On planar graphs arbitrarily decomposable into closed trails*

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Abstract

A graph G is arbitrarily decomposable into closed trails (ADCT) if the following is true: Whenever $(l_1, \dots, l_p)$ is a sequence of integers adding up to $|E(G)|$ and there is a closed trail of length l_i in G for $i = 1, \dots, p$, then there is a sequence $(T_1, \dots, T_p)$ of pairwise edge-disjoint closed trails in G such that T_i is of length l_i for $i = 1, \dots, p$. In the paper it is proved that a $2n$-vertex bipyramid is ADCT for any integer $n \geq 3$. Further, if G is a 4-connected planar graph that is ADCT, it contains at most four edges incident only with faces of degree at least 4. There are examples showing that the bound of four edges is tight.

1 Introduction

In the paper we deal with simple finite nonoriented graphs and we use almost exclusively the standard terminology and notations of graph theory.

For $p, q \in \mathbb{Z}$ set $[p, q] := \{z \in \mathbb{Z} : p \leq z \leq q\}$ and $[p, \infty) := \{z \in \mathbb{Z} : p \leq z\}$. Let G be a graph. A *closed trail* of length $p \in [3, |E(G)|]$ (a *p-trail* for brevity) in G is a sequence $(v_0, v_1, \dots, v_{p-1}, v_p)$ of vertices of G in which $v_0 = v_p$, $v_i v_{i+1} \in E(G)$ and $v_i v_{i+1} \neq v_j v_{j+1}$ for $i, j \in [0, p-1]$, $i \neq j$. The set of edges $\{v_i v_{i+1} : i \in [0, p-1]\}$ induces an Eulerian subgraph of G and we shall identify that subgraph with T. For formal reasons the empty sequence ( ) will be considered to be a closed trail of length 0. If G is *even* (i.e., all vertices of G are of even degrees), G can be written as an edge-disjoint union of closed trails in G. Let $\bigcup_{i=1}^{k} T_i$ be such a union and let $l_i := |E(T_i)|$ be the length of T_i for $i \in [1, k]$; we say that the sequence $L := (l_1, \dots, l_k)$ is realisable in G and that $(T_1, \dots, T_k)$ is a G-realisation

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of L. Let $\mathrm{Lct}(G)$ be the set of all l's such that G contains a closed trail of length l and let $\mathrm{Sct}(G)$ be the set of all finite sequences with terms from $\mathrm{Lct}(G)$ whose sum equals $|E(G)|$. If $L = (l_1, \dots, l_k)$ is realisable in G, then $L \in \mathrm{Sct}(G)$. One can pose the following natural question: Given $L \in \mathrm{Sct}(G)$, is it G-realisable? If the answer is positive for any $L \in \mathrm{Sct}(G)$, G is said to be *arbitrarily decomposable into closed trails* (ADCT for short).

The first achievement on the topic of ADCT graphs is due to Balister, who proved in [1] that for odd n the complete graph K_n is ADCT and the same is true for even n and the graph $K_n - M_n$, where M_n is a perfect matching in K_n; the motivation came from chromatic graph theory, see Balister et al. [4]. A complete bipartite graph $K_{m,n}$ is even if and only if both m and n are even; all such $K_{m,n}$'s are ADCT (Horňák and Woźniak [10]). The situation becomes more complicated when passing to complete tripartite graphs. Namely, according to our paper [8], if $K_{p,q,r}$ with $p \le q \le r$ is ADCT, then $(p, q, r) \in \{(1, 1, 3), (1, 1, 5)\}$ or $p = q = r$; moreover, the graphs $K_{1,1,3}$, $K_{1,1,5}$ and $K_{p,p,p}$, where $p = 5 \cdot 2^l$, $l \in [0, \infty)$, are ADCT. Balister has shown in [2] that there are positive constants n and ε such that whenever G is an even graph with $|V(G)| \ge n$ and $\delta(G) \ge (1 - \varepsilon)|V(G)|$, then G is ADCT.

There are also natural analogues of ADCT graphs in the case of digraphs (Balister [3], Cichacz [5]) and pseudographs (Cichacz et al. [6]).

The *concatenation* of sequences $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_n)$ is the sequence $AB = (a_1, \dots, a_m, b_1, \dots, b_n)$. The concatenation is associative, and this fact justifies the use of $\prod_{i=1}^k A_i$ for the concatenation of $k \in [2, \infty)$ sequences $A_1, \dots, A_k$ in the order given by the sequence $(A_1, \dots, A_k)$. A sequence $A = (a_1, \dots, a_m)$ is *changeable* to a sequence $\hat{A} = (\hat{a}_1, \dots, \hat{a}_m)$ of the same length m, in symbols $A \sim \hat{A}$, if there is a bijection $\beta : [1, m] \to [1, m]$ such that $\hat{a}_i = a_{\beta(i)}$ for any $i \in [1, m]$. For a finite sequence S of real numbers we use $\sigma(S)$ to denote the sum of terms of S.

