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Rainbow Faces in Edge Colored Plane Graphs

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Abstract

A face of an edge colored plane graph is called *rainbow* if all its edges receive distinct colors. The maximum number of colors used in an edge coloring of a connected plane graph G with no rainbow face is called *the edge-rainbowness* of G. In this paper we prove that the edge-rainbowness of G equals to the maximum number of edges of a connected bridge face factor H of G, where a *bridge face factor* H of a plane graph G is a spanning subgraph H of G in which every face is incident with a bridge and the interior of any one face $f \in F(G)$ is a subset of the interior of some face $f' \in F(H)$. We also show upper and lower bounds on the edge-rainbowness of graphs based on edge connectivity, girth of the dual G^* and other basic graph invariants. Moreover, we present infinite classes of graphs where these equalities are attained.

Keywords: edge coloring, rainbowness, rainbow coloring, factors,

1 Introduction

A subgraph of an edge colored plane graph is said to be *monochromatic* if all its edges have the same color. On the other hand if no two edges have the same color then such a subgraph is called *rainbow*, which is sometimes

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said *heterochromatic* or *multicolored*. Jamison, Jiang and Ling in [12], and Chen considered the parameter $f(G, H)$, the minimum n such that every edge-colored K_n contains either a monochromatic copy of G or a rainbow copy of H. Alon et al in [3] studied the function $f(H)$ which is the minimum integer n such that any proper edge coloring of K_n has a rainbow copy of H. Thomason and Wagner [24] considered colorings of the edges of complete graphs K_n that contain no rainbow paths P_{t+1} of lengths t. They showed that if at least t colors are used, then very few colorings are possible if $t \leq 5$ and these can be described precisely, whereas the situation for $t \geq 6$ is qualitatively different. Keevash et al in [16] considered the rainbow Turán number $ex^*(n, H)$, which is the largest integer m such that there exists a properly edge-colored graph with n vertices and m edges and which has no rainbow copy of H. Yuster in [25] has shown that for any fixed graph H on n vertices there is a constant σ such that every properly colored graph on $n\sigma$ vertices and minimum degree $(1 - \frac{1}{\chi(H)})h\sigma + o(n)$ has a rainbow H-factor. Fu and Woolbright in [10] proved the existence of a rainbow 1-factor of K_{2n}, $(n \geq 3)$ under an edge coloring with the property that every one color induces a perfect matching. Schiermayer in [22] determined a sufficient number of colors for the existence of a rainbow k-matching, $(k \geq 2)$, in the edge colored complete graph K_n, $(n \geq 3k+3)$. Suzuki in [23] gives a necessary and sufficient condition for the existence of a rainbow spanning tree in a connected graph. Long rainbow paths in graphs were investigated by Chen and Li in [11]. Similarly, with a probabilistic approach, Albert, Frieze and Reed in [2] proved that the edge colored complete graph K_n has a rainbow Hamiltonian cycle for a sufficiently large n provided that no one color appears more than $\lceil cn \rceil$ times, where $c \leq \frac{1}{32}$ is a constant. Some other related results can be found in Akbari and Alipour [1], Erdős and Rado [8] Faudree et al [9], Král' [17], Rödl and Tuza [21].

This paper is also motivated by a recent research of the vertex colored plane graphs having no vertex rainbow faces. A face of a vertex colored plane graph is called *vertex rainbow* if all its vertices receive distinct colors. The following is of interest

Question 1 *What is the maximum number of colors $\chi_f(G)$ that can be used in a vertex coloring of a connected plane graph G with no vertex rainbow face?*

We now survey some results dealing with this question. It is easy to see that for each connected plane graph G it holds $\chi_f(G) \geq \alpha(G) + 1$ where $\alpha(G)$ is an independence number of G. Actually, Ramamurthi and West in [19] and [20] noticed that every plane graph G of order n has a coloring with at least

$\lceil \frac{n}{4} \rceil + 1$ colors by the Four Color Theorem. They in [20] had conjectured and Jungič et al in [15] proved that this bound can be improved to the bound $\lceil \frac{n}{2} \rceil + 1$ for triangle-free plane graphs. More generally, Jungič et al in [15] proved that every planar graph of order n with girth $g \geq 5$ has a non-rainbow coloring with at least $\lceil \frac{g-3}{g-2} n - \frac{g-7}{2(g-2)} \rceil$ colors if g is odd and $\lceil \frac{g-3}{g-2} n - \frac{g-6}{2(g-2)} \rceil$ if g is even. It is also shown that these bounds are best possible. Dvořák and Král' [5] proved that every plane graph with at least five vertices has a coloring with two colors as well as a coloring with three colors that avoid both monochromatic and rainbow faces.

There are also results concerning upper bounds on $\chi_f(G)$. Negami [18] investigated plane triangulations and showed that $\chi_f(G) \leq 2\alpha(G)$. Dvořák et al in [7] proved that for every n vertex 3-connected plane graph G it holds $\chi_f(G) \leq \lfloor \frac{7n-8}{9} \rfloor$, and for every 4-connected graph G it holds $\chi_f(G) \leq \lfloor \frac{5n-6}{8} \rfloor$ if $n \not\equiv 3 \pmod 8$ and $\chi_f(G) \leq \lfloor \frac{5n-6}{8} \rfloor + 1$ if $n \equiv 3 \pmod 8$ and for every 5-connected plane graph G $\chi_f(G) \leq \lfloor \frac{43}{100} n - \frac{19}{25} \rfloor$. Moreover the bounds for the 3- and 4-connected graphs are best possible.

