisfied by a boolean valuation φ. Take the first true literal c in each clause and partition the corresponding clause cell according to Lemma 1, getting a cycle of the form $(c, f(c), v(c), f(v(c)))$ (called a *decisive cycle*). Now, if a variable cell $\Gamma(x_j)$ is crossed by no decisive cycle, partition it according to Lemma 5. If a variable cell $\Gamma(x_j)$ is crossed by one decisive cycle, partition it according to Lemma 3. It is easy to see that no variable cell can be crossed by more than two decisive cycles and if it is crossed by two, it must be one of the cases described by Lemma 4. It is easy to see that each player of this game has a cycle of length less than 6 and what has been said above implies that $\pi \in Core(\Gamma)$.

Conversely, let $Core(\Gamma)$ contain a permutation π giving each player a cycle with length less than 6. Then each variable cell is partitioned according either to Lemma 3, or Lemma 4 or Lemma 5. Let us define the truth values by the rule: if π contains a decisive cycle containing player x_j^1 or player x_j^2, we make variable x_j true; if π contains a decisive cycle containing player z_j^1 or player z_j^2, we make variable x_j false; the truth values of other variables may be defined arbitrarily. Previous Lemmas imply that this definition is not contradictory. Further, Lemma 1 implies that each clause cell is crossed by at least one decisive cycle and hence the corresponding clause is satisfied. ∎

Another decision problem is:

Name: 3-CYCLES-KE

Instance: Arbitrary instance $\Gamma = (V, G, \mathcal{O})$ of the KE game with strict preferences

Question: Does $Core(\Gamma)$ contain a permutation π with $|C^\pi(i)| \leq 3$ for each $i \in V$?

Theorem 2 *Problem* 3-CYCLES-KE *is NP-complete.*

Proof. NP-completeness will be demonstrated by a polynomial transformation from the following well-known NP-complete problem (see [9]):

Name: EXACT 3-COVER

Instance: A finite set X, $|X| = 3q$ and a family $\mathcal{F}$ of three-element subsets of X.

Question: Does there exist a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that each element of X belongs to exactly one set from $\mathcal{F}'$?

Our proof is inspired by the proof of Theorem 3.7 in [9]. Suppose that the elements of X are ordered $x_1, x_2, \ldots, x_n$, $n = 3q$, $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ and that $F_i = \{x_i^1, x_i^2, x_i^3\}$. There will be 9 players a_i^k, b_i^k, c_i^k, $k = 1, 2, 3$ for each $F_i \in \mathcal{F}$ (forming the i^{th} cell) and one player denoted by x_j for each $x_j \in X$. Preferences of players are given in Figure 3:

$P(a_i^k) : b_i^k$	for $i = 1, 2, \ldots, m$ and $k = 1, 2, 3$
$P(b_i^k) : a_i^{k+1}, x_i^k, c_i^k$	for $i = 1, 2, \ldots, m$ and $k = 1, 2$
$P(b_i^3) : a_{i+1}^1, x_i^3, c_i^3$	for $i = 1, 2, \ldots, m$
$P(c_i^k) : a_i^k, c_i^{k+1}$	for $i = 1, 2, \ldots, m$ and $k = 1, 2, 3$ (modulo 3)
$P(x_j) : A(j), x_{j+1}$	for $j = 1, 2, \ldots, n$ (modulo n)

Figure 3: Preferences of players in Theorem 2

Symbols $A(j)$ in the preference lists of players x_j represent the a-players corresponding to those sets $F_i \in \mathcal{F}$ that contain element x_j (in arbitrary order).

In the TTC permutation for this game there is one cycle with $6m$ players: $(a_1^1, b_1^1, a_1^2, b_1^2, a_1^3, b_1^3, a_2^1, b_2^1, \ldots, a_m^1, b_m^1, a_m^2, b_m^2, a_m^3, b_m^3)$, the second cycle contains all the $x-$players in the basic order: $(x_1, x_2, \ldots, x_n)$ and there are m further cycles for the triples of c-players: $(c_1^1, c_1^2, c_1^3), \ldots, (c_m^1, c_m^2, c_m^3)$.

Now we show that the constructed game admits a core permutation π with cycles of length at most 3 if and only if the corresponding instance of EXACT 3-COVER has a solution.