Consider a planar graph G and its plane embedding $\tilde{G}$ with sets $V(\tilde{G})$, $E(\tilde{G})$ and $F(\tilde{G})$ of vertices, edges and faces; throughout the whole paper we supose (without loss of generality) that $V(G) = V(\tilde{G})$. We shall denote by $\pi(\tilde{G})$ the plane of the embedding $\tilde{G}$. If $x \in V(\tilde{G}) \cup E(\tilde{G})$ and $f \in F(\tilde{G})$, $x \sim f$ (and vice versa $f \sim x$) will denote the fact that x and f are incident with each other. Let $V(f) := \{v \in V(\tilde{G}) : v \sim f\}$, and, for $e \in E(\tilde{G})$, let $F(e) := \{f \in F(\tilde{G}) : f \sim e\}$. The *degree* of f is $\deg(f) := \sum_{e \sim f}(3 - |F(e)|)$; if G is 2-connected, f is bounded by a cycle (called *facial*) and $\deg(f) = |V(f)|$. A *d-face* (a *d-vertex*) is a face (a vertex) of degree d; by $f_d(\tilde{G})$ and $v_d(\tilde{G})$ we denote the number of d-faces and that of d-vertices, respectively, of $\tilde{G}$. If G is 3-connected, then, by a well known result of Whitney, a plane embedding $\tilde{G}$ of G is unique in such a sense that for any edge $e \in E(G)$ there is a (unique) multiset $\{d_1, d_2\}$ (*degree multiset* of e, in symbols $\mathrm{dms}(e)$) such that (the image of) e is in $\tilde{G}$ incident with faces f_1 and f_2 satisfying $\deg(f_i) = d_i$, $i = 1, 2$. In such a case we define $E_{4+}(G) := \{e \in E(G) : 3 \notin \mathrm{dms}(e)\}$.

Suppose that T_1, T_2 are edge-disjoint closed trails in a graph G and denote by $T_1 + T_2$ the set of all closed trails T in G with $E(T) = E(T_1) \cup E(T_2)$. Evidently, the set $T_1 + T_2$ is nonempty if and only if $V(T_1)$ and $V(T_2)$ are non-disjoint.

2 Preparatory results

The following two easy statements are taken from our paper [8].

Lemma 1 *If G is a graph, $L_1, L_2 \in \mathrm{Sct}(G)$ and $L_1 \sim L_2$, then L_1 is G-realisable if and only if L_2 is G-realisable.* ■

Lemma 2 *If G is an even graph, then* $\mathrm{Lct}(G) \subseteq [3, |E(G)| - 3] \cup \{|E(G)|\}$. ■

Fijavž et al. proved in [7] the following theorem:

Theorem 3 *If G is a planar graph of minimum degree at least four, then G contains a 3-cycle, a 5-cycle, and a 6-cycle.* ■

The additional assumption of 2-connectedness of G enables us to prove a little bit more.

Theorem 4 *If G is a 2-connected planar graph of minimum degree at least four, then G contains a 4-cycle or a 7-cycle.*

Proof. Suppose that G contains neither a 4-cycle nor a 7-cycle and consider a plane embedding $\tilde{G}$ of G.

Consider a 5-face $p \in F(\tilde{G})$ with a boundary cycle $(w_1, w_2, w_3, w_4, w_5, w_1)$. Let us first show that if $g \in F(\tilde{G})$ is a 3-face with $|V(p) \cap V(g)| \geq 2$, then p is adjacent to g and $|V(p) \cap V(g)| = 2$. If p is not adjacent to g, then we may suppose without loss of generality that $V(p) \cap V(g) = \{w_1, w_3\}$. Then (w_1, w_2, w_3, w, w_1), where $w \in V(g) - V(p)$, is a 4-cycle in $\tilde{G}$, a contradiction. Thus p is adjacent to g, say $\{w_1, w_2\} \subseteq V(g)$. If $|V(p) \cap V(g)| = 3$, then, since $\deg(w_i) > 2$, $i = 1, 2$, the third vertex of g must be w_4. In such a case, however, $(w_1, w_2, w_3, w_4, w_1)$ is a 4-cycle in $\tilde{G}$, a contradiction.

Our next claim is that p is adjacent to at most one 3-face. Suppose that p is adjacent to 3-faces g_1, g_2, $g_1 \neq g_2$. Then $|V(p) \cap V(g_1)| = |V(p) \cap V(g_2)| = 2$ and we may suppose without loss of generality that $V(g_1) = \{w_1, w_2, x_1\}$ and $V(g_2) = \{w_i, w_{i+1}, x_2\}$, where $i \in [2, 3]$ and $x_1, x_2 \notin V(p)$. If $x_1 = x_2$, then $(w_1, w_2, w_3, x_1, w_1)$ is a 4-cycle in $\tilde{G}$; on the other hand, if $x_1 \neq x_2$, then the subgraph of $\tilde{G}$ induced by $V(p) \cup \{x_1, x_2\}$ contains a 7-cycle, in both cases a contradiction.

Since the graph $\tilde{G}$ is 2-connected, any face $f \in F(\tilde{G})$ is incident with $\deg(f)$ vertices. Therefore, $\sum_{v\sim f} \frac{1}{\deg(f)} = 1$ and

$$|F(\tilde{G})| = \sum_{f\in F(\tilde{G})} \sum_{v\sim f} \frac{1}{\deg(f)} = \sum_{v\in V(\tilde{G})} \sum_{f\sim v} \frac{1}{\deg(f)}.$$

Moreover, $|V(\tilde{G})| = \sum_{v\in V(\tilde{G})} 1$ and $2|E(\tilde{G})| = \sum_{v\in V(\tilde{G})} \deg(v)$, hence Euler's formula $|V(\tilde{G})| - |E(\tilde{G})| + |F(\tilde{G})| = 2$ can be rewritten as

$$\sum_{v\in V(\tilde{G})} c(v) = 2, \tag{1}$$

where $c : V(\tilde{G}) \to \mathbb{Q}$ is a rational valued map defined by

$$c(v) := 1 - \frac{1}{2}\deg(v) + \sum_{f\sim v} \frac{1}{\deg(f)}.$$

Consider a vertex $v \in V(\tilde{G})$ of degree d. Let $f_1, \dots, f_d$ be faces incident with v in a cyclic order around v and suppose that $\deg(f_1) \le \deg(f_i)$ for any $i \in [2, d]$. For $i \in [1, d]$ let vv_i be the common edge of f_i and f_{i+1} (where indices are taken modulo d in the set $[1, d]$). If $\deg(f_1) \ge 5$, then $c(v) \le 1 - \frac{d}{2} + \frac{d}{5} = 1 - \frac{3d}{10} \le -\frac{1}{5}$.

