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Abstract: A face of a vertex coloured plane graph is called *loose* if the number of colours used on its vertices is at least three. A k-colouring of a graph G is called the *loose k-colouring* if it involves a loose face. The *looseness* of a plane graph G, $ls(G)$, is the minimum k such that any surjective k-colouring is a loose k-colouring of G. In this paper we prove that the looseness of a connected plane graph G equals 2 plus the maximum number of vertex disjoint cycles in the dual graph G^*.

We also show upper bounds on the looseness of graphs based on the edge connectivity, the girth of the dual graphs and other basic graph invariants. Moreover, we present infinite classes of graphs where these equalities are attained.

Keywords: vertex colouring, loose colouring, looseness, plane graph, dual graph

2000 Mathematics Subject Classification: 05C15

1 Introduction

We use the standard terminology according to Bondy and Murty [1], except for few notations defined throughout. However, we recall some frequently used terms. All considered graphs are finite, loops and multiple edges are allowed.

Let $G = (V, E, F)$ be a connected plane graph with the vertex set V, the edge set E and the face set F. The *degree* of a vertex v is the number of edges incident with v, each loop counting as two edges. For a face f, the *size* of f, $deg(f)$, is defined to be the length of the shortest closed walk containing all edges from the boundary of f.

We write $v \in f$ if a vertex v is incident with a face f. The vertices u and v are *face independent* if there is no face f such that $v \in f$ and $u \in f$. A set S of vertices is face independent if any two vertices from S are face independent. The *face independence number* of a graph G, $\alpha_2(G)$, is defined to be the number of vertices in the maximum face independent set of vertices.

A *k-colouring* of a graph G is a mapping $\varphi : V(G) \to \{1, \ldots, k\}$. For a set $X \subseteq V$ we define $\varphi(X)$ to be the set of colours $\{\varphi(v); v \in X\}$. Particularly, if f is a face of G then $\varphi(f)$ denotes the set of colours used on the vertices incident with the face f. A face $f \in F$ is called *loose* if $|\varphi(f)| \geq 3$ (*rainbow* if $|\varphi(f)| = deg(f)$), otherwise it is called nonloose (nonrainbow, respectively). A k-colouring of a graph G is called the *nonloose k-colouring* (*nonrainbow k-colouring*) if it does not involve any loose (rainbow) face, otherwise it is a *loose k-colouring* (*rainbow k-colouring*, respectively).

An edge in a vertex coloured graph is called *heterochromatic* if its ends are coloured with different colours. Otherwise it is called *monochromatic.*

This paper is motivated by recent papers [3], [4], [5], [6], [7], [8], [12], [13] dealing with the vertex coloured plane graphs having rainbow faces. The above mentioned papers try to answer the following Ramsey type question:

Question 1 *What is the minimum number of colours $rb(G)$ that any surjective vertex colouring of a connected plane graph G with $rb(G)$ colours enforces a rainbow face?*

The parameter $rb(G)$ is called the *rainbowness* of G, see [5]. We now survey some results dealing with Question 1. Ramamurthi and West in [12], [13] noticed that every plane graph G of order n has a nonrainbow colouring with at least $\lceil \frac{n}{4} \rceil + 1$ colours by the Four Colour Theorem. They in [13] had conjectured and Jungič et al. in [6] proved that this bound can be improved to the bound $\lceil \frac{n}{2} \rceil + 1$ for triangle-free plane graphs. More generally, Jungič et al. in [6] proved that every plane graph of order n with girth $g \geq 5$ has a nonrainbow colouring with at least $\lceil \frac{g-3}{g-2} n - \frac{g-7}{2(g-2)} \rceil$ colours if g is odd and $\lceil \frac{g-3}{g-2} n - \frac{g-6}{2(g-2)} \rceil$ if g is even. It is also shown that these bounds are best possible.