Let $\mathcal{F}'$ be an exact cover. Consider permutation π defined as follows: For each i with $F_i \in \mathcal{F}'$ we have in the i^{th} cell four 3−cycles:

$$(c_i^1, c_i^2, c_i^3) \text{ and } (a_i^k, b_i^k, x_i^k) \text{ for } k = 1, 2, 3.$$

For each i such that $F_j \notin \mathcal{F}'$ the i^{th} cell contains three 3−cycles:

$$(a_i^k, b_i^k, c_i^k), k = 1, 2, 3.$$

Notice that each player is on a cycle of length 3 and since the constructed graph does not contain shorter cycles, the only possibility for a player y to improve would be by a permutation σ such that y prefers $\sigma(y)$ to $\pi(y)$. However, this is impossible for a-players, as $\pi(a_i^k) = f(a_i^k)$ for all i and k. Moreover, for all other players a better cycle must contain an a-player, hence no blocking coalition can arise.

Conversely, suppose that there exists a permutation giving each player a cycle of length 3. The proof of Theorem 3.7 in [9] implies that in this case there exists an exact cover. ■

Our third decision problem is:

Name: FULL-COVER-KE

Instance: Arbitrary instance $\Gamma = (V, G, \mathcal{O})$ of the KE game with strict preferences

Question: Does $Core(\Gamma)$ contain a permutation π that covers all players in V?

Theorem 3 *Problem* FULL-COVER-KE *is NP-complete.*

Proof. NP-completeness will be demonstrated by a polynomial transformation from RESTRICTED 3-SAT.

For each formula B we construct an instance $\Gamma = (V, G, \mathcal{O})$ of the KE game with the following properties. For each variable x_j there will be a *variable cell* of 4 *variable players* $x_j^1, x_j^2, x_j^3, x_j^4$. Players x_j^1, x_j^3 correspond to the first and to the second occurence of literal x_j, while players x_j^2, x_j^4 correspond to the first and to the second occurence of literal $\overline{x_j}$. For each clause K_k there is one *clause player* c_k. By v_k^1, v_k^2, v_k^3 we denote the variable players corresponding to the first, second and third literal of K_k. Conversely, the symbol $c(v)$ for a variable player v will denote the clause player corresponding to the clause containing the literal corresponding to player v.

Preferences of players are given in Figure 4.

$P(x_j^t) : x_j^{t+1}, c(x_j^{t-1}), x_j^{t-1}$	for $j = 1, 2, \ldots, n$, $t = 1, 2, 3, 4$ (modulo 4)
$P(c_k) : v_k^1, v_k^2, v_k^3$	for $k = 1, 2, \ldots, m$

Figure 4: Preferences of players in Theorem 3

The TTC permutation for this game consists of m cycles in variable cells of the form $(x_j^1, x_j^2, x_j^3, x_j^4)$, while all the clause players remain uncovered.

Now suppose that B is satisfied by a boolean valuation φ, we shall create a permutation $\pi \in Core(\Gamma)$ that covers all players. For each $k = 1, 2, \ldots, m$, take the first true literal (say the one in position s) in each clause K_k and take the corresponding entry v_k^s in the preference list of player c_k. Then create m cycles of the form $(v_k^s, f(v_k^s), c(v_k^s))$, called again *decisive* cycles.

In a variable cell which is not crossed by a decisive cycle, let π equal the TTC cycle. This variable cell will be denoted as type 0 cell. If a variable cell is crossed by one decisive cycle (type 1 variable cell), say $(x_j^t, x_j^{t+1}, c(x_j^t))$, we add to π cycle (x_j^{t+2}, x_j^{t+3}). As φ makes all clauses true, it is easy to see that π covers all players, since if the variable cell is crossed by two decisive cycles (type 2 variable cell), all its players are already covered. To show that $\pi \in Core(\Gamma)$ let us consider all types of variable cells in turn. (See Figure 5.)