Besides results on non-rainbow colorings of plane graphs with no short cycles and non-trivially connected plane graphs, there are also results on specific families of plane graphs, e.g. the numbers $\chi_f(G)$ were also determined for semiregular polyhedra by Jendrol' and Schrötter [14]. Moreover, connected cubic plane graphs were studied by Jendrol' in [13] and tight estimations on $\chi_f(G)$ were found there.

All the above mentioned papers lead us to find an answer to the following natural modification of Questions 1.

Question 2 *What is the maximum number of colors that can be used in a coloring of edges of a plane graph G with no rainbow face, i.e. a face with edges of mutually disjoint colors?*

2 Notation and preliminaries

We use the standard terminology according to Bondy and Murty [4], except for few notations defined throughout. However we recall some frequently used terms. All considered graphs are finite, loops and multiedges are allowed. Let $G = (V, E, F)$ be a connected plane graph with the vertex set V, the edge set E and the face set F. The *degree* of a vertex v is the number of edges incident with v, each loop counting as two edges. For a face f, the *size* (or degree) of f is defined to be the length of the shortest closed walk containing all edges from the boundary of f. We write $e \in f$ if an edge e is incident with a face f. A face f of a plane graphs G is *involved* in a face h of a plane graph H if the interior of f is a subset of the interior of h. The graph H is said to be a *face factor* of a plane graph G, if $V(H) = V(G)$, $E(H) \subseteq E(G)$ and for every face $f \in F(G)$ there is a face $f' \in F(H)$, such that f is involved in f'. An edge e of a connected graph G is called *a bridge* if $G - e$ is disconnected graph. A face factor H of a connected plane graph G is called a *bridge face factor* if every face of H is incident with a bridge. The bridge face factor H of a plane graph G is *maximum* if there is no bridge face factor H' of G having more edges than H. A *edge k-coloring* of the graph G is a mapping $\varphi : E \to \{1, 2, ..., k\}$. For a set $X \subseteq E$ we define $\varphi(X)$ to be the set of colors $\{\varphi(e); e \in X\}$. Particularly, if f is a face of G then $\varphi(f)$ denotes the set of colors used on the edges incident with the face f. A face $f \in F$ is called *rainbow* if $|\varphi(f)| = deg(f)$, otherwise is called non-rainbow. A k-coloring of a graph G is called a *non-rainbow k-coloring* if it does not involve any rainbow face, otherwise it is called a rainbow k-coloring. Let us define the *edge-rainbowness* of a plane graph G, $erb(G)$, to be the maximum k such that there is a surjective non-rainbow edge k-coloring of G. For a plane graph G with a face of size one we define the edge-rainbowness of G to be 0. Therefore, in sequel, we consider only graphs without faces of size one. Observe, that if the connected plane graph G has no face of size one then no connected face factor H of G has such a face. The main goal of this paper is to determine the edge-rainbowness of connected plane graphs, i.e. to contribute to a solution of the Question 2.

The rest of the paper is organized as follows. In the third Section we present a result concerning a relation between the edge-rainbowness and the number of edges in a maximum bridge face factor of G. The fourth Section deals with upper and lower bounds on the edge-rainbowness of graphs in terms of basic graph invariants. We show that if the edge connectivity of G is $\kappa'(G) = \kappa'$ then the edge-rainbowness is at most

$$|E(G)| - \left\lceil \frac{\kappa' - 1}{\kappa'} |F(G)| \right\rceil ,$$

moreover, this bound is tight. Next, in this Section, we prove that if H is a spanning subgraphs of G^* and $\mathcal{P}(H) = \{E_1, \ldots, E_t\}$ is a partition of its edge set into disjoint subsets such that for each vertex $v \in V(G^*)$, $deg_{G^*[E_i]} \geq 2$ for at least one $i \in \{1, \ldots, t\}$, then the edge-rainbowness can be bounded from below by

$$|E(G)| - |E(H)| + t ,$$

moreover, this bound is tight. Note that $G^*[E_i]$ denotes a subgraph of G^*

induced by the edge set E_i. We also show that if G is a 3-connected cubic plane n-vertex graph with the independent set of vertices D that dominates all the faces of a graph G then

$$\frac{3}{2}n - 2|D| \le erb(G) \le \frac{7n-8}{6} \quad ,$$

moreover, these bounds are tight.

Finally in the fifth and in the sixth Sections we present several classes of graphs that show the tightness of the presented estimations.