There are also results concerning upper bounds on $rb(G)$. Dvořák et al. in [4] proved that for every n vertex 3-connected plane graph G it holds $rb(G) \leq \lfloor \frac{7n+1}{9} \rfloor$, for every 4-connected graph G it holds $rb(G) \leq \lfloor \frac{5n+2}{8} \rfloor$ if $n \not\equiv 3 \pmod 8$, $rb(G) \leq \lfloor \frac{5n-6}{8} \rfloor$ if $n \equiv 3 \pmod 8$ and for every 5-connected plane graph G $rb(G) \leq \lfloor \frac{43}{100} n + \frac{6}{25} \rfloor$. Moreover, the bounds for the 3- and 4-connected graphs are best possible.

There are also results on specific families of plane graphs, e.g. the numbers $rb(G)$ were also determined for all, up to three semiregular polyhedra by Jendroľ and Schrötter [7]. Connected cubic plane graphs were studied by Jendroľ in [5] and tight estimations on $rb(G)$ were found there. In the paper Dvořák et al. [3] it is proved that for 3-connected cubic plane graph G $rb(G) = \frac{n}{2} - \mu^* - 1$, where n is the order of G and μ^* is the size of the maximum matching of the dual graph G^*.

In this paper we consider a relaxation of Question 1. Namely we are interested in finding an answer to the following

Question 2 *What is the minimum number of colours $ls(G)$ that any surjective vertex colouring of a connected plane graph G with $ls(G)$ colours enforces a loose face?*

The invariant $ls(G)$ of a plane graph G is called the *looseness* of G and it has been introduced by Negami and Midorikawa [11], see also [10]. The looseness is well defined for all plane graphs having at least one face incident with at least three different vertices. Throughout the paper, we will consider only such graphs. Observe that if a plane graph G has only triangular faces then $rb(G) = ls(G)$.

On the other hand, in general, the difference between the rainbowness and the looseness of a graph can be arbitrary large. For example, it is proved in [3] that the rainbowness of the d-sided prism, $d \geq 4$, is $\lfloor \frac{3d}{2} \rfloor$ but it is easy to see that the looseness of the same d-sided prism is 4.

Negami [9] proved that for plane triangulation G

$$\alpha_0(G) + 2 \leq ls(G) \leq 2\alpha_0(G) + 1 \ ,$$

where $\alpha_0(G)$ is the vertex independence number of G.

The rest of the paper is organized as follows. In the Section 2 we prove that the looseness of every connected plane graph G is equal to 2 plus the maximum number of vertex disjoint cycles in the dual graph G^*. The problem of determining the maximum number of vertex disjoint cycles in plane graphs is known to be NP-complete, see Bodlaender et al. [2], so good estimations for this parameter are welcome. Thus the remainder of the paper is devoted to finding some estimations for looseness of plane graphs. The Section 3 deals with upper bounds on the looseness of plane graphs in terms of basic graph invariants, namely the girth and the edge connectivity. We show that if the girth of the dual graph G^* is g then the looseness is at most

$$\frac{1}{g}|F(G)| + 2 \ ,$$

moreover, this bound is tight. We use the fact that the edge connectivity κ' of a plane graph G has a close relation to the girth of the dual graph G^* to prove that

$$ls(G) \leq \frac{1}{\kappa'}|F(G)| + 2 \ ,$$

moreover, the bound is the best possible. We also show that if G is a 3-connected cubic plane n-vertex graph then

$$ls(G) \leq \frac{1}{6}n + \frac{8}{3} \ .$$

Finally, in the Section 4 we prove that if G is a connected simple plane graph on n vertices, then the looseness of G is at most $\frac{2n+2}{3}$. This bound is the best posssible for classes of 1-, 2-, and 3-connected plane graphs.

2 General properties of the loose colourings

Consider a surjective k-colouring of a connected plane graph $G = (V, E, F)$.

Proposition 2.1 *Let G be a plane graph. Then $ls(G) \geq \alpha_2(G) + 2$.*

Proof
The existence of a suitable nonloose $(\alpha_2(G) + 1)$-colouring of G is an easy observation. ■

We proceed with the other estimations on looseness of plane graphs.