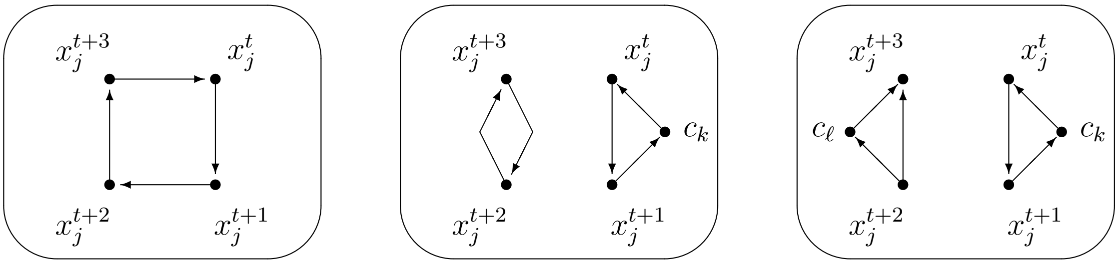

Figure 5: Variable cells of types 0,1, and 2

Type 0. For each player x_j^t we have $\pi(x_j^t) = f(x_j^t)$, hence these players can only improve by a shorter cycle, but in this case player $f(x_j^t)$ would be assigned his second or third choice and hence will be worse off. Consequently, no player of a variable cell of type 0 can be in a blocking coalition.

Type 1. Let the cycles of this variable cell be of the form given in the middle part of Figure 5. For player x_j^{t+2} we have $\pi(x_j^{t+2}) = f(x_j^{t+2})$ and the shortest possible cycle, hence he cannot improve. Further as $\pi(x_j^t) = f(x_j^t)$, player x_j^t could only improve on cycle (x_j^t, x_j^{t+1}), but in this case player x_j^{t+1} would be worse off. Player x_j^{t+1} needs for improvement player x_j^{t+2}, who, as we already know, will not cooperate. Finally, x_j^{t+3} could improve either on cycle $(x_j^{t+3}, c(x_j^{t+2}), x_j^{t+2})$ or on (x_j^{t+3}, x_j^t), but both cycles contain a player who would become worse off, hence no blocking coalition containing player x_j^{t+3} is possible.

Type 2. The argument for players of this variable cell is similar to the previous case.

Finally, as for the clause players only variable players are acceptable and no one of them will be in a blocking coalition, no blocking coalition at all is possible.

Conversely, let π be a core permutation covering all players of this game. Without loss of generality let us suppose that no clause contains simultaneously a literal as well as its negation. We shall show that each clause player is contained in a decisive 3-cycle. For, if say a clause player c_k were in a longer cycle C, then C must contain a part of one of the forms:

- $(\ldots, c_k, x_j^t, x_j^{t+1}, x_j^{t+2}, c_\ell, \ldots)$ with $\ell \neq k$. However, in this case player x_j^{t+3} must be alone – a contradiction with the assumption that π covers all players.
- $(\ldots, c_k, x_j^t, x_j^{t+1}, x_j^{t+2}, x_j^{t+3}, c_\ell, \ldots)$ with $\ell \neq k$. Now we have a blocking coalition $(x_j^t, x_j^{t+1}, x_j^{t+2}, x_j^{t+3})$.

Hence the decisive cycles create type-1 and type-2 variable cells, moreover, for type-2 variable cells the decisive cycles contain the opposite arcs of the variable cell. Therefore we set variable x_j to be true if $\pi(x_j^2) = c_k$ or $\pi(x_j^4) = c_k$ and we set x_j to be false if $\pi(x_j^1) = c_k$ or $\pi(x_j^3) = c_k$ for some k. It is easy to see that this definition of truth values is not contradictory and makes all clauses true. ■

5 Simulations

The results of the previous section give very little hope for an efficient algorithm able to give some information on the structure of $Core(\Gamma)$. On the other hand, examples presented in Section 3 show that the TTC algorithm may fail to find an 'optimal' permutation. Therefore we wanted to get at least some picture about the percentage of cases when the TTC permutation can be improved on.

Hence we performed some simulations. However, unlike in [19, 21, 22] our primary concern was not to simulate real patients' situation, we wanted rather to look at the KE game as a general cooperative game on graphs. Still we wanted to see how the size and an additional structure of the graph, which can be present in real-life data, influences the obtained results.

We generated 1000 samples, each containing $n = 20$ players and then we generated 1000 samples with $n = 60$ players, for each model considered. We used three types of models and in each one the value of parameter r, the *probability of rejection*, could vary.

The Random model represents general graphs. For each player i we first randomly ordered the remaining $n-1$ players and then for each pair (i, j) the arc (i, j) was added to the graph with probability $1 - r$.