In the sequel we suppose that $\deg(f_1) = 3$. A *v-section* is a subsequence $(f_i, \dots, f_j)$ of the sequence $(f_1, \dots, f_d)$ such that $\deg(f_i) = \deg(f_{j+1}) = 3$ and $\deg(f_k) \ge 5$ for any $k \in [i+1, j]$. If $\deg(f_i) = 3$, then $\deg(f_{i+1}) \ge 5$, for otherwise $(v, v_{i-1}, v_i, v_{i+1}, v)$ would be a 4-cycle in $\tilde{G}$. Thus, any v-section is of length at least two. Let s be the number of v-sections and let $(S_1, \dots, S_s)$ be the natural sequence of v-sections: S_1 starts with f_1, and, if S_l ends with f_m, then S_{l+1} starts with f_{m+1}. Provided that $S_p = (f_q, \dots, f_r)$, we have $\sum_{k=q}^{r} \frac{1}{\deg(f_k)} = (r + 1 - q)\sigma_p$, where σ_p is the mean value of the fraction $\frac{1}{\deg(f_k)}$ for $k \in [q, r]$. Let $l_p := r + 1 - q$ denote the length of the v-section S_p. If l_p=2, then from the claim above we know that $\deg(f_r) = \deg(f_{q+1}) \ge 6$, and so $\sigma_p \le \frac{1}{2}(\frac{1}{3} + \frac{1}{6}) = \frac{1}{4}$. On the other hand, if $l_p \ge 3$, then $\sigma_p \le \frac{1}{l_p}(\frac{1}{3} + \frac{l_p-1}{5}) = \frac{1}{5} + \frac{2}{15l_p} \le \frac{11}{45} < \frac{1}{4}$. Therefore, $\sum_{k=1}^{d} \frac{1}{\deg(f_k)} = \sum_{k=1}^{s} l_k\sigma_k \le \sum_{k=1}^{s} \frac{l_k}{4} = \frac{d}{4}$ and $c(v) = 1 - \frac{d}{2} + \sum_{k=1}^{d} \frac{1}{\deg(f_k)} \le 1 - \frac{d}{2} + \frac{d}{4} = 1 - \frac{d}{4} \le 0$.

Since $c(v) \le 0$ for each $v \in V(\tilde{G})$, we have obtained a contradiction with (1). ■

3 Four-connected planar graphs

Proposition 5 *If G is a 4-connected planar graph and T is a closed trail of length 3 in G, then $E_{4+}(G) \cap E(T) = \emptyset$.*

Proof. Consider a plane embedding $\tilde{G}$ of G and the closed trail $\tilde{T}$ (of length 3) in $\tilde{G}$ corresponding to T. Assume that $xy \in E_{4+}(\tilde{G}) \cap E(\tilde{T})$ and let $x_1, x_2 = y, \dots, x_d$

be neighbours of x in a cyclic order around x. Then $V(\tilde{T}) = \{x, x_2, x_k\}$ for some $k \in [1, d] - \{2\}$. Consider the closed Jordan curve $J := V(\tilde{T}) \cup \bigcup_{e \in E(\tilde{T})} e$ in $\pi(\tilde{G})$ and the regions π_1, π_2 of $\pi(\tilde{G})$ cut off by J. If $k \in \{1, 3\}$, then the face f of $\tilde{G}$ incident with xy and xx_k is of degree at least 4; there is $i \in [1, 2]$ such that all $\deg(f) - 3$ vertices of $V(f) - \{x, x_2, x_k\}$ lie in the region π_i and x_d lies in π_{3-i}. On the other hand, if $k \in [4, d]$, there is $j \in [1, 2]$ such that x_1 lies in π_j and x_3 in π_{3-j}. Thus, in both cases $\tilde{G} - V(\tilde{T})$ is disconnected in contradiction with 4-connectedness of $\tilde{G}$. ■

Proposition 6 *Suppose that a* 4*-connected planar graph* G *is* ADCT *and* $(k)(3)^r \in \mathrm{Sct(G)}$*. Then the following hold:*

1. *If* $k \geq 4$*, then* $|E_{4+}(G)| \leq k$*.*
2. *If* $k = 3$*, then* $E_{4+}(G) = \emptyset$*.*

Proof. There exists a G-realisation $(T_1, \ldots, T_{r+1})$ of the sequence $(k)(3)^r$ (in which T_1 is of length k). By Proposition 5 we have $E_{4+}(G) \cap \bigcup_{j=2}^{r+1} E(T_j) = \emptyset$ and $E_{4+}(G) \subseteq E(T_1)$. If $k \geq 4$, then $k = |E(T_1)| \geq |E_{4+}(G)|$. If $k = 3$, then we have also $E_{4+}(G) \cap E(T_1) = \emptyset$, and so $E_{4+}(G) = \emptyset$. ■