3 General properties of the edge-rainbow colorings

Consider a surjective edge k-coloring of a connected plane graph $G = (V,E,F)$. If a face f is incident with a bridge then $|\varphi(f)| < deg(f)$. This immediately gives

Lemma 3.1 *Let G be a connected plane graph. If each face of G is incident with a bridge then $erb(G) = |E(G)|$.*

□

Lemma 3.2 *Let G be a connected plane graph and H its connected face factor such that $H = G - e$ for some edge $e \in E(G)$. Then $erb(H) \le erb(G)$.*

Proof. Consider a non-rainbow k-coloring of H with $k = erb(H)$. We show that this coloring can be extended to a non-rainbow k-coloring of G. An edge e of G that is not present in H separates two faces f_1 and f_2 of G. Let f be a face of H, such that f_1 and f_2 are involved in it. Let the non-rainbowness of f in H be caused by two edges a and b with $\varphi(a) = \varphi(b)$. If $a \in f_1$ and $b \in f_2$ then we put $\varphi(e) = \varphi(a)$ and both f_1 and f_2 are non-rainbow faces in G. If w.l.o.g. $a \in f_1$ and $b \in f_1$ then we put $\varphi(e) = \varphi(c)$ for some $c \in f_2$, G has no face of degree one and therefore such c exists, and both f_1 and f_2 are non-rainbow faces in G. In all the cases we obtain a non-rainbow k-coloring of G. □

Corollary 3.3 *Let G be a connected plane graph. Let H be its connected bridge face factor. Then $erb(G) \ge |E(H)|$.*

Corollary 3.4 *Let G be a connected plane graph with n vertices. Then*

$$erb(G) \ge n - 1 = |V(G)| - 1 \ .$$

Lemma 3.5 *Let G be a connected plane graph and let φ be a non-rainbow surjective $erb(G)$-edge coloring of G. Let for an edge e, $G-e$ be a connected face factor of G. If $\varphi(e)$ appears on at least two different edges of G then $erb(G) \leq erb(G-e)$.*

Proof. Let $H = G-e$. We show that the coloring φ induces a suitable $erb(G)$-coloring of H. Clearly the edge e is incident with two faces f_1 and f_2 of G. Deleting the edge e a new face f, in H, appears. If non-rainbowness of at least one of faces f_1, f_2 was enforced by a color different from $\varphi(e)$ then f is also non-rainbow. Otherwise the non-rainbowness of both faces f_1, f_2 was caused by the color $\varphi(e)$ and therefore there is an edge e_i on f_i with $\varphi(e_i) = \varphi(e)$ for $i = 1, 2$. Now f contains both e_1 and e_2 and is non-rainbow in H. We have a surjective $erb(G)$-coloring of the subgraph H which is non-rainbow. □

Lemma 3.6 *Let G be a connected plane graph having a face f which is not incident with any bridge. Then there is an edge e on f such that $erb(G) \leq erb(G-e)$.*

Proof. Let φ be a surjective non-rainbow $erb(G)$-coloring of G. By the coloring φ the face f is non-rainbow and therefore it contains two edges e_1 and e_2 with $\varphi(e_1) = \varphi(e_2)$. Set $e = e_1$ and apply Lemma 3.5. □

Using Corollary 3.3 and Lemma 3.6 we immediately obtain the first main result

Theorem 3.7 *Let G be a connected plane graph. Let H be its maximum connected bridge face factor. Then $erb(G) = erb(H) = |E(H)|$.*

Proof. By Lemma 3.6 there is a sequence of edges $e_1, e_2, \ldots, e_t$ such that $H' = G - \{e_1, \ldots, e_t\}$ is a connected spanning subgraph of G, in which every face of H' is incident with a bridge and $erb(G) \leq erb(H') = |E(H')|$. From the maximality of H we have $|E(H')| \leq |E(H)|$ and, because of Corollary 3.3, it holds $erb(G) \geq erb(H)$. Then $erb(G) = erb(H) = |E(H)|$. Note, that constructed H' is also the maximum connected bridge face factor of G. □

4 Estimations on edge-rainbowness

In this section we prove several estimations on edge-rainbowness for connected plane graphs. First let us recall that the (geometric) dual of the plane graph $G = (V, E, F)$ is the graph $G^* = (V^*, E^*, F^*)$ with the following property: for each face f of graph G there is a vertex f^* of G^*, and corresponding to each edge e there is an edge e^* of G^*, two vertices f^* and h^* are joined by an edge

e^* in G^* if and only if the corresponding faces f and h are separated by the edge e in G, see Bondy and Murty [4].

4.1 2+factor of the dual graph

For a connected graph $G = (V, E)$ its spanning subgraph $H = (V, E')$ is called the *2+factor* of G if there is a partition $\mathcal{P}(H) = \{E_1, \ldots, E_t\}$ of the edge set $E(H)$ into t disjoint subsets such that each vertex v has degree $deg_{G[E_i]} \geq 2$ for at least one $i \in \{1, \ldots, t\}$. Let us recall that $G[E_i]$ is a subgraph of G induced by the edge set E_i. First we show a lower bound on the edge-rainbowness.

Theorem 4.1 *Let $G = (V, E, F)$ be a connected plane graph such that the dual G^* of G has 2+factor H with a partition $\mathcal{P}(H) = \{E_1, \ldots, E_t\}$. Then*

$$erb(G) \geq |E(G)| - |E(H)| + t \ .$$

Moreover, there exists 2+factor H_0 with partition $\mathcal{P}(H_0)$ with $|\mathcal{P}(H_0)| = t_0$ such that

$$erb(G) = |E(G)| - |E(H_0)| + t_0 \ .$$

Proof. We create a non-rainbow edge coloring using

$$|E(G)| - |E(H)| + t$$

colors. Since there is the 2+factor H of G^*, each vertex of G^* has a positive degree in at least one of the edge disjoint subgraphs $G[E_i]$, $i \in \{1, \ldots, t\}$. First we color edges of G corresponding to the edges in E_i with the color i. Next we color remaining edges of G with distinct colors. Because H is the 2+factor each vertex $v \in G^*$ is incident with at least two edges colored with the same color. Non-rainbowness of G is therefore fulfilled. We have used exactly $|E(G)| - |E(H)| + t$ colors and have obtained the non-rainbow coloring. The bound immediately follows.

