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Abstract

Let G be the "$ax+b$"-group with the left invariant Haar measure $d\nu$ and ψ be a fixed real-valued admissible wavelet on $L_2(\mathbb{R})$. The structure of the space of Calderón (wavelet) transforms $W_\psi(L_2(\mathbb{R}))$ inside $L_2(G, d\nu)$ is described after identifying the group G with the upper half-plane Π in $\mathbb{C}$. Using this result some properties and the Wick calculus of the Calderón-Toeplitz operators T_a acting on $W_\psi(L_2(\mathbb{R}))$ whose symbols $a = a(\zeta)$ depend on $v = \Im\zeta$ for $\zeta \in G$ are investigated.

1 Introduction

Let G be the noncompact group of shifts and dilations acting on $L_2(\mathbb{R})$ with the left invariant Haar measure $d\nu$, the so called "$ax+b$"-group. In the first part of this paper we study the associated space of Calderón (or wavelet) transforms $W_\psi(L_2(\mathbb{R}))$ and show its structure inside the space $L_2(G,\, d\nu)$ after identifying the group G with the upper half-plane Π in the complex plane $\mathbb{C}$. The main idea is based on Vasilevski's paper [20], where the structure of Bergman and poly-Bergman spaces in $L_2(\Pi)$ was obtained (and its complete decomposition onto them).

The representation of the space of Calderón transforms $W_\psi(L_2(\mathbb{R}))$ is especially important in the study of the Calderón-Toeplitz operators with symbols depending only on $v = \Im\zeta$ for $\zeta \in G$ (as in the case of Bergman spaces and Toeplitz operators acting on them, cf. [9]). For a given bounded function a on G and an admissible wavelet ψ on $L_2(\mathbb{R})$, the *Calderón-Toeplitz operator T_a with symbol a* is defined to be the map of $L_2(\mathbb{R})$ to $L_2(\mathbb{R})$ given by

$$T_a f = \int_G a(\zeta)\langle f, \psi_\zeta\rangle \psi_\zeta \, d\nu(\zeta), \quad f \in L_2(\mathbb{R}).$$

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Taking the inner product with a function $g \in L_2(\mathbb{R})$, the definition of T_a may be written in a weak sense, namely,

$$\langle T_a f, g\rangle = \int_G a(\zeta)\langle f, \psi_\zeta\rangle\langle \psi_\zeta, g\rangle \, d\nu(\zeta), \quad f, g \in L_2(\mathbb{R}), \tag{1}$$

according to the Calderón reproducing formula (see Section 2).

Alternatively, the Calderón-Toeplitz operator T_a may be viewed as acting on $L_2(G, d\nu)$ and given as $P_\psi M_a P_\psi$, where P_ψ is the orthogonal projection from $L_2(G, d\nu)$ onto $W_\psi(L_2(\mathbb{R}))$ and M_a is the operator of pointwise multiplication by a on $L_2(G, d\nu)$. Thus, $P_\psi M_a P_\psi$ is a Toeplitz operator acting on the Hilbert space $W_\psi(L_2(\mathbb{R}))$ and it has the matrix representation

$$\begin{bmatrix} W_\psi T_a W_\psi^* & 0 \\ 0 & 0 \end{bmatrix}$$

with respect to the decomposition $L_2(G,\, d\nu) = W_\psi(L_2(\mathbb{R})) \oplus W_\psi(L_2(\mathbb{R}))^\perp$, where W_ψ, resp. W_ψ^*, is the continuous wavelet transform operator, resp. its adjoint (see Section 2). The realization on $L_2(\mathbb{R})$ is of intrinsic interest as an alternative quantization to classical pseudodifferential calculus. The realization using $W_\psi(L_2(\mathbb{R}))$ is of our present interest.

Clearly, it is also possible to define a more general class of Toeplitz-type operators including these of Calderón-Toeplitz. Such operators are based on the group theoretical approach (the group representations): let U be a representation of a group G acting on functions defined on $\mathbb{R}^n$, a be a function defined on G and ϕ be a function defined on $\mathbb{R}^n$. The operator $T_{a,\phi}$ acting on functions on $\mathbb{R}^n$ by

$$T_{a,\phi} f = \int_G a(g)\langle f, U_g\phi\rangle U_g\phi \, dg,$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $L_2(\mathbb{R}^n)$ and dg is the left invariant measure on G, is called the *Toeplitz operator based on the representation U*. Note that such operators are, in fact, examples of a more general A-Toeplitz operator defined in [7]. The particular case of Toeplitz-type operators with respect to specific representations (the Schrödinger representation of the reduced Heisenberg group and the natural representation of the "$ax + b$"-group) was investigated in [13].

The study of Toeplitz-type operators based on the Calderón reproducing formula is a relatively new area of operator research which began in the early 1990's. These operators were introduced by R. Rochberg in [17] as a wavelet counterpart of Toeplitz operators defined on Hilbert spaces of holomorphic functions. They are the model operators that fit nicely in the context of wavelet decomposition of function spaces and almost diagonalization of operators. This class of operators includes as Toeplitz operators on classical Fock spaces and weighted Bergman spaces of the unit disk as many interesting classes of Fourier multiplier operators, singular integral operators, paracommutators and paraproducts. Also, these operators are an effective time-frequency localization tool, cf. [6], which provides ways of analyzing signals by describing their frequency content as it varies over time, cf. [10].

As far as we know, there is only a few papers of R. Rochberg and K. Nowak on this topic investigating the Calderón-Toeplitz operators mainly with respect to their eigenvalues estimates and Schatten ideal criteria, cf. [12], [13], [17], [18]. However, the deeper results on the properties of Calderón-Toeplitz operators (e.g. compactness, boundedness, spectra, etc.) are not known in general.