Theorem 7 *If a* 4*-connected planar graph is* ADCT*, it contains a* 4*-cycle.*

Proof. Suppose G is a 4-connected planar graph that does not contain any 4-cycle and let $\tilde{G}$ be a plane embedding of G. From Euler's formula for $\tilde{G}$ one can easily derive that $\sum_{i=3}(4-i)(f_i(\tilde{G})+v_i(\tilde{G})) = 8$, hence $f_3(\tilde{G}) = 8+\sum_{i=5}^{\infty}(i-4)(f_i(\tilde{G}) + v_i(\tilde{G})) \geq 8$. A 3-face of $\tilde{G}$ is adjacent only to faces of degrees at least five, and so $\sum_{i=5}^{\infty} i f_i(\tilde{G}) \geq 3f_3(\tilde{G}) \geq 24$. Thus, $|E(\tilde{G})| = \frac{1}{2}\sum_{i=3}^{\infty} i f_i(\tilde{G}) \geq 24$. By Theorems 3 and 4 we have $3, 5, 7 \in \mathrm{Lct}(G)$. Let $m \in [0, 2]$ be such that $|E(G)| \equiv m \pmod 3$, put $(e_0, e_1, e_2) := (0, 7, 5)$ and $r_m := \frac{|E(G)|-e_m}{3} \geq 6$. Then the set Sct(G) contains the sequence R_m, where $R_0 := (3)^{r_0}$ and $R_j := (e_j)(3)^{r_j}$, $j = 1, 2$. By Proposition 6 we have $|E_{4+}(G)| \leq e_m$.

Let $(p_0, p_1, p_2) := (3, 2, 4)$ and $(s_0, s_1, s_2) := (r_0 - 5, r_1 - 1, r_2 - 5) \in [2, \infty)^3$. The set $\mathrm{Sct}(G)$ contains also the sequence $S_m := (5)^{p_m}(3)^{s_m}$. Let $\mathcal{T} = (T_1^5, \ldots, T_{p_m}^5, T_1^3, \ldots, T_{s_m}^3)$ be a G-realisation of S_m. Because of $|E_{4+}(G)| \leq e_m$ we may suppose without loss of generality that $|E(T_1^5) \cap E_{4+}(G)| \leq l_m := \lfloor \frac{e_m}{p_m} \rfloor$. Thus, if $T_1^5 = (x_1, x_2, x_3, x_4, x_5, x_1)$, there is a decomposition $\{I_1, I_2\}$ of the set $[1, 5]$ such that $|I_1| \leq l_m$ and $x_i x_{i+1} \in E_{4+}(G) \Leftrightarrow i \in I_1$ (with indices taken modulo 5 in the set [1,5]). Moreover, the 5-cycle T_1^5 has no chords (otherwise there would be a 4-cycle in G). Therefore, for any $j \in I_2$ there is a vertex $v_j \notin V(T_1^5)$ such that $\{(x_j, x_{j+1}, v_j, x_j) : j \in I_2\}$ is a system of pairwise edge-disjoint facial 3-cycles in G (to see that $i \neq j$ for $i, j \in I_2$ implies $v_i \neq v_j$ we repeat the reasoning concerning the 5-face p in the proof of Theorem 4), and so $|\bigcup_{i \in I_2}\{v_i\}| = |I_2|$. Since G contains no 4-cycle, it is easy to see that $E_i^- := \{v_i x_j : j \in [1, 5] - \{i\}, j \not\equiv i + 1 \pmod 5\}$ with $i \in I_2$ and $E^- := \{v_i v_j : i, j \in I_2, i \neq j\}$ are sets of non-edges of G. Thus, the set of edges $E := \{v_i x_{i+j} : i \in I_2, j \in [0,1]\}$ is such that $E \cap E(T_i^5)$ induces a connected subgraph G_i of G for any $i \in [2, p_m]$. Indeed, provided that $e_1, e_2 \in E \cap E(T_i^5)$ belong to distinct components of G_i, at least one of the remaining three edges of T_i^5, say e, is such that both its vertices are in $\{x_j : j \in [1,5]\} \cup \bigcup_{j \in I_2} \{v_j\}$; thus, either e is a chord of T_1^5 or $e \in E^- \cup \bigcup_{j \in I_2} E_j^-$, in both cases a contradiction. Since the subgraph of G induced by the set of edges E is a union of paths or C_{10}, G_i must be a path for every $i \in [2, p_m] \neq \emptyset$. If G_i is of length 3 or 4, a 4-cycle in G can easily be found. So, $|E \cap E(T_i^5)| \leq 2$ for all $i \in [2, p_m]$, and, since $|I_2| = 5 - |I_1| \geq 5 - l_m$, the number of edges that do not belong to 5-trails of $\mathcal{T}$ is at least $2(5 - l_m) - 2(p_m - 1) \geq 2$. Hence, there is $i \in I_2$ and $j \in [0,1]$ such that the edge $v_i x_{i+j}$ belongs to a 3-trail (v_i, x_{i+j}, x, v_i) of $\mathcal{T}$ with $x \neq x_{i+1-j}$ so that $(v_i, x_{i+j}, x, x_{i+1-j}, v_i)$ is a 4-cycle in G, a contradiction. ■

Theorem 8 *Let G be a 4-connected planar graph that is* ADCT *and let* $|E(G)| \equiv m \pmod 3$, $m \in [0,2]$. *Then the following hold:*

1. *If $m = 0$, then $E_{4+}(G) = \emptyset$.*
2. *If $m \in [1,2]$, then $|E_{4+}(G)| \leq 4$ and the bound is tight.*

Proof. By Theorems 3 and 7 we have $3, 4, 5 \in \mathrm{Lct}(G)$. Let $\tilde{G}$ be a plane embedding of G. As in the proof of Theorem 6 we obtain $f_3(\tilde{G}) \geq 8$, and so $q := |E(G)| = \frac{1}{2} \sum_{i=3}^{\infty} i f_i(\tilde{G}) \geq 12$.