To find a desired 2+factor observe that every non-rainbow coloring φ of graph G naturally induces a 2+factor H of G^*. The subgraph H of G^* is induced by the edges associated with those edges of G which colors are not alone among the colors of the edges of G under the coloring φ. Desired partition $\mathcal{P}(H)$ consists of the edge sets E_i, $i \in I$, associated to the edges of G with color i used on at least two edges of G. Let H_0 with a partition $\mathcal{P}(H_0)$ be a 2+factor induced by a non-rainbow $erb(G)$-coloring of a graph G. Hence, there is a 2+factor H_0 with partition $\mathcal{P}(H_0) = \{E_1, \ldots, E_{t_0}\}$ for which $erb(G) = |E(G)| - |E(H_0)| + t_0$. □

Next let us consider a set of vertices of G that dominates all faces of the graph G. Let us recall that a vertex u dominates a face f if u is incident with f.

Corollary 4.2 *Let $G = (V, E, F)$ be a plane graph and D be an independent set of vertices that dominates all faces of the graph G. Let E_D be the set of edges incident with the vertices in D. Then the following holds*

$$erb(G) \geq |E(G)| - |E_D| + |D| \ .$$

Moreover, the bound is sharp.

Proof. One can easy observe that the edges of G^* corresponding to those incident with the vertices of D form a 2+factor P of the dual graph G^* where a partition of its edge set $E(P)$ has cardinality $|D|$. Hence, $erb(G) \geq |E(G)| - |E_D| + |D|$.

For the sharpness of the bound see the graphs of a bipyramid B_{2t}, $t \in \mathbb{N}$, in Section 5, with $D = \{v_1, v_3, v_5, \ldots, v_{2t-1}\}$.

□

Let us also mention an estimation for the class of connected cubic plane graphs. Using Corollary 4.2 we obtain a lower bound on the edge-rainbowness of a cubic plane graph.

Corollary 4.3 *Let G be a cubic plane graph on n vertices and D be an independent set of vertices that dominates all faces of G. Then*

$$erb(G) \geq \frac{3}{2}n - 2|D|$$

Moreover, the bound is sharp.

Proof. One can easy observe that $|E_D| \leq \sum_{v \in D} deg_G(v) = 3|D|$. Hence we have $erb(G) \geq |E(G)| - 3|D| + |D| = |E(G)| - 2|D|$. Since the graph is cubic we have $3|V(G)| = 2|E(G)|$. Hence $|E(G)| = \frac{3}{2}n$ and the corollary follows.

For the sharpness of the bound see the graph of the generalized dodecahedron R_{3t+2}, $t \in \mathbb{N}$, in Section 5, with $D = \{v_1, u_4, u_7, \ldots, u_{6t+1}, w_{3t+2}\}$. □

4.2 Girth of the graph

Let us remind that the girth of a graph G is the length of its shortest cycle.

Theorem 4.4 *Let $G = (V, E, F)$ be a connected plane graph, let g be the girth of the dual graph G^* of G. Then*

$$erb(G) \le |E(G)| - \frac{g-1}{g}|F(G)|.$$

Moreover, the bound is sharp.

Proof. Let $\varphi : E(G) \to I$ be a non-rainbow coloring of the edges of the graph G, where $I \subseteq \mathbb{N}$ is the set of colors. This coloring induces the coloring φ' of the edges of the dual graph G^* in a natural way, i.e. the corresponding edges of G and G^* have the same color. For a color $i \in I$ we define G_i^* to be a subgraph of G^* induced by the edges colored with the color i under the coloring φ'. A vertex u of G^* is said to be non-rainbow if it is incident with two edges of the same color. Clearly each vertex of G^* is non-rainbow under φ'. Let $\psi : V(G^*) \to I$ be a mapping defined as follows: $\psi(v) = i$ provided that the vertex $v \in V(G^*)$ is non-rainbow due to color i. Note that if there are two or more colors that appear at least twice at vertex v we choose exactly one color among them. The degree of any one vertex in $\psi^{-1}(i)$ is at least 2 in G_i^*, and hence

$$|\psi^{-1}(i)| \le |E(G_i^*)|$$

For a vertex $v \in V(G^*)$ with $\psi(v) = i$ put $a(v) = \frac{|E(G_i^*)|-1}{|\psi^{-1}(i)|}$. Next consider two cases.

Case 1: Let $|E(G_i^*)| < g$ then G_i^* is a forest. Hence $|\psi^{-1}(i)| \le |E(G_i^*)| - 1$. So it holds $a(v) \ge 1 \ge \frac{g-1}{g}$.

Case 2: Let $|E(G_i^*)| \ge g$ then $a(v) \ge \frac{|E(G_i^*)|-1}{|E(G_i^*)|} \ge \frac{g-1}{g}$.