Moreover, the related operators show up in physics when working with coherent states, see e.g. [1], and [11]. Thus, recall the necessary ingredients of the Berezin theory, cf. [2], [3] (for a connection with Toeplitz operators on Bergman spaces see [21]). Let $\{k_g\}_{g\in S}$ be a system of coherent states parametrized by elements g of some set S carrying a measure $d\mu$ in a Hilbert space H. If $V : H \to L_2(S)$ is the isomorphic inclusion

$$V : f \in H \mapsto \langle f, k_g\rangle_H \in L_2(S),$$

and the integral operator

$$(Pf)(g) = \int_S f(h)\langle k_h, k_g\rangle_H \, d\mu(h)$$

is the orthogonal projection from $L_2(S)$ onto $V(H)$, then the function $a(g)$, $g \in S$ is called the *anti-Wick* (or *contravariant*) *symbol* of an operator $A : H \to H$, if

$$VAV^{-1}|_{V(H)} = Pa(g)P = Pa(g)I|_{V(H)} : V(H) \to V(H),$$

or, in the other terminology, the operator $VAV^{-1}|_{V(H)}$ is the *Toeplitz operator*,

$$A_{a(g)} = Pa(g)I|_{V(H)} : V(H) \to V(H),$$

with the symbol $a(g)$. For a given bounded linear operator $A : H \to H$, introduce the (Wick) function

$$\widetilde{a}_A(g, h) = \frac{\langle Ak_h, k_g\rangle_H}{\langle k_h, k_g\rangle_H}, \quad g, h \in S.$$

The restriction of the function $\widetilde{a}_A(g, h)$ onto the diagonal

$$\widetilde{a}_A(g) = \widetilde{a}_A(g, g) = \frac{\langle Ak_g, k_g\rangle_H}{\langle k_g, k_g\rangle_H}, \quad g \in S,$$

is called the *Wick* (or *covariant*) *symbol* of the operator A. Note that the Wick and anti-Wick symbols of an operator $A : H \to H$ are connected by the *Berezin transform*,

$$\widetilde{a}(g) = \int_S a(h) \, \frac{\langle k_g, k_h\rangle_H \langle k_h, k_g\rangle_H}{\langle k_g, k_g\rangle_H} \, d\mu.$$

In our particular case $A = T_a$, $H = L_2(\mathbb{R})$, and $S = G = \Pi$ equipped with the measure $d\mu = d\nu(\zeta) = v^{-2}dudv$. Note that admissible wavelets in $L_2(\mathbb{R})$ forming a dense vector subspace of $L_2(\mathbb{R})$ are, in fact, nothing but coherent states related to "$ax+b$"-group G, and therefore $k_g = \psi_\zeta(x) = v^{-1/2}\,\psi(\frac{x-u}{v})$.

The key result, which gives an access to the properties of Calderón-Toeplitz operators studied in the second part of this paper, is established in Section 4. Namely, we prove that the Calderón-Toeplitz operator T_a with symbol $a(v)$ depending only on $v = \Im\zeta$, $\zeta \in G$, is unitarily equivalent to the multiplication operator $\gamma_a I$ acting on $L_2(\mathbb{R})$, where the function γ_a is given by

$$\gamma_a(\xi) = \int_{\mathbb{R}_+} a(v)|\hat{\psi}(v\xi)|^2\,\frac{dv}{v}, \quad \xi \in \mathbb{R}.$$

In this context we also mention the Wick symbol $\widetilde{a}(\zeta)$ of the Calderón-Toeplitz operator T_a which together with the star product carries as well many essential properties of the corresponding operator. The star product defines the composition of two Wick symbols $\widetilde{a}_A$ and $\widetilde{a}_B$ of the operators A and B, respectively, as the Wick symbol of the composition AB, i.e., $\widetilde{a}_A * \widetilde{a}_B = \widetilde{a}_{AB}$. In Section 4 we also state the formulas for the Wick symbols of Calderón-Toeplitz operators T_a whose symbols depend only on $v = \Im\zeta$, as well as the formulas for the star product in terms of our function γ_a.

2 Preliminaries

Here we use the obvious notations: $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+$ ($\mathbb{R}_-$) is the positive (negative) half-line and χ_+ (χ_-) is the characteristic function of $\mathbb{R}_+$ ($\mathbb{R}_-$).

The *Calderón reproducing formula* is the following resolution of unity on $L_2(\mathbb{R})$,

$$\langle f, g\rangle = \int_G \langle f, \psi_\zeta\rangle\langle\psi_\zeta, g\rangle\, d\nu(\zeta), \quad f, g \in L_2(\mathbb{R}), \tag{2}$$

where $G = \{\zeta = (u,v);\, u \in \mathbb{R}, v > 0\}$ is the "$ax+b$"-group with the left invariant Haar measure $d\nu(\zeta) = v^{-2}\,du\,dv$. The group G acts on $L_2(\mathbb{R})$ via translations and dilations, i.e. for $\zeta = (u,v) \in G$, the unitary representation U_ζ of G is given by

$$(U_\zeta\psi)(x) = \psi_{u,v}(x) = \frac{1}{\sqrt{v}}\,\psi\left(\frac{x-u}{v}\right),$$

where $\psi \in L_2(\mathbb{R})$ is an *admissible wavelet* satisfying

$$\int_{\mathbb{R}_+} |\hat{\psi}(x\xi)|^2 \frac{d\xi}{\xi} = 1, \tag{3}$$

for almost every $x \in \mathbb{R}$, and $\hat{\psi}$ stands for the Fourier transform $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ given by

$$\mathcal{F}\{g\}(\xi) = \hat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{-2\pi i x\xi}\, dx. \tag{4}$$

Thus, $\psi_{u,v}$ is the function ψ translated to be centered at u, scaled to a width of v, and renormalized to be a unit vector ($\|\psi\| = 1$). Recall that $(\hat{U}_\zeta \hat{\psi})(\xi) = \hat{\psi}_{u,v}(\xi) = \sqrt{v}\, \hat{\psi}(v\xi) e^{-2\pi i u\xi}$ on Fourier transform side. The integral (2) is understood in a weak sense. It is not hard to check that the *admissibility condition* (3) is not only sufficient, but also necessary for the Calderón reproducing formula to hold.

In what follows we identify the group G with the upper half-plane $\Pi = \{\zeta = u + iv;\ u \in \mathbb{R}, v > 0\}$ in the complex plane $\mathbb{C}$ and consider an arbitrary *real-valued admissible wavelet* $\psi \in L_2(\mathbb{R})$. Also, $\langle \cdot, \cdot \rangle$ always means the inner product on $L_2(\mathbb{R})$, whereas $\langle \cdot, \cdot \rangle_G$ denotes the inner product on $L_2(G, d\nu)$.