Let us recall that the (geometric) *dual* $G^* = (V^*, E^*, F^*)$ of the plane graph $G = (V, E, F)$ one can define as follows (see [1], p.252): Corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*; two vertices f^* and g^* are joined by the edge e^* in G^* if and only if their corresponding faces f and g are separated by the edge e in G (an edge separates the faces incident with it).

First we show a lower bound on the looseness.

Theorem 2.1 *Let $G = (V, E, F)$ be a connected plane graph such that the dual G^* of G has t vertex disjoint cycles. Then*

$$ls(G) \geq t + 2 \ .$$

Proof
Let $\mathcal{C} = \{C_1, \ldots, C_t\}$ be a set of vertex disjoint cycles in G^*. Each C_i divides the plane into two parts, let D_i be the bounded one.

Since the cycles in $\mathcal{C}$ are vertex disjoint, for each $1 \leq i < j \leq t$ we have either $D_i \cap D_j = \emptyset$, or $D_i \subsetneq D_j$, or vice versa. We say that C_i is inside C_j (and write $C_i \preceq C_j$) if $D_i \subseteq D_j$, for $1 \leq i \leq j \leq t$. It is easy to see that $\preceq$ is reflexive, antisymmetric and transitive relation on $\mathcal{C}$, thus $(\mathcal{C}, \preceq)$ is a partially ordered set. We can extend the ordering into a linear ordering. Therefore, without loss of generality we can assume that the cycles in $\mathcal{C}$ are indexed in such a way that $C_i \preceq C_j$ implies $i \leq j$.

Given the cycles $\mathcal{C}$, we colour the faces of G^* in the following way: Faces contained in $D_i \setminus \bigcup_{s=1}^{i-1} D_s$ receive the colour i; faces not contained in any D_i receive the colour $t+1$. It is clear that this colouring induces the colouring of the vertices of the graph G using precisely $t+1$ colours. Observe that all the faces inside C_i (the faces forming D_i) received the colours at most i.

We show that this colouring is a nonloose $(t+1)$-colouring. Let us suppose for a contradiction that the graph G contains a face f which is incident with three vertices v_i, v_j, v_k having three different colours $1 \leq i < j < k \leq t+1$. These correspond to three faces v_i^*, v_j^*, v_k^* of G^* coloured i, j, k, all incident with a vertex f^*. Since the face v_i^* has colour i, $v_i^* \subseteq D_i$. On the other hand, the face v_j^* has colour $j > i$, therefore v_j^* is not inside C_i. The vertex f^* is incident both with a face inside and outside C_i, hence $f^* \in C_i$. Analogously, v_j^* is inside C_j, v_k^* is outside C_j, hence $f^* \in C_j$. The vertex f^* lies on two vertex disjoint cycles, a contradiction. ■

Theorem 2.2 *Let $G = (V, E, F)$ be a connected plane graph and let G^* be its dual. Then there are t_0 vertex disjoint cycles in G^* such that*

$$ls(G) = t_0 + 2 \ .$$

Proof

Let φ be a nonloose k-colouring of the graph G, such that $k = ls(G) - 1$. Let $G^* = (V^*, E^*, F^*)$ be the dual of G.

Let E_{ij} denotes a set of edges of a graph G such that their ends are coloured with colours i and j, $i \neq j$. Let E_{ij}^* be a set of edges in the dual graph G^* which correspond to E_{ij} in G.