The number of colors in the coloring ψ' (and also in ψ) is

$$|I| = |E(G^*)| - \sum_{i\in I} |E(G_i^*)| + |I| = |E(G^*)| - \sum_{i\in I} (|E(G_i^*)| - 1) =$$

$$= |E(G^*)| - \sum_{i\in I, |\psi^{-1}(i)|>0} |\psi^{-1}(i)| \frac{|E(G_i^*)| - 1}{|\psi^{-1}(i)|} - \sum_{i\in I, |\psi^{-1}(i)|=0} (|E(G_i^*)| - 1) \le$$

$$\le |E(G^*)| - \sum_{i\in I, |\psi^{-1}(i)|>0} |\psi^{-1}(i)| \frac{|E(G_i^*)| - 1}{|\psi^{-1}(i)|} = |E(G^*)| - \sum_{i\in I} \sum_{v\in\psi^{-1}(i)} \frac{|E(G_i^*)| - 1}{|\psi^{-1}(i)|} =$$

$$= |E(G^*)| - \sum_{i\in I} \sum_{v\in\psi^{-1}(i)} a(v) = |E(G^*)| - \sum_{v\in V(G^*)} a(v) \le$$

$$\leq |E(G^*)| - \sum_{v \in V(G^*)} \frac{g-1}{g} = |E(G^*)| - \frac{g-1}{g}|V(G^*)| =$$

$$= |E(G)| - \frac{g-1}{g}|F(G)|$$

colors. So we have

$$erb(G) \leq |E(G)| - \frac{g-1}{g}|F(G)|.$$

For the sharpness of the upper bound see Corollary 4.5 and Theorem 4.6. □

If the dual graph G^* of the graph G has a special spanning subgraph then the equality in Theorem 4.4 holds.

Corollary 4.5 *Let $G = (V, E, F)$ be a plane graph. If the dual G^* has girth g and there are disjoint cycles C_1, C_2, ..., C_t of order g that cover all the vertices of G^* then $erb(G) = |E(G)| - (g-1)t = |E(G)| - |F(G)| + t$.*

Proof. The lower bound follows from Theorem 4.1, because the cycles $C_1, C_2, \dots, C_t$ form a 2+factor of the graph G^* having the edge partition $\{E(C_1), E(C_2), \dots, E(C_t)\}$. The upper bound is obtained from Theorem 4.4, because the number of vertices of G^* is tg and hence $erb(G) \leq |E(G)| - (g-1)t$. For an infinite class of graphs where such disjoint cycles exist see the case $\kappa' = 5$ in the proof of Theorem 6.1 below, where the dual graph G_i^* of G_i has suitable cycles of length 5. □

Now we show how the edge connectivity of a plane graph G attacks the edge-rainbowness of G. Observe that if the edge connectivity of a graph G is $\kappa'(G) = \kappa'$ then the girth of the dual graph G^* is κ'. Hence we can obtain an upper bound for the edge-rainbowness of plane graphs in terms of the edge connectivity of G. The proof of sharpness of this bound can be seen in Section 6.

Theorem 4.6 *Let $G = (V, E, F)$ be a connected plane graph with the edge connectivity κ' then*

$$erb(G) \leq |E(G)| - \frac{\kappa'-1}{\kappa'}|F(G)| \ .$$

Moreover, the bound is sharp.

□

Using Theorem 4.6 we can receive an upper bound for the edge-rainbowness of 3-connected cubic plane graphs.

Corollary 4.7 *Let G be an n-vertex 3-connected cubic plane graph. Then*

$$erb(G) \leq \frac{7n-8}{6}$$

Moreover, the bound is sharp.

Proof. We apply the Euler's polyhedral formula $|V(G)|+|F(G)|-|E(G)| = 2$ and the fact that $|E(G)| = \frac{3}{2}|V(G)|$. We obtain $|F(G)| = \frac{1}{2}n + 2$. Edge connectivity of G is three and after applying Theorem 4.6 we have

$$erb(G) \leq |E(G)| - \frac{2}{3}|F(G)| = \frac{3}{2}n - \frac{2n+8}{6} = \frac{7n-8}{6} \ .$$

For sharpness see the graph of a generalized dodecahedron R_r, $r \in \mathbb{N}$, in Section 5. □

4.3 Independent set in the dual graph

Let us recall that $\alpha(G^*)$, the independence number of G^*, is the cardinality of the maximum independence set of G^*. Observe, that an independent set corresponds to the edge disjoint set of faces in G. We have

Theorem 4.8 *Let G be a connected plane graph. Then*

$$erb(G) \leq |E(G)| - \alpha(G^*)$$

.

Proof. Let $\alpha(G^*) = \alpha^*$ and let ψ be a non-rainbow coloring of G. Let $\{v_1, v_2, ..., v_{\alpha^*}\}$ be the maximum independent set in G^*. Let the set of corresponding faces in G be $J = \{f_1, f_2, ..., f_{\alpha^*}\}$. Each face in J has at least two edges colored with the same color under the coloring ψ. Since these faces are edge disjoint there are at most $\sum_{i=1}^{\alpha^*}(deg(f_i) - 1)$ distinct colors on edges incident with these faces. Hence the maximum number of used colors under a coloring ψ is at most $\sum_{i=1}^{\alpha^*}(deg(f_i) - 1) + |E(G)| - \sum_{i=1}^{\alpha^*}(deg(f_i)) = |E(G)| - \alpha^*$ and the result follows. □

5 Four classes of graphs

In this Section we determine the exact values of the edge-rainbowness of four classes of graphs. These four examples show the tightness of boundaries in theorems from the previous Section.