For a fixed $\psi \in L_2(\mathbb{R})$, the functions $W_\psi f$ on G of the form

$$(W_\psi f)(\eta) = \langle f, \psi_\eta \rangle, \quad f \in L_2(\mathbb{R}), \tag{5}$$

form a reproducing kernel Hilbert space $W_\psi(L_2(\mathbb{R}))$ called the *space of Calderón (wavelet) transforms* (since $W_\psi f$ is called the *continuous wavelet transform* of a function f with respect to the analyzing wavelet ψ). The space $W_\psi(L_2(\mathbb{R}))$ is a closed subspace of $L_2(G,\, d\nu)$. Now, for admissible ψ, the Calderón reproducing formula (2) reads as follows:

$$\langle f, g \rangle = \langle W_\psi f, W_\psi g \rangle_G, \quad f, g \in L_2(\mathbb{R}),$$

which means that the continuous wavelet transform operator $W_\psi : L_2(\mathbb{R}) \to L_2(G,\, d\nu)$ given by (5) is an isometry. Consequently, for an admissible wavelet $\psi \in L_2(\mathbb{R})$ and for all $f \in L_2(\mathbb{R})$ holds

$$\int_G |(W_\psi f)(\eta)|^2\, d\nu(\eta) = \|f\|^2, \tag{6}$$

and the integral operator $P_\psi : L_2(G,\, d\nu) \to L_2(G,\, d\nu)$ given by

$$(P_\psi F)(\eta) = \int_G F(\zeta) K_\zeta(\eta)\, d\nu(\zeta), \quad F \in L_2(G,\, d\nu), \tag{7}$$

is the orthogonal projection onto $W_\psi(L_2(\mathbb{R}))$, where the reproducing kernel $K_\zeta(\eta) = \langle \psi_\zeta, \psi_\eta \rangle$ in $W_\psi(L_2(\mathbb{R}))$ is the autocorrelation function of ψ. Since ψ is a real-valued admissible wavelet, then $K_\zeta(\eta)$ is a real symmetric (reproducing) kernel. Obviously, $\langle K_\zeta, K_\eta \rangle_G = K_\zeta(\eta)$, and thus $|\langle K_\zeta, K_\eta \rangle_G| = |\langle \psi_\zeta, \psi_\eta \rangle|$. Easily we have

Lemma 2.1 *If $F \in W_\psi(L_2(\mathbb{R}))$, then $P_\psi F = F$, i.e. for all $\zeta \in G$,*

$$F(\zeta) = \int_G F(\eta) K_\zeta(\eta)\, d\nu(\eta). \tag{8}$$

This means that a function $f \in L_2(G, d\nu)$ is the wavelet transform of a certain signal if and only if it satisfies the reproducing property (8) with $(W_\psi f)(\cdot) = F(\cdot)$. A special case of the reproducing formula above is the following identity:

$$1 = \int_G |K_\zeta(\eta)|^2 \, d\nu(\eta), \quad \zeta \in G.$$

As it was stated in introduction we may identify the operators T_a and $P_\psi M_a P_\psi$ on the Hilbert space $W_\psi(L_2(\mathbb{R}))$. Thus for $F \in W_\psi(L_2(\mathbb{R}))$ we get

$$(T_a F)(\zeta) = \langle a P_\psi F, K_\zeta \rangle_G = \langle a F, K_\zeta \rangle_G. \tag{9}$$

The following easy result implies that the Calderón-Toeplitz operator acting on $W_\psi(L_2(\mathbb{R}))$ is the integral operator with kernel $(T_a K_\eta)(\zeta)$.

Theorem 2.2 *Let a be a real bounded integrable function on G, and $F \in W_\psi(L_2(\mathbb{R}))$. Then the Calderón-Toeplitz operator acting on $W_\psi(L_2(\mathbb{R}))$ has the form*

$$(T_a F)(\zeta) = \int_G F(\eta)(T_a K_\eta)(\zeta) \, d\nu(\eta). \tag{10}$$

Moreover, if T_a is bounded on $W_\psi(L_2(\mathbb{R}))$, then T_a is self-adjoint.

Proof. According to (9) and (8) we may write

$$\begin{aligned}
(T_a F)(\zeta) &= \int_G a(\theta) F(\theta) K_\zeta(\theta) \, d\nu(\theta) \\
&= \int_G a(\theta) \left(\int_G F(\eta) K_\theta(\eta) \, d\nu(\eta) \right) K_\zeta(\theta) \, d\nu(\theta) \\
&= \int_G F(\eta) \left(\int_G a(\theta) K_\eta(\theta) K_\zeta(\theta) \, d\nu(\theta) \right) d\nu(\eta) \\
&= \int_G F(\eta)(T_a K_\eta)(\zeta) \, d\nu(\eta).
\end{aligned}$$

If T_a is bounded on $W_\psi(L_2(\mathbb{R}))$, then T_a^* is bounded on $W_\psi(L_2(\mathbb{R}))$ as well. Thus, for $F \in W_\psi(L_2(\mathbb{R}))$ we have

$$(T_a^* F)(\zeta) = \langle T_a^* F, K_\zeta \rangle_G = \langle F, T_a K_\zeta \rangle_G = \langle F, a K_\zeta \rangle_G = (T_a F)(\zeta).$$

This completes the proof. □

3 Representation of $W_\psi(L_2(\mathbb{R}))$

First, let us recall the following well known results describing the relation between weighted Bergman spaces on the upper half-plane and the space of Calderón transforms of Hardy space functions with respect to Bergman wavelets, cf. [8], [16]:

Let $H_2(\mathbb{R})$ be the Hardy space of all square integrable functions whose Fourier transform is supported on $\mathbb{R}_+$, i.e.

$$H_2(\mathbb{R}) = \{f \in L_2(\mathbb{R});\ \hat{f}(\xi) = 0 \text{ a.e. } \xi \leq 0\},$$

and take the specific (Bergman) wavelet ψ^α, i.e. for $\alpha > 0$,

$$\hat{\psi^\alpha}(\xi) = \begin{cases} c_\alpha \xi^\alpha e^{-2\pi\xi} & \text{for } \xi > 0, \\ 0 & \text{for } \xi \leq 0, \end{cases}$$

where

$$c_\alpha = \left(\int_{\mathbb{R}_+} |\xi^\alpha e^{-2\pi\xi}|^2 \frac{d\xi}{\xi} \right)^{-\frac{1}{2}} = \frac{(4\pi)^\alpha}{\sqrt{\Gamma(2\alpha)}}$$

is the normalization factor ($\Gamma(z)$ is the Euler gamma function). Denote by $W_{\psi^\alpha}(H_2(\mathbb{R}))$ the space of wavelet transforms of functions in the Hardy space $H_2(\mathbb{R})$ with respect to Bergman wavelet ψ^α. Clearly, $W_{\psi^\alpha}(H_2(\mathbb{R}))$ is a reproducing kernel Hilbert space with the following kernel

$$\langle \psi^\alpha_\eta, \psi^\alpha_\zeta \rangle = \frac{\alpha}{2\pi} (tv)^{\alpha+1/2} \left(\frac{2i}{\zeta - \overline{\eta}} \right)^{2\alpha+1},$$

where $\zeta = (u, v), \eta = (s, t) \in G$. Let $A_\beta(G)$, $\beta > -1$, stand for the weighted Bergman space of holomorphic functions on G satisfying

$$\|F\|^2_{A_\beta} = \int_{\mathbb{R}_+} \int_{\mathbb{R}} |F(u + iv)|^2\, v^\beta\, dudv < \infty.$$