The graph $G^*[E_{ij}^*]$ induced by the edges from E_{ij}^* has minimum degree at least two, because if a face $f \in F(G)$ is incident with a two coloured edge then its boundary walk contains at least two such edges. Hence, each graph $G^*[E_{ij}^*]$ contains at least one cycle, say C_{ij}. Moreover, each vertex $v^* \in V(G^*)$ can be incident with at most one type of heterochromatic edges, otherwise the face corresponding to v^* would be incident with at least three vertices coloured differently. Hence,

$$G^*[E_{ij}^*] \cap G^*[E_{lk}^*] = \emptyset \text{ for } \{i, j\} \neq \{k, l\}. \tag{1}$$

From (1) follows that the cycles C_{ij} and C_{kl} are vertex disjoint for $\{i, j\} \neq \{k, l\}$. Let t_0 denote the number of such cycles. Applying Theorem 2.1 we have $t_0 \leq ls(G) - 2$.

The colouring φ uses k colours, therefore the graph G has at least $k-1$ types of heterochromatic edges. Hence, $t_0 \geq k - 1 = ls(G) - 2$. So $t_0 = ls(G) - 2$. ■

3 Looseness, girth and edge-connectivity

Let us remind that the girth of a graph G is the length of its shortest cycle.

Theorem 3.1 *Let $G = (V, E, F)$ be a connected plane graph, let g be the girth of the dual graph G^* of G. Then*

$$ls(G) \leq \frac{1}{g}|F(G)| + 2 \ .$$

Moreover, the bound is sharp.

Proof
From Theorem 2.2 results that the dual graph G^* of a graph G contains t vertex disjoint cycles such that $ls(G) = t + 2$. Clearly, each cycle contains at least g vertices. Hence, we get $t \leq \frac{|F(G)|}{g}$. So we are done.

To prove the sharpness of the bound, it suffices to find examples of plane graphs such that

$$g \cdot \alpha_2(G) \geq |F(G)|. \tag{2}$$

Then, by Proposition 2.1 we have $ls(G) \geq \alpha_2(G) + 2 \geq \frac{1}{g}|F(G)| + 2$.

Let $g \geq 2$ be fixed. Let G_0 be a plane drawing of the complete bipartite graph $K_{2,k}$ ($k \geq \frac{g}{2}$) such that all the faces are quadrangles. Let u_1 and u_2 be the two vertices of degree k and $v_1, \ldots, v_k$ be the vertices of degree 2.

For $i = 1, \ldots, k$ we insert a vertex w_i into the face bounded by u_1, v_i, u_2, v_{i+1} ($v_{k+1} = v_1$), add an edge joining w_i to u_1 and to u_2, $\lfloor \frac{g-2}{2} \rfloor$ parallel edges joining w_i and v_i and $\lceil \frac{g-2}{2} \rceil$ parallel edges joining w_i and v_{i+1}, see Figure 1 for illustration.

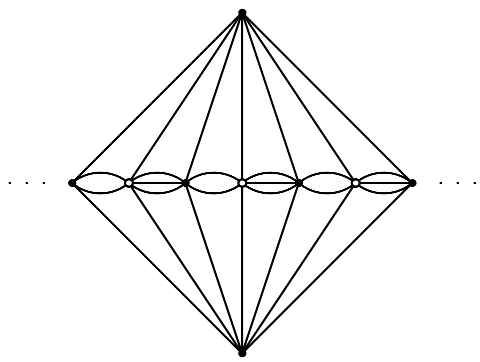

Figure 1: An example of a graph where inequality (2) is satisfied, for $g = 7$.

This way we obtain the graph G_g having $2k$ vertices of degree g and 2 vertices of degree $2k$. It is easy to see that $g(G_g^*) = g$ and $|F(G_g)| = g \cdot k$. The vertices $w_1, \ldots w_k$ are face independent, thus $\alpha_2(G_g) \geq k$. Therefore, the inequality (2) holds.

For $g = 1$ it suffices to add a vertex of degree one into each face of G_0. The resulting graph G_1 has k faces, the girth of G_1^* is 1 and the k new vertices form a face independent set, therefore, the graph G_1 satisfies (2).

For $g = 1, 2, 3$, and 4, the graphs G_g are simple. Graphs with $g > 5$ cannot be simple, since every simple plane graph contains a vertex of degree at most 5. For $g = 5$ we can use a different construction to find examples of *simple* plane graphs such that the upper bound is attained.