1. If $m = 0$, then $(3)^{\frac{q}{3}} \in \mathrm{Sct}(G)$, hence we are done by Proposition 6.2.

2. If $m \in [1,2]$, then $\frac{q-m-3}{3} \geq 3$, $(m+3)(3)^{\frac{q-m-3}{3}} \in \mathrm{Sct}(G)$, and so $|E_{4+}(G)| \leq m + 3$. Provided that $m = 1$, we have obtained the desired inequality. In the case $m = 2$ suppose $|E_{4+}(G)| = 5$ and consider $\tilde{G}$-realisations $(T_1^1, \dots, T_r^1)$ and $(T_1^2, \dots, T_r^2)$ of the sequences $(5)(3)^{r-1}$ and $(4)^2(3)^{r-2}$ with $r := \frac{q-5}{3} \geq 3$. From Proposition 5 it follows that $E(T_1^1) = E_{4+}(\tilde{G}) \subseteq E(T_1^2) \cup E(T_2^2)$, hence there is $i \in [1,2]$ such that $3 \leq |E_{4+}(\tilde{G}) \cap E(T_i^2)| = |E(T_1^1) \cap E(T_i^2)|$. Thus, if $T_1^1 = (x_1, x_2, x_3, x_4, x_5, x_1)$, without loss of generality we may suppose that $T_i^2 = (x_1, x_2, x_3, x_4, x_1)$. Since $x_4 x_5 \in E_{4+}(\tilde{G})$, any region cut off from $\pi(\tilde{G})$ by the closed Jordan curve $J := \{x_1, x_4, x_5\} \cup \{x_1 x_4, x_4 x_5, x_5 x_1\}$ (here $x_i x_j$ is an open arc between points x_i and x_j) contains at least one vertex (if f is the face of $\tilde{G}$ incident with $x_4 x_5$ and lying in that region, then $\deg(f) \geq 4$, and so $V(f) - \{x_1, x_4, x_5\} \neq \emptyset$). Thus $\tilde{G} - \{x_1, x_4, x_5\}$ is disconnected in contradiction with 4-connectedness of $\tilde{G}$.

Consider the planar graph G_m, $m = 1, 2$, whose plane embedding is presented in Fig. $m+1$ in such a way that grey regions are isomorphic to a disc embedding of the graph F_{23} depicted in Fig. 1 (and the vertices denoted by v in Fig. 2 are to be identified). In our paper [9] it is proved that G_m is a 4-connected planar graph that is ADCT and satisfies $|E_{4+}(G_m)| = 4$ (the edges in $E_{4+}(G_m)$ are dashed). Since $|E(G_1)| = 850 \equiv 1 \pmod 3$ and $|E(G_2)| = 629 \equiv 2 \pmod 3$, the bound $|E_{4+}(G)| \leq 4$ in our Theorem is indeed tight. ■

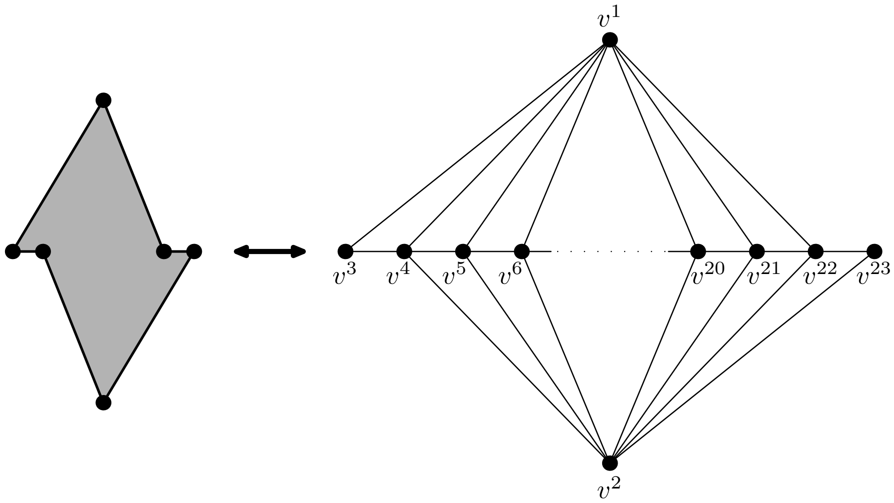

Figure 1: The graph F_{23}

4 Bipyramids

Let B_m denote the m-vertex bipyramid, $m \in [5, \infty)$. If $m = 2n$ is even, $B_{2n} = \bigcup_{i=1}^{2n-2} C_3^i$ is the edge-disjoint union of $2n-2$ C_3's, see Fig. 4. For $p, q \in [1, 2n-2]$ the graph $C(p, q) := \bigcup_{i=p}^{q} C_3^i$ is connected and even (possibly empty, if $q < p$). We have the following evident assertions:

Proposition 9 *If $n \in [3, \infty)$, $r \in [1, \infty)$ and $(p_1, \dots, p_{2r})$ is a sequence of integers from $[1, 2n-2]$ satisfying $p_{2i} \geq p_{2i-1} - 1$ for every $i \in [1, r]$ and $p_{2i+1} = p_{2i} + 1$ for every $i \in [1, r-1]$, then $C(p_1, p_{2r})$ is the edge-disjoint union of r graphs $C(p_{2i-1}, p_{2i})$ with $i \in [1, r]$.* ■

Proposition 10 *If $n \in [3, \infty)$, $p_1, q_1, p_2, q_2 \in [1, 2n-2]$ and $q_1 - p_1 = q_2 - p_2$, then $C(p_1, q_1)$ is isomorphic to $C(p_2, q_2)$.* ■

Lemma 11 *If $n \in [3, \infty)$, $l_1, l_2, l_3 \in [4, 6n-14]$, there is $j \in [1, 2]$ such that $l_i \equiv j \pmod 3$, $i = 1, 2, 3$, and $l_1 + l_2 + l_3 = 3l \leq 6n - 6$, then the sequence (l_1, l_2, l_3) is realisable in $C(1, l)$.*

Proof. Because of Lemma 1 we may suppose without loss of generality that if the sequence $L := (l_1, l_2, l_3)$ contains the term $j+6$, then all such terms are at the beginning of L. Put $p_1 := 1$, $p_2 := j+3$, $p_{2i+1} := j+3+\sum_{k=1}^{i-1} \frac{l_k-j-3}{3}+1$ and $p_{2i+2} := j+3+\sum_{k=1}^{i} \frac{l_k-j-3}{3}$, $i = 1, 2, 3$, and consider a $C(1, j+3)$-realisation (T_1^j, T_2^j, T_3^j) of the sequence $(j+3)^3$, where $T_1^1 := (v^1, v^3, v^4, v^5, v^1)$, $T_2^1 := (v^2, v^5, v^6, v^7, v^2)$, $T_3^1 := (v^1, v^4, v^2, v^6, v^1)$, $T_1^2 := (v^2, v^6, v^1, v^3, v^4, v^2)$, $T_2^2 := (v^1, v^5, v^6, v^7, v^8, v^1)$, $T_3^2 := (v^2, v^7, v^1, v^4, v^5, v^2)$. Further, let T_i' be a closed Eulerian trail in the graph $C(p_{2i+1}, p_{2i+2})$ (of length $l_i - j - 3$), $i = 1, 2, 3$. From our assumption it follows