The identity (6) motivates the definition of the (Bergman) transform

$$(B^\alpha F)(u, v) = v^{-\alpha-\frac{1}{2}} F(u, v),$$

which leads to the following

Theorem 3.1 *The unitary operator B^α gives an isometrical isomorphism of the space $L_2(G,\, d\nu)$ onto $L_2(G,\, v^{2\alpha-1}dudv)$ under which the space of Calderón transforms $W_{\psi^\alpha}(H_2(\mathbb{R}))$ is mapped onto the weighted Bergman space $A_{2\alpha-1}(G)$.*

The unitary map B^α from C_{ψ^α} to $A_{2\alpha-1}$ provides a unitary equivalence between commutators defined by the wavelet ψ^α and their Bergman space analogues, cf. [14]. Also, this result provides a good tool for studying Toeplitz operators on weighted Bergman spaces on the upper half-plane which (in this case) become unitarily equivalent to Calderón-Toeplitz operators, see [12].

Our next aim is to give some more general results involving an arbitrary real-valued admissible wavelet $\psi \in L_2(\mathbb{R})$ and obtain new representation of the space $W_\psi(L_2(\mathbb{R}))$. A direct Fourier transform calculation yields the following useful lemma.

Lemma 3.2 *Let $\zeta = (u,v) \in G$ and $\eta = (s,t) \in G$. Then the Fourier transform (with respect to $s = \Re\eta$) of the kernel function $K_\eta(\zeta)$ is equal to*

$$\sqrt{tv}\,\hat{\psi}(v\xi)\,\overline{\hat{\psi}(t\xi)}e^{-2\pi iu\xi}.$$

Introduce the unitary operator $U_1 = (\mathcal{F} \otimes I) : L_2(G, d\nu(\zeta)) = L_2(\mathbb{R}, du) \otimes L_2(\mathbb{R}_+, v^{-2}dv) \to L_2(\mathbb{R}, du) \otimes L_2(\mathbb{R}_+, v^{-2}dv)$, with $\zeta = (u,v) \in G$. Let us denote by A_1 the image of the space $W_\psi(L_2(\mathbb{R}))$ in the mapping U_1. The space A_1 consists of all functions

$$F(u,v) = \sqrt{v}\,f(u)\overline{\hat{\psi}(uv)}, \tag{11}$$

where $f \in L_2(\mathbb{R})$ and $\psi \in L_2(\mathbb{R})$ is a real admissible wavelet. Moreover, $\|F(u,v)\|_{A_1} = \|f(u)\|_{L_2(\mathbb{R},\,du)}$. The orthogonal projection $B_1 : L_2(G, d\nu) \to A_1$ has obviously the form

$$B_1 = U_1 P_\psi U_1^{-1} = (\mathcal{F} \otimes I) P_\psi (\mathcal{F}^{-1} \otimes I),$$

and therefore

$$(\mathcal{F}^{-1} \otimes I) B_1 F = P_\psi (\mathcal{F}^{-1} \otimes I) F = \langle (\mathcal{F}^{-1} \otimes I) F, K_\eta \rangle_G = \langle F, (\mathcal{F} \otimes I) K_\eta \rangle_G.$$

According to Lemma 3.2 we get

$$\begin{aligned}
(\mathcal{F}^{-1} \otimes I) B_1 F &= \langle F, (\mathcal{F} \otimes I) K_\eta \rangle_G \\
&= \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}} \sqrt{tv}\,F(\xi,t)\,\hat{\psi}(t\xi)\,\overline{\hat{\psi}(v\xi)}e^{2\pi iu\xi}\,d\xi \right) \frac{dt}{t^2} \\
&= \sqrt{v} \int_{\mathbb{R}} \overline{\hat{\psi}(v\xi)} \left(\int_{\mathbb{R}_+} F(\xi,t)\,\hat{\psi}(t\xi)\,\frac{dt}{t^{3/2}} \right) e^{2\pi iu\xi}\,d\xi \\
&= (\mathcal{F}^{-1} \otimes I) \left(\sqrt{v}\,\overline{\hat{\psi}(uv)} \int_{\mathbb{R}_+} F(u,t)\,\hat{\psi}(ut)\,\frac{dt}{t^{3/2}} \right),
\end{aligned}$$

or equivalently,

$$(B_1 F)(u,v) = \sqrt{v}\,\overline{\hat{\psi}(uv)} \int_{\mathbb{R}_+} F(u,t)\,\hat{\psi}(ut)\,\frac{dt}{t^{3/2}}.$$

Thus for each function $F \in L_2(G, d\nu)$ its image $(B_1F)(u,v)$ has the form (11) with

$$f(u) = \int_{\mathbb{R}_+} F(u,t)\hat{\psi}(ut)\,\frac{dt}{t^{3/2}},$$

and therefore for each function F_0 of the form (11) we have $(B_1F_0)(u,v) = F_0(u,v)$. Indeed

$$\begin{aligned}(B_1F_0)(u,v) &= \sqrt{v}\,\overline{\hat{\psi}(uv)}\int_{\mathbb{R}_+} F_0(u,t)\,\hat{\psi}(ut)\,\frac{dt}{t^{3/2}}\\ &= \sqrt{v}\,\overline{\hat{\psi}(uv)}f(u)\int_{\mathbb{R}_+} |\hat{\psi}(ut)|^2\,\frac{dt}{t}\\ &= \sqrt{v}\,\overline{\hat{\psi}(uv)}f(u) = F_0(u,v).\end{aligned}$$

Thus A_1 coincides with the space of all functions of the form (11).