Let us start with a k-sided antiprism A_k, subdivide each triangular face into three new triangles by adding a new vertex of degree three, delete the edges of the antiprism originally incident with two triangles and insert the other of the two diagonals in the new quadrangular faces. This way we get a 5-regular plane graph H on $4k$ vertices (see Figure 2 for illustration). Take the dual H^* of H and insert a new vertex of degree 5 into each face of H^* and join it to all five vertices incident with the face. Let the resulting graph be G'_5. It is easy to see that the girth of the dual of G'_5 is 5 and that $|F(G'_5)| = 5 \cdot 4k = 20k$. The $4k$ newly inserted vertices form a face independent set in G'_5, therefore G'_5 satifies (2).

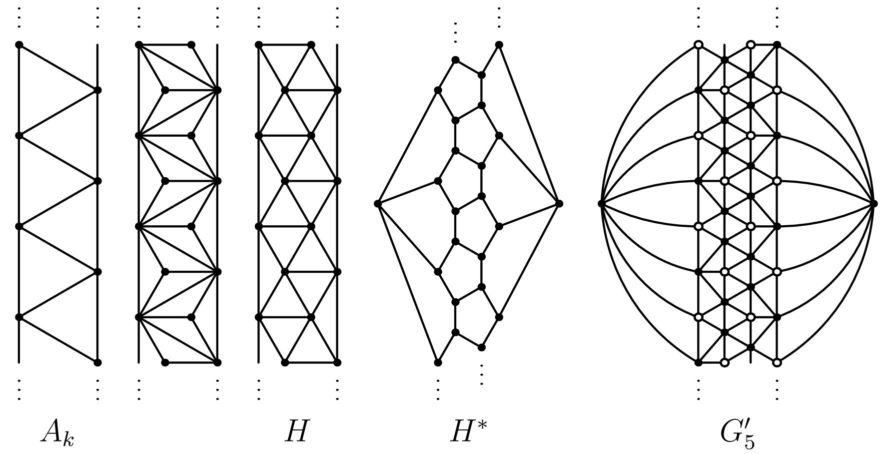

Figure 2: The construction of the graph G'_5 from the graph of an antiprism.

■

Corollary 3.1 *Let $G = (V, E, F)$ be a connected plane graph. If the dual G^* has girth g and there are disjoint cycles $C_1, \ldots, C_t$ of order g that cover all the vertices of G^* then $ls(G) = t + 2$.*

Proof

The lower bound follows from Theorem 2.1. The upper bound is obtained from Theorem 3.1, because the number of vertices of G^* is tg. ■

The edge connectivity of a plane graph G plays an important role in the concept of the looseness of G. Observe, that each minimum edge-cut of size g in

G corresponds to a cycle in G^* and vice versa, therefore, the edge connectivity of a graph G is equal to the girth of the dual graph G^*. Hence, we can obtain an upper bound for the looseness of plane graphs in terms of the edge connectivity of G.

Theorem 3.2 *Let $G = (V, E, F)$ be a connected plane graph with the edge connectivity κ'. Then*

$$ls(G) \leq \frac{1}{\kappa'}|F(G)| + 2 \ .$$

Moreover, the bound is sharp.

Using Theorem 3.2 we can receive an upper bound for the looseness of 3-connected cubic plane graphs.

Corollary 3.2 *Let G be an n-vertex 3-connected cubic plane graph. Then*

$$ls(G) \leq \frac{1}{6}n + \frac{8}{3} \ .$$

Moreover, the bound is sharp.