The ABO model takes into account that players correspond to patient-donor pairs. Therefore for each vertex the ABO blood type of the patient as well as the ABO type of the donor was randomly and independently generated according to the frequency distribution given by the table in the left part of Figure 6 (this table was taken from [21] and rounded up to the nearest integer).

Blood type	Frequency (%)
O	48
A	34
B	14
AB	4

Blood type	Donor O	Donor A	Donor B	Donor AB
Patient O	14.0	37.8	12.0	2.0
Patient A	6.3	6.8	5.1	2.8
Patient B	2.4	6.1	1.2	2.1
Patient AB	0.5	0.5	0.2	0.1

Figure 6: Blood type frequences for ABO and ABO2 models

Based on the blood type of the patient corresponding to vertex i, the set of preliminary acceptable vertices $A(i)$ was defined according to the blood types of the donors corresponding to other vertices. Vertices in $A(i)$ were randomly permuted and the final preference list of i was obtained by including each $j \in A(i)$ with probability $1 - r$.

The ABO2 model does not choose the blood type of the patient and the donor corresponding to a given vertex independently, rather the ABO-type probability of the pair is chosen according to the table taken from [22] (the right half of Figure 6). Then the graph and preferences were constructed analogously as in the ABO model.

The ABO2 model represents the best approximation of the real data from all three considered models.

Afterwards, the TTC permutation π was computed for each generated preference profile. Then we performed two heuristics:

Cut-cycle heuristics. This heuristics tries to shorten the TTC cycles. We took each TTC cycle C of length at least 4 and tested each pair i, j of vertices on C, such that the TTC permutation π gave $\pi(i) \neq j$ and $\pi(j) \neq i$. Then a new permutation σ was created by the rule (see Figure 7)

$$\sigma(i) = \pi(j),\ \sigma(j) = \pi(i),\ \sigma(k) = \pi(k) \text{ for all } k \neq i, j.$$

Using the algorithm from [4] we tested whether $\sigma \in Core(\Gamma)$. If for at least one pair the answer was possitive, the instance was recorded as a success.

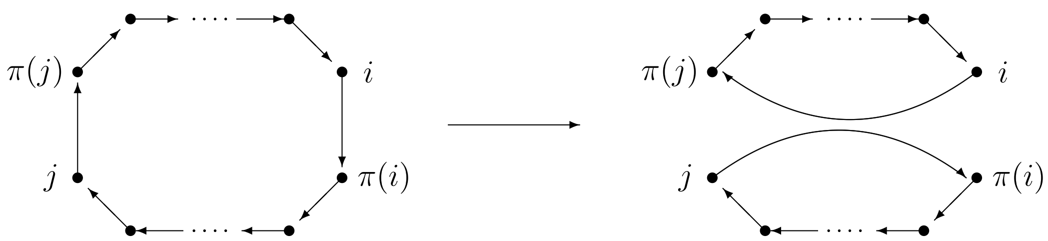

Figure 7: Cut-cycle heuristics

Cut-and-add heuristics. This heuristics tries to cover more vertices than the TTC permutation π does. We again considered each TTC cycle C with length at least 3. If the length of C was at least 4, we tested each triple i, j, k of vertices on C such that $\pi(i) \neq j$ and $\pi(j) \neq i$ and each uncovered vertex u. A new permutation σ was created by the rule

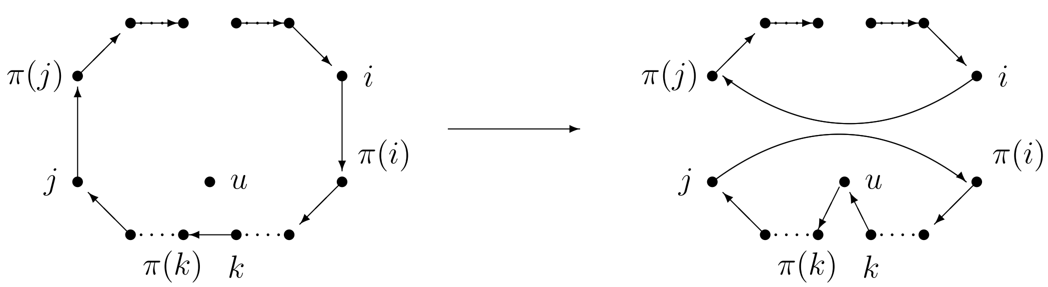

Figure 8: Cut-and-add heuristics

$$\sigma(i)=\pi(j),\ \sigma(j)=\pi(i),\ \sigma(k)=u,\ \sigma(u)=\pi(k), \sigma(\ell)=\pi(\ell) \text{ for all } \ell\neq i,j,k,u$$