Introduce now the unitary operator $U_2 : L_2(\mathbb{R}, du)\otimes L_2(\mathbb{R}_+, v^{-2}dv) \to L_2(\mathbb{R}, dx)\otimes L_2(\mathbb{R}_+, dy)$ by the rule

$$U_2 : F(u,v) \mapsto \frac{\sqrt{|x|}}{y}F\left(x, \frac{y}{|x|}\right).$$

Then the inverse operator $U_2^{-1} = U_2^* : L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}_+, dy) \to L_2(\mathbb{R}, du) \otimes L_2(\mathbb{R}_+, v^{-2}dv)$ is given by

$$U_2^{-1} : F(x,y) \mapsto \sqrt{|u|}v\,F(u, |u|v).$$

Denote by $A_2 = U_2(A_1)$. Then the operator $B_2 = U_2B_1U_2^{-1}$ is obviously the orthogonal projection of $L_2(G, d\nu)$ onto A_2, and

$$\begin{aligned}(B_2F)(x,y) &= (U_2B_1U_2^{-1}F)(x,y)\\ &= U_2\left(\sqrt{|u|}v\,\overline{\hat{\psi}(uv)}\int_{\mathbb{R}_+} F(u,|u|\theta)\,\hat{\psi}(u\theta)\,\frac{d\theta}{\sqrt{\theta}}\right)\\ &= \frac{1}{\sqrt{y}}\,\overline{\hat{\psi}(\mathrm{sgn}(x)y)}\int_{\mathbb{R}_+} F(x,\tau)\,\hat{\psi}(\mathrm{sgn}(x)\tau)\,\frac{d\tau}{\sqrt{\tau}}.\end{aligned}$$

Now with $F \in A_1$ the space $A_2 \subset L_2(\mathbb{R}, dx)\otimes L_2(\mathbb{R}_+, dy)$ consists of all functions of the form

$$(U_2F)(x,y) = \frac{\sqrt{|x|}}{y}\,F\left(x, \frac{y}{|x|}\right) = \frac{1}{\sqrt{y}}\,\overline{\hat{\psi}(\mathrm{sgn}(x)y)}\,f(x).$$

Introducing

$$l_\pm(y) = \frac{1}{\sqrt{y}}\,\overline{\hat{\psi}(\pm y)},$$

we obviously have that $l_\pm(y) \in L_2(\mathbb{R}_+,\, dy)$ and $\|l_\pm(y)\| = 1$ (from the admissibility condition (3)). For each $f \in L_2(\mathbb{R})$, let $f_\pm(x) = \chi_\pm(x)f(x)$ be its restriction onto positive and negative half-lines. Then

$$(U_2F)(x, y) = f_+(x)l_+(y) + f_-(x)l_-(y).$$

Denote by $L_\pm$ the one-dimensional subspaces of $L_2(\mathbb{R}_+,\, dy)$ generated by functions $l_\pm(y)$. Then

$$A_2 = L_2(\mathbb{R}_+) \otimes L_+ \oplus L_2(\mathbb{R}_-) \otimes L_-,$$

and the one-dimensional projections $P_\pm$ of $L_2(\mathbb{R}_+,\, dy)$ onto $L_\pm$ have the form

$$(P_\pm H)(y) = \langle H, l_\pm\rangle l_\pm = \frac{1}{\sqrt{y}}\,\overline{\hat{\psi}(\pm y)} \int_{\mathbb{R}_+} H(\tau)\hat{\psi}(\pm\tau)\,\frac{d\tau}{\sqrt{\tau}}. \tag{12}$$

Thus, $B_2 = \chi_+ I \otimes P_+ \oplus \chi_- I \otimes P_-$. This leads immediately to the following theorem which describes the structure of the space of wavelet transforms $W_\psi(L_2(\mathbb{R}))$ inside $L_2(G, d\nu)$.

Theorem 3.3 *The unitary operator $U = U_2U_1$ gives an isometrical isomorphism of the space $L_2(G, d\nu) = L_2(\mathbb{R},\, du) \otimes L_2(\mathbb{R}_+,\, v^{-2}dv)$ onto $L_2(\mathbb{R},\, dx) \otimes L_2(\mathbb{R}_+,\, dy)$ under which*

(i) the space of Calderón transforms $W_\psi(L_2(\mathbb{R}))$ is mapped onto $L_2(\mathbb{R}_+) \otimes L_+ \oplus L_2(\mathbb{R}_-) \otimes L_-$ with

$$U : W_\psi(L_2(\mathbb{R})) \mapsto L_2(\mathbb{R}_+) \otimes L_+ \oplus L_2(\mathbb{R}_-) \otimes L_-,$$

where $L_\pm$ are the one-dimensional subspaces of $L_2(\mathbb{R}_+,\, dy)$ generated by

$$l_\pm(y) = \frac{1}{\sqrt{y}}\,\overline{\hat{\psi}(\pm y)};$$

(ii) the projection P_ψ is unitarily equivalent to

$$UP_\psi U^{-1} = \chi_+ I \otimes P_+ \oplus \chi_- I \otimes P_-,$$

where $P_\pm$ are the one-dimensional projections of $L_2(\mathbb{R}_+,\, dy)$ onto $L_\pm$ given in (12).

Introduce the operator

$$R_0 : L_2(\mathbb{R}) \to L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_-) \otimes L_2(\mathbb{R}_+)$$

by the rule

$$(R_0 f)(x, y) = f_+(x)l_+(y) + f_-(x)l_-(y).$$

Obviously, the image of R_0 coincides with the space A_2. The adjoint operator

$$R_0^* : L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_-) \otimes L_2(\mathbb{R}_+) \to L_2(\mathbb{R})$$

is given by

$$(R_0^* F)(x) = \chi_+(x) \int_{\mathbb{R}_+} F(x,\tau)\overline{l_+(\tau)}\, d\tau + \chi_-(x) \int_{\mathbb{R}_+} F(x,\tau)\overline{l_-(\tau)}\, d\tau,$$

and

$$\begin{aligned} R_0^* R_0 &= I \ : L_2(\mathbb{R}) \to L_2(\mathbb{R}), \\ R_0 R_0^* &= B_2 : L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_-) \otimes L_2(\mathbb{R}_+) \to A_2. \end{aligned}$$

Summarizing the above construction we have

Theorem 3.4 *The operator $R = R_0^* U$ maps the space $L_2(G, d\nu)$ onto $L_2(\mathbb{R})$, and the restriction*

$$R|_{W_\psi(L_2(\mathbb{R}))} : W_\psi(L_2(\mathbb{R})) \to L_2(\mathbb{R})$$

is an isometrical isomorphism. The adjoint operator

$$R^* = U^* R_0 : L_2(\mathbb{R}) \to W_\psi(L_2(\mathbb{R})) \subset L_2(G, d\nu)$$

is an isometrical isomorphism of the space $L_2(\mathbb{R})$ onto the subspace $W_\psi(L_2(\mathbb{R}))$ of the space $L_2(G, d\nu)$.