Proof

We apply the Euler's polyhedral formula $|V(G)| + |F(G)| - |E(G)| = 2$ and the fact that $2|E(G)| = 3|V(G)|$. We obtain $|F(G)| = \frac{1}{2}n + 2$. The edge connectivity of G is three and after applying Theorem 3.2 we have

$$ls(G) \leq \frac{1}{3}|F(G)| + 2 = \frac{1}{3}\left(\frac{1}{2}n + 2\right) + 2 = \frac{1}{6}n + \frac{8}{3} \ .$$

The sharpness follows from the following theorem. ■

Theorem 3.3 *For any integer $t \geq 4$ there exists a 3-connected cubic plane graph $G = (V, E, F)$ on n vertices such that*

$$t = ls(G) = \frac{1}{6}n + \frac{8}{3} \ . \qquad (3)$$

Proof

For $t = 4$ the equation (3) yields $n = 8$; it suffices to consider the graph of a cube. For $t = 5$ we get $n = 14$, one of the possible examples is depicted in Figure 3.

Let $t \geq 6$ and let H be an arbitrary simple 3-connected cubic plane graph on $2t - 8$ vertices (for example, K_4 or $(t-4)$-sided prism). The graph H is cubic plane graph therefore $|F(H)| = t - 2$.

We insert a star $K_{1,3}$ into each face of H in the following way. Let f be a face of H. We choose three edges on the boundary walk of the face f and we put

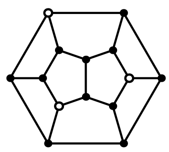

Figure 3: An example of graph on 14 vertices with $ls(G) = 5$.

one vertex on each of them and put one "central" vertex to the interior of f. We join this "central" vertex with the other three added vertices. If we perform this operation on each face of H we get a new graph G. The graph G is 3-connected cubic plane graph, furthermore, it has $n = 2t - 8 + 4 \cdot (t-2) = 6t - 16$ vertices.

Colour the "central" vertices of G with distinct colours $1, \ldots, t-2$ and the remaining vertices with the same colour $t-1$. We obtain a nonloose $(t-1)$-colouring of G. Hence, $ls(G) \geq t$. On the other hand, from Corollary 3.2 we have

$$ls(G) \leq \frac{1}{6}(6t - 16) + \frac{8}{3} = t,$$

which proves the claim. ■

4 Looseness and number of vertices

Proposition 4.1 *Let G be a plane graph on n vertices which contains a face incident with at least three vertices. Then $ls(G) \leq n$.*

If G is a simple plane graph then this trivial upper bound is tight if and only if G is a triangle. If multiple edges or loops are allowed, suitable graphs could be constructed easily.

Now we show that if G is a simple connected plane graph on n vertices, then the number of colours in a nonloose colouring does not exceed $\frac{2n+2}{3} - 1$. This bound is the best posssible for classes of 1-, 2-, and 3-connected plane graphs.

Let $\mathcal{A}_n$ be a set of connected simple plane graphs on n vertices. Let

$$\mathcal{B}_n = \{G \in \mathcal{A}_n : \quad \forall H \in \mathcal{A}_n \quad ls(G) \geq ls(H)\}$$

be the set of the graphs on n vertices with the maximal possible looseness; let its value be denoted by t_n. Let

$$\mathcal{C}_n = \{G \in \mathcal{B}_n : \quad \forall H \in \mathcal{B}_n \quad |E(G)| \leq |E(H)|\}$$

be the set of the graphs on n vertices with the maximal possible looseness and minimal number of edges.

Lemma 4.1 *Let G be a graph from the set $\mathcal{C}_n$ and let φ be a nonloose $(t_n - 1)$-colouring of G. Then every heterochromatic edge of G is a bridge.*

Proof
Suppose there is a heterochromatic edge e which is not a bridge. Let $H = G - e$ be a graph obtained from G by deleting the edge e. Clearly, the graph H is a connected plane graph on n vertices and $ls(H) = ls(G)$. We have a contradiction with the fact that $G \in \mathcal{C}_n$. ■

Lemma 4.2 *There is a graph $W \in \mathcal{C}_n$ and its nonloose $(t_n - 1)$-colouring φ such that all the vertices contained in at least one cycle have the same colour.*