5.1 Prisms

The r-sided prism D_r, $r \geq 3$, consists of the vertex set $V = \{u_1, u_2, ..., u_r, v_1, v_2, ..., v_r\}$ and the edge set $E = \{\{u_i, u_{i+1}\} \cup \{v_i, v_{i+1}\} \cup \{u_i, v_i\}$, for $i = 1, ..., r$, indices modulo $r\}$. The set of faces of D_r consists of two r-gonal faces $f = [u_1, ..., u_r]$ and $h = [v_1, ..., v_r]$ and r quadrangles $[u_i u_{i+1} v_{i+1} v_i]$ for any for $i = 1, ..., r$, indices modulo r.

Theorem 5.1 *Let D_r be an r-sided prism. Then $erb(D_3) = 5$ and $erb(D_r) = 2r$ for $r \geq 4$.*

Proof. Let H be a bridge face factor of D_r. The face f of D_r has to be involved in a face f' of H. Therefore at least one edge from the boundary of f is missing in H. Similarly at least one edge from the boundary of h is missing in H. Every face of D_r is involved in a face f' or h'. If not, then there is a face, say m, distinct from f' and h' (f' and h' need not to be different). Thus neither $u_i u_{i+1}$ nor $v_i v_{i+1}$ is a bridge in m and therefore there is $j \in \{1, \ldots, r\}$ such that $u_j v_j$ is a bridge in m. Hence all the quadrangular faces have to be involved in m. Thus all edges from the boundary of f are in H, a contradiction. Hence H has at most two faces, and from the Euler's polyhedral formula we have $|E(H)| \leq |V(H)| = |V(D_r)|$. Hence $erb(D_r) \leq 2r$.

Let $r \geq 4$. Now we choose a 2-factor of the dual graph D_r^* (which is also a 2+factor of D_r^*). First a cycle of the length three with edges incident with the face associated in the dual D_r^* with a vertex $u_1 \in V(D_r)$. Next a cycle containing all other vertices of D_r^* is chosen. These cycles form a 2+factor of D_r^* and from Theorem 4.1 we have $erb(D_r) \geq 3r - 3 - (r-1) + 2 = 2r$

If $r = 3$ then applying Theorem 4.4 we have $erb(D_3) \leq \frac{34}{6}$ and from the Corollary 3.4 we have $erb(D_3) \geq |V(D_3)| - 1 = 5$. So we are done. □

5.2 Antiprisms

An r-sided antiprism A_r is defined as follows. The vertex set $V(A_r) = \{u_1, u_2, ..., u_r, v_1, v_2, ..., v_r\}$, $r \geq 3$. The edge set $E(A_r) = \{\{u_i u_{i+1}\} \cup \{v_i v_{i+1}\} \cup \{u_i v_i\} \cup \{u_i v_{i-1}\}$, $i = 1, ..., r$, indices modulo $r\}$. The face set of A_r consists of two r-gonal faces f and h where $f = [u_1, ..., u_r]$, $h = [v_1, ..., v_r]$ and $2r$ faces $f_i = [u_i u_{i+1} v_i]$ and $h_i = [v_i v_{i+1} u_{i+1}]$, $i = 1, ..., r$, indices modulo r.

Theorem 5.2 *Let A_r be an r-sided antiprism. Then $erb(A_3) = 5$ and $erb(A_r) = 2n$ for $r \geq 4$.*

Proof. Let $r \geq 4$. Let H be a bridge face factor of A_r. The face f of A_n has to be involved in a face f' of H. Therefore at least one edge from the boundary of f is missing in H. Similarly at least one edge from the boundary of h is missing in H. Every one face of A_n is involved in a face f' or h'. If not, then there is another face, say m, distinct from f' and h'. Thus neither u_iu_{i+1} nor v_iv_{i+1} is a bridge in m and therefore there is $j \in \{1, \ldots, r\}$ such that u_jv_j or $u_{j+1}v_j$ is a bridge in m. Hence all the triangular faces have to be involved in m and therefore all edges from the boundary of f are in H, a contradiction. Therefore H has at most two faces and we have $|E(H)| \leq |V(H)| = |V(A_r)|$, and hence $erb(A_r) \leq 2r$.

To get the opposite inequality we choose 2+factor in the dual graph consisting of two cycles C_4 and C_{2r-2}. First a cycle C_4 of length 4, a boundary cycle of the face associated with the vertex $u_1 \in V(A_r)$ is chosen. The second cycle contains all other vertices of the dual graph A_r^*. Using Theorem 4.1 we have $erb(A_r) \geq 4r - 4 - (2r - 2) + 2 = 2r$

If $r = 3$ then applying Theorem 4.4 we have $erb(A_3) \leq \frac{17}{3}$ and from the Corollary 3.4 we have $erb(A_3) \geq |V(A_3)| - 1 = 5$. □

The above two examples show the tightness of the bound in Theorem 4.1.