Corollary 3.5 According to the construction of operators R and R^* we have:

$$\begin{aligned} RR^* &= I : L_2(\mathbb{R}) \to L_2(\mathbb{R}), \\ R^* R &= P_\psi : L_2(G, d\nu) \to W_\psi(L_2(\mathbb{R})). \end{aligned}$$

Theorem 3.6 *The isometrical isomorphism*

$$R^* = U^* R_0 : L_2(\mathbb{R}) \to W_\psi(L_2(\mathbb{R}))$$

is given by

$$(R^* f)(z) = \sqrt{y} \int_{\mathbb{R}} f(\xi)\overline{\hat{\psi}(y\xi)} e^{2\pi i x\xi}\, d\xi, \quad z = (x,y) \in G. \tag{13}$$

Proof. The direct calculation yields

$$\begin{aligned} (R^* f)(z) &= (U^* R_0 f)(z) = (U_1^* U_2^* R_0 f)(z) \\ &= (\mathcal{F}^{-1} \otimes I) \left(\sqrt{|\xi| y}\ (f_+(\xi) l_+(y|\xi|) + f_-(\xi) l_-(y|\xi|)) \right) \\ &= (\mathcal{F}^{-1} \otimes I) \left(\sqrt{y}\ \left(f_+(\xi)\overline{\hat{\psi}(y|\xi|)} + f_-(\xi)\overline{\hat{\psi}(-y|\xi|)} \right) \right) \\ &= (\mathcal{F}^{-1} \otimes I) \left(\sqrt{y}\, f(\xi)\overline{\hat{\psi}(y\xi)} \right) \\ &= \sqrt{y} \int_{\mathbb{R}} f(\xi)\overline{\hat{\psi}(y\xi)} e^{2\pi i x\xi}\, d\xi. \end{aligned}$$

□

Corollary 3.7 The inverse isomorphism

$$R = R_0^* U : W_\psi(L_2(\mathbb{R})) \to L_2(\mathbb{R})$$

is given by

$$(RF)(\xi) = \int_G F(\zeta)\hat{\psi}(v\xi)e^{-2\pi i u\xi}\,\frac{dudv}{v^{3/2}}, \quad \zeta = (u,v) \in G. \tag{14}$$

4 Calderón-Toeplitz operators with symbols depending on $v = \Im\zeta$

Now, using the representation of $W_\psi(L_2(\mathbb{R}))$ from the previous section we may state the following theorem describing the important tool for studying the properties of Calderón-Toeplitz operator T_a with symbol a depending only on $v = \Im\zeta$.

Theorem 4.1 *Let $a = a(v)$ be a measurable function on $L_2(\mathbb{R}_+)$. Then the Calderón-Toeplitz operator T_a acting on $W_\psi(L_2(\mathbb{R}))$ is unitarily equivalent to the multiplication operator $\gamma_a I = RT_aR^*$ acting on $L_2(R)$, where R and R^* are given by (14) and (13), respectively, and $\gamma_a(u) = \chi_+(u)\gamma_a^+(u) + \chi_-(u)\gamma_a^-(u)$, where*

$$\gamma_a^\pm(u) = \chi_\pm(u)\int_{\mathbb{R}_+} a\left(\frac{\tau}{|u|}\right)|l_\pm(\tau)|^2\,d\tau, \quad u \in \mathbb{R}.$$

Proof. The operator T_a is obviously unitarily equivalent to the operator

$$\begin{aligned} RT_aR^* &= RP_\psi aP_\psi R^* = R(R^*R)a(R^*R)R^* \\ &= (RR^*)RaR^*(RR^*) = RaR^* \\ &= R_0^*U_2(\mathcal{F}\otimes I)a(v)(\mathcal{F}^{-1}\otimes I)U_2^{-1}R_0 \\ &= R_0^*U_2a(v)U_2^{-1}R_0 \\ &= R_0^*a\left(\frac{y}{|x|}\right)R_0. \end{aligned}$$

Finally,

$$\begin{aligned} \left(R_0^*a\left(\frac{y}{|x|}\right)R_0 f\right)(x) &= f_+(u)\chi_+(u)\int_{\mathbb{R}_+} a\left(\frac{\tau}{|u|}\right)|l_+(\tau)|^2\,d\tau \\ &+ f_-(u)\chi_-(u)\int_{\mathbb{R}_+} a\left(\frac{\tau}{|u|}\right)|l_-(\tau)|^2\,d\tau \\ &= f(u)\Big(\chi_+(u)\gamma_a^+(u) + \chi_-(u)\gamma_a^-(u)\Big) \\ &= f(u)\gamma_a(u), \end{aligned}$$

where for $u \in \mathbb{R}$

$$\gamma_a^{\pm}(u) = \chi_{\pm}(u) \int_{\mathbb{R}_+} a\left(\frac{\tau}{|u|}\right) |l_{\pm}(\tau)|^2 \, d\tau,$$

and $\gamma_a(u) = \chi_+(u)\gamma_a^+(u) + \chi_-(u)\gamma_a^-(u)$. □

Corollary 4.2 The Calderón-Toeplitz operator T_a with symbol $a = a(v)$ is bounded on $W_\psi(L_2(\mathbb{R}))$ if and only if the corresponding function $\gamma_a(u)$ is bounded.

Reverting the statement of Theorem 4.1 we come to the following spectral-type representation of a Calderón-Toeplitz operator (which is much easier than the representation (10)). Its proof goes directly from Theorem 4.1, Theorem 3.6 and Corollary 3.7.

Theorem 4.3 *Let $\zeta = (u, v) \in G$. If $a(\zeta) = a(v)$ is a measurable function on $L_2(\mathbb{R}_+)$, the Calderón-Toeplitz operator T_a acting on $W_\psi(L_2(\mathbb{R}))$ has the following representation*

$$(T_aF)(\zeta) = \sqrt{v} \int_{\mathbb{R}} \gamma_a(\xi)\overline{\hat{\psi}(v\xi)} f(\xi) e^{2\pi i u\xi} \, d\xi, \tag{15}$$

where $f(\xi) = (RF)(\xi)$.

At the same time it is instructive to give a direct proof of the theorem which does not use the results of the previous section.