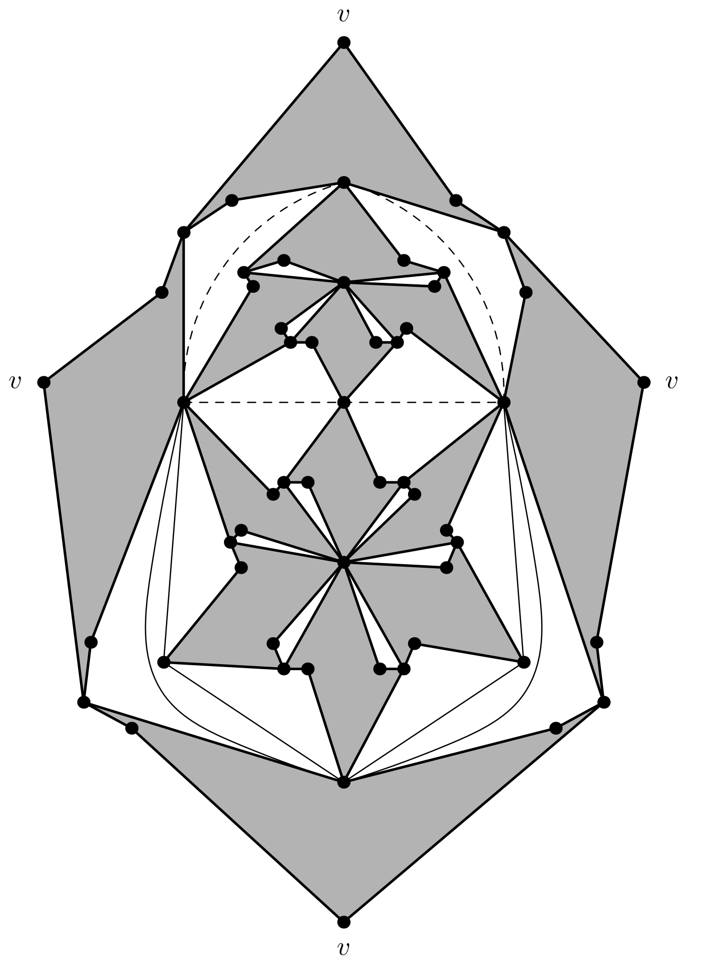

Figure 2: The graph G_1

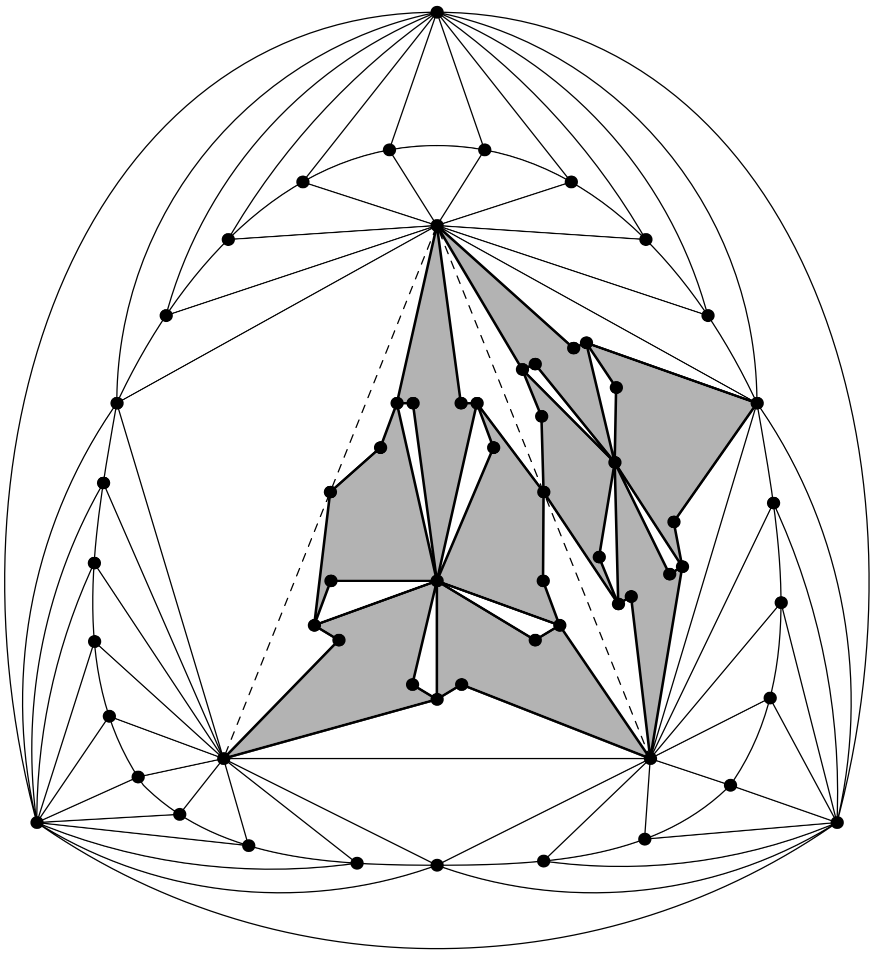

Figure 3: The graph G_2

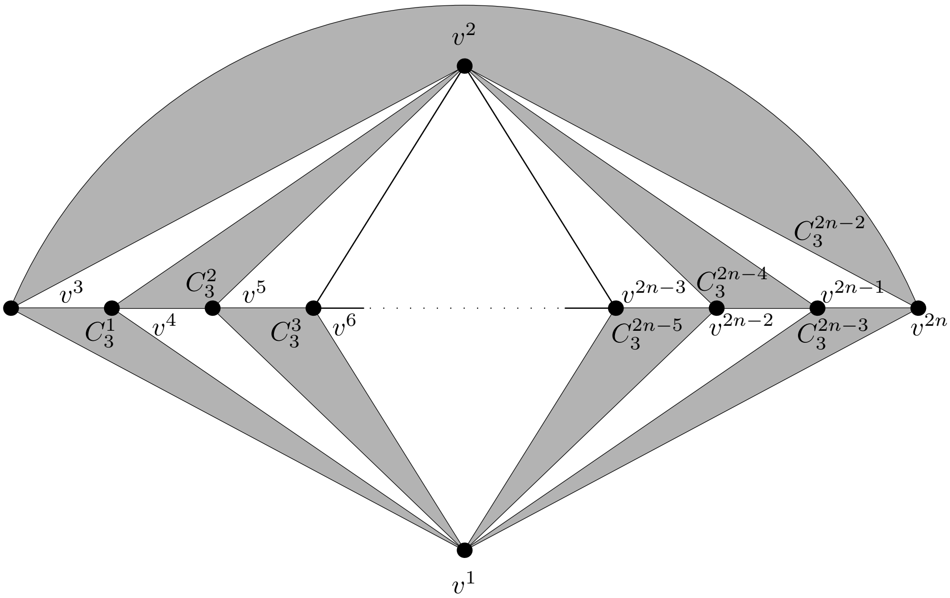

Figure 4: The graph B_{2n}

that if the trail T'_i is non-empty and $i \equiv k \pmod 2$, $k \in [0,1]$, then T'_i contains the vertex v^{j+1-k}; note that if T'_i is of length at least 6, it contains both v^1 and v^2. Since $v^j \in V(T_1^j)$, $v^{3-j} \in V(T_2^j)$ and $v^j \in V(T_3^j)$, there is a trail $T_i \in T_i^j + T'_i$ of length $j + 3 + (l_i - j - 3) = l_i$, $i = 1, 2, 3$. We have $p_2 > p_1$, $p_{2i+2} = p_{2i+1} + \frac{l_i - j - 3}{3} - 1 \geq p_{2i+1} - 1$, $p_{2i+1} = p_{2i} + 1$, $i = 1, 2, 3$, and $p_8 = l$, and so, by Proposition 9, (T_1, T_2, T_3) is a $C(1, l)$-realisation of L. ■

Lemma 12 *If $n \in [3, \infty)$, $l_1, l_2 \in [4, 6n - 10]$ are such that $l_i \not\equiv 0 \pmod 3$, $i = 1, 2$, and $l_1 + l_2 = 3l \leq 6n - 6$, then the sequence (l_1, l_2) is realisable in $C(1, l)$.*