5.3 Bipyramids

The r-sided bipyramid is a plane graph B_r, $r \geq 3$, with the vertex set $V(B_r) = \{\ u,\ v_1,\ ...,\ v_r,\ w\ \}$ and the edge set $E(B_r) = \{\{uv_i\} \cup \{v_iw\} \cup \{v_iv_{i+1}\},\ i = 1, ..., r$, indices modulo $r\}$. Hence B_r consists of $r + 2$ vertices, $3r$ edges and $2r$ triangular faces. In fact B_n is the dual graph to D_n.

Theorem 5.3 *Let B_r be bipyramid. Then $erb(B_3) = 5$ and $erb(B_r) = \left\lfloor \frac{3}{2} r \right\rfloor$ for $r \geq 4$.*

Proof. For the case $r \geq 4$ the bipyramid B_r is edge 4-connected. Applying Theorem 4.6 we have $erb(B_r) \leq 3r - \frac{3}{4} \cdot 2r = \frac{3}{2}r$. Consider 2 cases.

Case 1. Let $r = 2t$. Observe that the set of boundaries of quadrangular faces of the dual B_r^* associated with the vertices $v_1, v_3, \ldots, v_{2t-1}$ form a 2-factor consisting of cycles of length 4 of the dual graph B_r^*, and in fact a 2+factor of B_r^*. By Theorem 4.1 we have $erb(G) \leq 3r - 4t + t = 3t$.

Case 2. Let $r = 2t + 1$. Observe that the set of cycles of length 4 that are boundaries of quadrangular faces of the dual graph B_r^* associated with the vertices $v_1, v_3, \ldots, v_{2t-1}$ cover all but 2 vertices of B_r^*. Color the mentioned cycles with distinct colors from 1 to $t - 1$, one for each cycle and three edges incident with the face associated with v_{2t} with color t, the remaining $6t + 3 - 4t - 4 = 2t - 1$ edges with colors from $t + 1$ to $3t - 1$. This non-rainbow coloring uses $3t - 1 = \left\lfloor \frac{3}{2} r \right\rfloor$ colors.

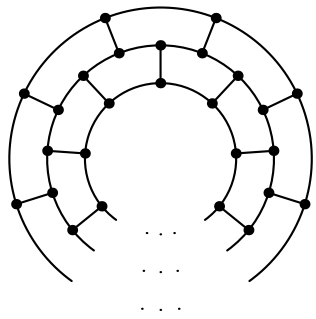

Fig. 1. Generalized dodecahedron

Hence $erb(G) = \left\lfloor \frac{3}{2}r \right\rfloor$.

For the case $r = 3$, we have $erb(B_3) \leq 9 - \frac{2}{3} \cdot 6 = 5$. To obtain suitable coloring we color the edges uv_1, uv_2, uv_3 with the color 1, the edges wv_1, wv_2, wv_3 with the color 2 and the remaining of edges with distinct colors. This forms non-rainbow 5-coloring.

□

5.4 Generalized dodecahedron

Let us denote the generalized dodecahedron by R_r, i.e. a cubic plane graph having exactly two r-gonal faces and $2r$ pentagonal faces in which these two r-gons are separated by two rings of pentagonal faces, see Fig. 1. Observe that R_r consists of $4r$ vertices, $6r$ edges and $2r + 2$ faces. Notice that if $r = 5$ we have a graph of the dodecahedron. We denote the vertices on the interior r-gonal face by $\{v_1, \ldots, v_r\}$, vertices on the exterior face by $\{w_1, \ldots, w_r\}$ and the remaining vertices by $\{u_1, \ldots, u_{2r}\}$ and the edge set $E(R_r) = \{\{v_iv_{i+1}\} \cup \{w_iw_{i+1}\} \cup \{u_ju_{j+1}\} \cup \{v_iu_{2i-1}\} \cup \{w_iu_{2i}\}\}$, $i = 1, \ldots, r$, $j = 1, \ldots, 2r$, indices i modulo r, indices j modulo $2r$.

Theorem 5.4 *Let R_r be a generalized dodecahedron. Then $erb(R_r) = \lfloor \frac{14r-4}{3} \rfloor$.*

Proof. Using Theorem 4.6 we have

$$erb(R_r) \leq 6r - \frac{2}{3}(2r + 2) = \frac{14n - 4}{3}$$

Consider three cases.

Case 1. Let $r = 3t$. We create a 2+factor of the graph R^*_{3t}. The 2+factor is created by the boundaries of triangular faces of R^*_{3t} associated with the vertices $\{u_1, u_4, \ldots, u_{6t-2}\}$, cycles of length 3, and two additional paths on three vertices where the edges of the first path are edges of dual graph R^*_{3t}

corresponding to the edges w_1w_2, w_2w_3 and the edges of the second path are edges of dual graph R^*_{3t} corresponding to the edges v_1v_2, v_2v_3. From Theorem 4.1 we have