Proof. Indeed, for a symbol $a = a(v)$ depending only on $v = \Im\zeta$ consider the Calderón-Toeplitz operator, see (9),

$$(T_aF)(\zeta) = \int_G a(t)F(\eta)K_\zeta(\eta) \, d\nu(\eta),$$

where $\eta = (s, t) \in G$. Represent the function $F(\eta)$ as follows

$$F(\eta) = \sqrt{t} \int_{\mathbb{R}} f(\xi)\overline{\hat{\psi}(t\xi)} e^{2\pi i s\xi} \, d\xi,$$

where $f \in L_2(\mathbb{R})$. Thus,

$$\begin{aligned}(T_aF)(\zeta) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}_+} \sqrt{t}\, a(t) \left(\int_{\mathbb{R}} f(\xi)\overline{\hat{\psi}(t\xi)} e^{2\pi i s\xi} \, d\xi \right) K_\zeta(\eta) \, \frac{dt}{t^2} \right) ds \\ &= \int_{\mathbb{R}} f(\xi) \left(\int_{\mathbb{R}_+} \sqrt{t}\, a(t)\overline{\hat{\psi}(t\xi)} \left(\int_{\mathbb{R}} K_\zeta(\eta) e^{2\pi i s\xi} \, ds \right) \frac{dt}{t^2} \right) d\xi.\end{aligned}$$

Using Lemma 3.2 we have

$$\int_{\mathbb{R}} K_\zeta(\eta) e^{2\pi i s\xi} \, ds = \overline{\mathcal{F}(K_\eta(\zeta))(\xi)} = \sqrt{tv}\, \hat{\psi}(t\xi)\, \overline{\hat{\psi}(v\xi)} e^{2\pi i u\xi},$$

and therefore

$$(T_aF)(\zeta) = \sqrt{v} \int_{\mathbb{R}} f(\xi)\overline{\hat{\psi}(v\xi)}e^{2\pi iu\xi} \left(\int_{\mathbb{R}_+} a(t)|\hat{\psi}(t\xi)|^2 \frac{dt}{t} \right) d\xi.$$

It remains to prove that $\int_{\mathbb{R}_+} a(t)|\hat{\psi}(t\xi)|^2 \frac{dt}{t} = \gamma_a(\xi)$. Indeed,

$$\begin{aligned}\gamma_a(\xi) &= \chi_+(\xi)\gamma_a^+(\xi) + \chi_-(\xi)\gamma_a^-(\xi) \\ &= \chi_+(\xi) \int_{\mathbb{R}_+} a\left(\frac{\tau}{|\xi|}\right) |l_+(\tau)|^2\, d\tau + \chi_-(\xi) \int_{\mathbb{R}_+} a\left(\frac{\tau}{|\xi|}\right) |l_-(\tau)|^2\, d\tau \\ &= \chi_+(\xi) \int_{\mathbb{R}_+} a\left(\frac{\tau}{|\xi|}\right) |\hat{\psi}(\tau)|^2 \frac{d\tau}{\tau} + \chi_-(\xi) \int_{\mathbb{R}_+} a\left(\frac{\tau}{|\xi|}\right) |\hat{\psi}(-\tau)|^2 \frac{d\tau}{\tau}.\end{aligned}$$

Using the substitution $\tau = t|\xi|$, we get

$$\begin{aligned}\gamma_a(\xi) &= \chi_+(\xi) \int_{\mathbb{R}_+} a(t)|\hat{\psi}(t|\xi|)|^2 \frac{dt}{t} + \chi_-(\xi) \int_{\mathbb{R}_+} a(t)|\hat{\psi}(-t|\xi|)|^2 \frac{dt}{t} \\ &= \chi_+(\xi) \int_{\mathbb{R}_+} a(t)|\hat{\psi}(t\xi)|^2 \frac{dt}{t} + \chi_-(\xi) \int_{\mathbb{R}_+} a(t)|\hat{\psi}(t\xi)|^2 \frac{dt}{t} \\ &= \int_{\mathbb{R}_+} a(t)|\hat{\psi}(t\xi)|^2 \frac{dt}{t},\end{aligned}$$

which completes the proof. □

It is straightforward to check that if the symbol a of the Calderón-Toeplitz operator T_a depends on u, i.e. $a(\zeta) = a(u)$ with $\zeta = (u, v) \in G$, then the operator T_a is closely related to the operator of multiplication by $a(u)$. For example, $\|T_a\| \sim \|a\|_\infty$, and such a T_a will never be compact. Also the case $a(\zeta) = a(v)$ was studied in [13] and the following statement was given therein without the proof. It may be seen that T_a is given by a convolution with the inverse Fourier transform of γ_a. For the sake of completeness we state it in our notation and give its short proof.

Lemma 4.4 *(cf. [13]) If $a(\zeta) = a(v)$ with $\zeta = (u, v) \in G$, then the Calderón-Toeplitz operator T_a has the form*

$$\langle T_a f, g\rangle = \int_{\mathbb{R}} \gamma_a(\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi.$$

Proof. By Fourier representation of W_ψ we have

$$W_\psi f(u, v) = \sqrt{v} \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{\psi}(v\xi)}e^{2\pi iu\xi}\, d\xi.$$

Then according to (1) we may write

$$\begin{aligned}\langle T_a f, g\rangle &= \langle aW_\psi f, W_\psi g\rangle_G = \int_{\mathbb{R}\times\mathbb{R}_+} a(v) W_\psi f(u,v)\overline{W_\psi g(u,v)}\, du\, \frac{dv}{v^2}\\ &= \int_{\mathbb{R}\times\mathbb{R}_+} a(v)\left(\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{\psi}(v\xi)} e^{2\pi i u\xi}\, d\xi\right)\cdot \overline{\left(\int_{\mathbb{R}} \hat{g}(\xi)\overline{\hat{\psi}(v\xi)} e^{2\pi i u\xi}\, d\xi\right)} \frac{dudv}{v}\\ &= \int_{\mathbb{R}\times\mathbb{R}_+} a(v)\left(\int_{\mathbb{R}} \overline{\hat{g}(\xi)}\hat{\psi}(v\xi) e^{-2\pi i u\xi}\, d\xi\right)\cdot \overline{\left(\int_{\mathbb{R}} \overline{\hat{f}(\xi)}\hat{\psi}(v\xi) e^{-2\pi i u\xi}\, d\xi\right)} \frac{dudv}{v}\\ &= \int_{\mathbb{R}_+} a(v)\left(\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, |\hat{\psi}(v\xi)|^2\, d\xi\right)\frac{dv}{v}.\end{aligned}$$

Applying Fubini theorem yields

$$\langle T_a f, g\rangle = \int_{\mathbb{R}}\left(\int_{\mathbb{R}_+} a(v)|\hat{\psi}(v\xi)|^2\,\frac{dv}{v}\right)\hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi.$$

Hence the result. □

Remark 4.5 In the other words, the Fourier transform computation shows that T_a is a (Fourier) multiplier with γ_a. This is in line with the informal analysis. The normalization of ψ ensures that $|\hat{\psi}(v\xi)|^2$ is a probability density with respect to the measure $\frac{dv}{v}$. Also, if we select $\psi \in L_2(\mathbb{R})$ with $\hat{\psi}$ concentrated near the unit sphere, then $|\hat{\psi}(v\xi)|^2$ will be concentrated near the sphere of radius v^{-1}.