(see Figure 8). For cycles C of length at least 3 we took any vertex $i \in C$ and an uncoverered vertex u and defined a new permutation σ by

$$\sigma(i)=\pi(\pi(i)),\ \sigma(\pi(i))=u,\ \sigma(u)=\pi(i),\ \sigma(\ell)=\pi(\ell) \text{ for all } \ell\neq i,\pi(i),u.$$

In both cases, a new vertex was covered and simultaneously an original TTC cycle was split into two shorter cycles. Then using the algorithm from [4] we tested whether $\sigma \in Core(\Gamma)$. Again, if in at least one case the obtained permutation was in $Core(\Gamma)$, the instance was recorded as a success.

	General graphs value of r			ABO value of r			ABO2 value of r		
	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6
Uncovered vertices (av. % per instance)	4,1	6,8	12,2	29,6	33,9	40,9	63,5	66,5	73,3
Cut-cycle (% of successes)	38,1	35,4	26,2	33,8	22,6	15,3	16,7	11,6	3,5
Cut-and-add (% of successes)	37,8	42,5	47,3	51,5	54,2	54,3	45,9	41,9	31,3

Figure 9: Average number of uncovered vertices and percentage of successes for Cut-cycle and Cut-and-add heuristics for samples with 20 players

	General graphs value of r			ABO value of r			ABO2 value of r		
	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6
Uncovered vertices (av. % per instance)	1,3	2,2	4,0	22,9	24,1	27,7	56,0	58,2	61,4
Cut-cycle (% of successes)	50,3	45,6	37,2	56,5	49,4	32,5	59,4	43,0	28,7
Cut-and-add (% of successes)	38,0	44,2	52,5	70,4	73,3	73,3	72,0	74,6	75,7

Figure 10: Average number of uncovered vertices and percentage of successes for Cut-cycle and Cut-and-add heuristics for samples with 60 players

The most important findings of our simulations are summarized in Figures 9 and 10. The simulations showed a relatively great number of cases when other than the TTC permutation in the core was found. As expected, the number of uncovered vertices grew as the number of arcs in the graphs decreased and consequently, the Cut-cycle heuristics proved to be in sparser graphs with a greater number of uncovered vertices less successful than the Cut-and-add heuristics. Also, if the number of vertices was greater, the percentage of cases when more core permutations were obtained was higher.

We would like also to remark that our implementation of these heuristics was far from optimal. When a new permutation σ was generated, we used the methods described in [4] to test whether $\sigma \in Core(\Gamma)$. However, as σ was a modification of another permutation in $Core(\Gamma)$, this test could be simplified and the computational complexity of the heuristics improved, but this was not the aim of this paper.

6 Conclusions

Section 4 contains some pessimistic results about the possibility to efficiently describe the core of the KE game. And still there are some other, equally interesting questions, which we left open, e.g.:

- Does $Core(\Gamma)$ contain *any* permutation different from the one given by TTC?
- Does $Core(\Gamma)$ contain a permutation π such that $|C^{\pi}(i)| = 2$ for each $i \in V$? This question is important especially because in the majority of real practical applications mostly paired kidney exchanges are used.

  Notice that if we consider a KE game as an instance of the Stable Roommates Problem [10], then π must be a stable matching, but not all stable matchings will belong to $Core(\Gamma)$. Consider e.g. the following preferences for 4 players:

  $$P(1) : 2, 3 \qquad P(2) : 3, 4 \qquad P(3) : 4, 1 \qquad P(4) : 1, 2$$

  The only stable matching in this example is $\{(1, 3), (2, 4)\}$, this is however blocked by the cycle (1,2,3,4).

Further, it turns out that it is even hard to approximate the maximum number of covered vertices by a core permutation [3].

7 Acknowledgement

We would like to thank to Dr. Ľuboslav Beňa, Transplantcentrum Košice, Slovakia, for detailed explanations of medical and ethical problems connected with transplantations of organs and to Jana Hajduková for valuable comments that helped to improve the presentation of the paper.

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