$$erb(R_{3t}) \geq 18t - 6t - 2 - 2 + 2t + 2 = 14t - 2$$

Case 2. Let $r = 3t + 1$. We create a 2+factor of the graph R^*_{3t+1}. This 2+factor is created by the boundaries of triangular faces of R^*_{3t+1} associated with the vertices $\{v_1, u_5, u_8, u_{11} \ldots, u_{6t-1}\}$ and one 4-cycle with edges of dual graph R^*_{3t+1} corresponding to the edges $w_{3t}w_{3t+1}$, $w_{3t+1}u_{6t+2}$, w_1u_2, w_1w_2. From Theorem 4.1 we have

$$erb(R_{3t+1}) \geq 18t + 6 - 6t - 4 + 2t + 1 = 14t + 3$$

Case 3. Let $r = 3t + 2$. We create a 2+factor of the graph R^*_{3t+2}. This 2+factor is created by the boundaries of triangular faces of R^*_{3t+2} associated with the vertices $\{v_1, u_4, u_7, u_{10} \ldots, u_{6t+1}, w_{3t+2}\}$. From Theorem 4.1 we have

$$erb(R_{3t+2}) \geq 18t + 12 - 6t - 6 + 2t + 2 = 14t + 8$$

The edge-rainbowness is an integer and the result follows. □

6 Edge rainbowness and edge connectivity

Classes of graphs presented in previous Section show the tightness of boundaries in Theorem 4.6 and consequently in Theorem 4.4 for some values of edge connectivity, or the girth of dual, respectively. Next we show the tightness of boundaries in Theorem 4.6 for all possible choices of the edge connectivity. According to the natural correspondence between the girth of G^* and the edge-connectivity of G this theorem also shows the tightness of Theorem 4.4 for all possible choices of the girth.

Theorem 6.1 *Let $\kappa' \in \{1, 2, 3, 4, 5\}$ and $t \in \mathbb{N}$. There exists a graph G on at least t vertices with the edge connectivity κ' such that*

$$erb(G) = |E(G)| - \left\lceil \frac{\kappa' - 1}{\kappa'} |F(G)| \right\rceil .$$

Proof. Consider next 5 cases.

Case 1: Let $\kappa' = 1$. Choose an arbitrary tree on t vertices. Theorem trivially holds.

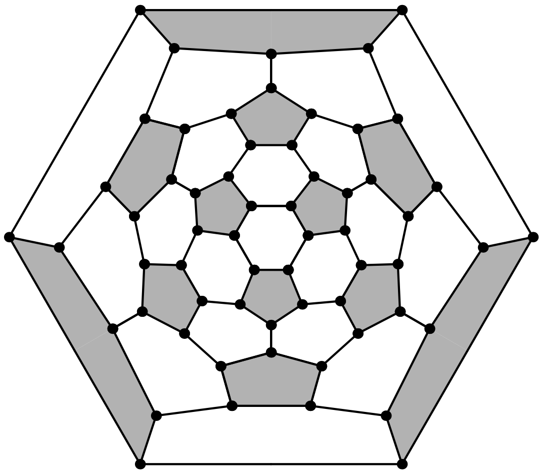

Fig. 2. Truncated icosahedron, (”The soccer ball graph”)

Case 2: Let $\kappa' = 2$. Choose a cycle on t vertices. Color two edges with the same color and the remaining edges with different colors. Hence a non-rainbow coloring is created and uses $|E(G)| - 1$ colors.

Case 3: Let $\kappa' = 3$. Choose the generalized dodecahedron R_t on $4t$ vertices.

Case 4: Let $\kappa' = 4$. Choose the $2t$-sided bipyramid B_{2t}.

Case 5: Let $\kappa' = 5$. We create an infinite sequence of edge 5-connected graphs G_i^*, $i \in \mathbb{N}$ where the equality is attained. First choose G_1^* to be the dual graph to the truncated icosahedron, (”the soccer ball graph”) G_1, see Fig. 2. One can easy check that G_1 has the 2+factor consisting of the cycles of length 5, (boundaries of gray faces in Fig. 2). By Theorem 4.1 we have $erb(G_1^*) \geq 90 - 12 \cdot 5 + 12 = 42$ and by Theorem 4.6 we have $erb(G_1^*) \leq 90 - \frac{4}{5} \cdot 60 = 42$ and hence $erb(G_0^*) = 42$.

To obtain the graph G_{i+1}^* for $i \geq 1$, take graph G_i and denote the vertices incident with an interior hexagonal face f' in G_i by the $u_1, u_2, u_3, u_4, u_5, u_6$. Let us denote the vertices incident with an exterior hexagonal face f in G_1 by the $v_1, v_2, v_3, v_4, v_5, v_6$. Now insert G_1 into a hexagonal face f' of G_i in such a way that the vertices v_i and u_i for $i = 1, 3, 5$ are joined by an edge. Denote the obtained graph by G_{i+1}. The graph G_{i+1}^* has $33i + 32$ vertices, $(i+1)60$ faces and $(i+1)90 + 3i$ edges. Observe that the obtained graph has desired 2+factor consisting of $(12i + 12)$ cycles of length 5. By Theorem 4.1 we have $erb(G_{i+1}^*) \geq 90(i+1) + 3i - 48(i+1) = 45i + 42$ and by Theorem 4.6 we have $erb(G_{i+1}^*) \leq 90(i+1) + 3i - \frac{4}{5} \cdot (60i + 60) = 45i + 42$.

To complete the proof of the Case 5 it is sufficient to take graph G_t^*. □

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