Remark 4.6 The results involving Calderón-Toeplitz operators with symbols depending on the individual coordinate functions describe an analogy between Calderón-Toeplitz operators and the calculus of pseudodifferential operators, cf. [13].

The following result gives the form of the Wick symbol of Calderón-Toeplitz operator T_a depending on $v = \Im\zeta$.

Theorem 4.7 *Let $a = a(v)$ be a measurable function on $L_2(\mathbb{R}_+)$. Then the Wick symbol $\widetilde{a}(\zeta)$ of the Calderón-Toeplitz operator T_a depends only on v as well, and has the form*

$$\widetilde{a}(v) = \widetilde{a}(\zeta,\zeta) = \frac{\langle T_a\psi_\zeta, \psi_\zeta\rangle}{\langle\psi_\zeta,\psi_\zeta\rangle} = v\int_{\mathbb{R}} \gamma_a(\xi)|\hat{\psi}(v\xi)|^2\, d\xi. \qquad (16)$$

The corresponding Wick function is given by the formula

$$\widetilde{a}(\zeta,\eta) = \frac{\langle T_a\psi_\zeta, \psi_\eta\rangle}{\langle\psi_\zeta,\psi_\eta\rangle} = \frac{\sqrt{tv}}{K_\zeta(\eta)}\int_{\mathbb{R}} \gamma_a(\xi)\hat{\psi}(v\xi)\overline{\hat{\psi}(t\xi)} e^{-2\pi i\xi(u-s)}\, d\xi, \qquad (17)$$

with $\zeta = (u,v)$, $\eta = (s,t) \in G$.

Proof. Consider the kernel $K_\zeta(\eta) = \langle \psi_\zeta, \psi_\eta \rangle$ with $\zeta = (u, v)$, $\eta = (s, t) \in G$. By Lemma 3.2,

$$U_1 K_\zeta(\eta) = \sqrt{tv}\, \hat{\psi}(t\xi)\overline{\hat{\psi}(v\xi)} e^{-2\pi i s\xi},$$

and therefore from representation (1) we get

$$\begin{aligned}
\langle T_a \psi_\zeta, \psi_\zeta \rangle &= \langle aK_\zeta, K_\zeta \rangle_G = \langle U_1(aK_\zeta), U_1 K_\zeta \rangle_G = \langle aU_1 K_\zeta, U_1 K_\zeta \rangle_G \\
&= v \int_{\mathbb{R}\times\mathbb{R}_+} a(t) |\hat{\psi}(v\xi)|^2 |\hat{\psi}(t\xi)|^2 \, \frac{d\xi dt}{t} \\
&= v \int_{\mathbb{R}} \left(\int_{\mathbb{R}_+} a(t) |\hat{\psi}(t\xi)|^2 \, \frac{dt}{t} \right) |\hat{\psi}(v\xi)|^2 \, d\xi \\
&= v \int_{\mathbb{R}} \gamma_a(\xi) |\hat{\psi}(v\xi)|^2 \, d\xi.
\end{aligned}$$

Thus, we have (16). The equality (17) may be verified by direct calculations, or using Lemma 4.4. □

Remark 4.8 Let $\zeta = (u, v), \eta = (s, t) \in G$, $a(\zeta) = a(v)$, and $F \in W_\psi(L_2(\mathbb{R}))$. Applying the general approach to coherent states, cf. [3], to our particular case it is immediate that

$$\widetilde{a}(\zeta, \eta) = \frac{\langle T_a \psi_\zeta, \psi_\eta \rangle}{\langle \psi_\zeta, \psi_\eta \rangle} = \frac{\langle aK_\zeta, K_\eta \rangle_G}{K_\zeta(\eta)},$$

and therefore by using (8) we have

$$\begin{aligned}
(T_a F)(\eta) &= \int_G a(\theta) F(\theta) K_\eta(\theta) \, d\nu(\theta) \\
&= \int_G a(\theta) \left(\int_G F(\zeta) K_\theta(\zeta) \, d\nu(\zeta) \right) K_\eta(\theta) \, d\nu(\theta) \\
&= \int_G F(\zeta) \left(\int_G a(\theta) K_\theta(\zeta) K_\eta(\theta) \, d\nu(\theta) \right) d\nu(\zeta) \\
&= \int_G F(\zeta) K_\zeta(\eta) \left(\frac{1}{K_\zeta(\eta)} \int_G a(\theta) K_\zeta(\theta) K_\eta(\theta) \, d\nu(\theta) \right) d\nu(\zeta) \\
&= \int_G \widetilde{a}(\zeta, \eta) F(\zeta) K_\zeta(\eta) \, d\nu(\zeta) \\
&= \sqrt{t} \int_{\mathbb{R}\times\mathbb{R}_+} F(\zeta) \left(\int_{\mathbb{R}} \gamma_a(\xi) \hat{\psi}(v\xi) \overline{\hat{\psi}(t\xi)} e^{-2\pi i \xi(u-s)} \, d\xi \right) \frac{dudv}{v^{3/2}} \\
&= \sqrt{t} \int_{\mathbb{R}} \gamma_a(\xi) \overline{\hat{\psi}(t\xi)} e^{2\pi i s\xi} \left(\int_{\mathbb{R}\times\mathbb{R}_+} F(\zeta) \hat{\psi}(v\xi) e^{-2\pi i u\xi} \, \frac{dudv}{v^{3/2}} \right) d\xi \\
&= \sqrt{t} \int_{\mathbb{R}} \gamma_a(\xi) \overline{\hat{\psi}(t\xi)} (RF)(\xi) e^{2\pi i s\xi} \, d\xi,
\end{aligned}$$

i.e. writing the Calderón-Toeplitz operator T_a in terms of its Wick symbol yields the representation (15).

Corollary 4.9 Let T_a and T_b be two Calderón-Toeplitz operators with symbols $a(v)$ and $b(v)$, where $a(v)$, $b(v)$ are measurable functions on $L_2(\mathbb{R}_+)$, and let $\widetilde{a}(v)$ and $\widetilde{b}(v)$ be their Wick symbols, respectively. Then the Wick symbol $\widetilde{c}(v)$ of the composition T_aT_b is given by

$$\widetilde{c}(v) = (\widetilde{a} \star \widetilde{b})(v) = v \int_{\mathbb{R}} \gamma_a(\xi)\gamma_b(\xi)|\hat{\psi}(v\xi)|^2 \, d\xi.$$

Proof. The result follows immediately from Theorem 4.1 and Theorem 4.7. □

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