Proof. Again by Lemma 1 we may assume that if the sequence $L := (l_1, l_2)$ contains the term 7 or the term 8, then all such terms are at the beginning of L. Let $j \in [1, 2]$ be such that $l_1 \equiv j \pmod 3$ and $l_2 \equiv 3 - j \pmod 3$. Put $p_1 := 1$, $p_2 := 3$, $p_3 := 4$, $p_4 := 3 + \frac{l_1 - j - 3}{3}$, $p_5 := 4 + \frac{l_1 - j - 3}{3}$ and $p_6 := 3 + \frac{l_1 - j - 3}{3} + \frac{l_2 + j - 6}{3} = l$ and let (T_1^1, T_2^1) be the $C(1, 3)$-realisation of the sequence $(j + 3, 6 - j)$, where $T_1^1 := (v^1)[\prod_{k=5-j}^{4}(v^k)](v^2, v^5, v^1)$ and $T_2^1 := (v^1)[\prod_{k=j+2}^{6}(v^k)](v^1)$. Moreover, consider a closed Eulerian trail T'_i in the graph $C(p_{2i+1}, p_{2i+2})$, $i = 1, 2$. If the trail T'_i is non-empty, it contains the vertex v^{3-i}, $i = 1, 2$. In such a case there is a trail $T_i \in T_i^1 + T'_i$, (note that $v^{3-i} \in V(T_i^1)$), $i = 1, 2$; T_1 is of length $j + 3 + (l_1 - j - 3) = l_1$ and T_2 is of length $6 - j + (l_2 + j - 6) = l_2$. Thus, similarly as in the proof of Lemma 11, (T_1, T_2) is a $C(1, l)$-realisation of L. ■

Lemma 13 *If $n \in [3, \infty)$, $s \in [1, 2n-2]$, $l_i \in [3, 6n-6]$ and $l_i \equiv 0 \pmod 3$ for every $i \in [1, s]$ and $\sum_{i=1}^{s} l_i = 3l \leq 6n-6$, then the sequence $\prod_{i=1}^{s}(l_i)$ is realisable in $C(1, l)$.*

Proof. Let T_i be a closed Eulerian trail in the graph $C(p_{2i-1}, p_{2i})$ where $p_{2i-1} := \sum_{k=1}^{i-1} \frac{l_k}{3} + 1$ and $p_{2i} := \sum_{k=1}^{i} \frac{l_k}{3}$ for each $i \in [1, s]$. Then T_i is of length l_i and $\prod_{i=1}^{s}(T_i)$ is a realisation of $\prod_{i=1}^{s}(l_i)$ in the graph $C(1, l)$. ■

Theorem 14 *The graph B_m is* ADCT *if and only if $m \equiv 0 \pmod 2$.*

Proof. If the graph B_m is ADCT, it is even, and so $\deg_{B_m}(v_1) = m - 2 \equiv 0 \pmod 2$. Suppose now that $m = 2n$ is even. By Lemma 2 we have $\mathrm{Lct}(B_{2n}) \subseteq [3, 6n-9] \cup \{6n-6\} =: \mathcal{B}_{2n}$. We are going to show that any sequence L of integers of $\mathcal{B}_{2n}$ adding up to $6n - 6 = |E(B_{2n})|$ is B_{2n}-realisable (which in fact implicitly proves that $\mathrm{Lct}(B_{2n}) = \mathcal{B}_{2n}$). Let m_i be the number of terms of L that are congruent to i modulo 3, $i = 0, 1, 2$. Since $m_1 - m_2 \equiv m_1 + 2m_2 \equiv |E(B_{2n})| \equiv 0 \pmod 3$, m_1 and m_2 are in the same congruence class modulo 3. Let $j \in [0, 2]$ be such that $m_i \equiv j \pmod 3$, $i = 1, 2$. Then $L \sim L_{1,2}L_0L_1L_2$ where $L_{1,2}$ consists of $2j$ terms belonging alternatingly to the congruence classes 1 and 2 modulo 3 and L_i is formed from remaining terms in the congruence class i modulo 3, $i = 0, 1, 2$.

Put $p_1 := 1$, $p_2 := \sigma(L_{1,2})/3$, $p_4 := \sigma(L_{1,2}L_0)/3$, $p_6 := \sigma(L_{1,2}L_0L_1)/3$, $p_8 := \sigma(L)/3$ and $p_{2i+1} := p_{2i} + 1$, $i = 1, 2, 3$. By Lemma 12 applied to j alternating sequences (in the above sense of length 2 concatenating to $L_{1,2}$ and by Proposition 10 we see that $L_{1,2}$ has a realisation $\mathcal{T}_{1,2}$ in the graph $C(p_1, p_2)$: if $L_{1,2} = (l_1, l_2, l_3, l_4)$, then (l_1, l_2) is realisable in the graph $C(p_1, (l_1 + l_2)/3)$ and (l_3, l_4) in the graph $C(1, (l_3 + l_4)/3)$ that is isomorphic to $C((l_1 + l_2)/3 + 1, p_2)$. By Lemma 13, the sequence L_0 has a realisation $\mathcal{T}_0$ in the graph $C(p_3, p_4)$. Further, by Lemma 11 applied to $(m_i - j)/3$ sequences of length 3 concatenating to L_i, and by Proposition 10, the sequence L_i has a $C(p_{2i+3}, p_{2i+4})$-realisation $\mathcal{T}_i$, $i = 1, 2$. Using Proposition 9 then $\mathcal{T}_{1,2}\mathcal{T}_0\mathcal{T}_1\mathcal{T}_2$ is a realisation of the sequence $L_{1,2}L_0L_1L_2 \sim L$ in the graph $C(1, \sigma(L)/3) = B_{2n}$ and we are done by Lemma 1. ■

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