Proof
Let $G \in \mathcal{C}_n$ and let φ be a nonloose $(t_n - 1)$-colouring of G. Assume that G contains at least two vertex disjoint cycles. We choose such cycles C_1 and C_2 that the distance between C_1 and C_2 is the smallest possible. From Lemma 4.1 follows that each vertex on C_1 has the same colour i and each vertex on C_2 has the same colour j. Assume that $i \neq j$. The graph G is connected, hence there is a path between C_1 and C_2. Observe that each vertex on the shortest path between C_1 and C_2 have a colour from the set $\{i, j\}$ (otherwise, there would be either a loose face or another pair of cycles in a smaller distance). Hence, every such path contains at least one heterochromatic edge uv. From Lemma 4.1 follows that e is a bridge in G.

Now we define a new graph H and a vertex colouring $\widetilde{\varphi}$ of H in the following way. Let G/uv be the graph obtained when contracting the edge uv and let z be a new vertex so obtained.

$$V(H) = V(G/uv) \cup \{w\}, \quad E(H) = E(G/uv) \cup \{zw\}.$$

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \neq z, w \text{ and } \varphi(x) \neq j, \\ i & \text{if } \varphi(x) = j, \\ i & \text{if } x = z, \\ j & \text{if } x = w. \end{cases}$$

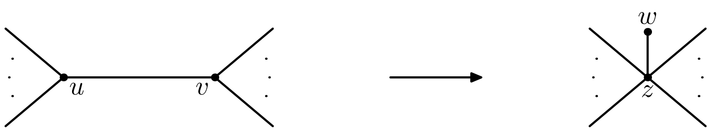

Figure 4: The new graph H from the graph G

Clearly, $ls(H) = ls(G) = t_n$. If we use this method of recolouring of cycles we can find a graph $W \in \mathcal{C}_n$ such that vertices of all cycles have the same colour. ■

Lemma 4.3 *There is a graph G in the set $\mathcal{C}_n$ and a nonloose $(t_n - 1)$-colouring φ of G such that each vertex on each cycle has the same colour 1 and for each colour from the set $\{2, \ldots, t_n - 1\}$ there is a leaf $v_i \in V(G)$ such that $\varphi(v_i) = i$.*

Proof
From Lemma 4.2 follows that there is a graph $W \in \mathcal{C}_n$ and a nonloose $(t_n - 1)$-colouring φ_1 of W such that each vertex on each cycle has the same colour, say 1.

Let i be a colour from the set $\{2, \dots, t_n - 1\}$. Each vertex with colour i, $i \neq 1$, does not lie on any cycle in W. Suppose that there is no leaf with colour i. Every vertex coloured by i is incident only with bridges, therefore, its neighbours have colours from the set $\{1, i\}$ (else there is a face which uses three different colours). Let uv be an edge such that $\varphi_1(u) = i$ and $\varphi_1(v) = 1$.

Now we define a new graph H and a vertex colouring φ_2 of H in the following way. Let z be a vertex which we get when conctracting the edge uv.

$$V(H) = V(G/uv) \cup \{w\}, \quad E(H) = E(G/uv) \cup \{zw\}.$$

$$\varphi_2(x) = \begin{cases} \varphi_1(x) & \text{if } x \neq z, w, \\ 1 & \text{if } x = z, \\ i & \text{if } x = w. \end{cases}$$

Observe that $ls(H) = ls(G) = t_n$. Moreover, the vertex w is a leaf and $\varphi_2(w) = i$. Using this method of recolouring of vertices we can find a graph G with the desired properties. ∎

Theorem 4.1 *Let G be a connected simple plane graph on n vertices. Then*

$$ls(G) \leq \frac{2n + 2}{3}.$$

Moreover, the bound is the best possible.

Proof
Let H be a graph from $\mathcal{C}_n$ and φ be a nonloose $(t_n - 1)$-colouring of H such that each vertex on each cycle has the same colour 1 and for each colour i, $i \in \{2, \dots, t_n - 1\}$ there is a leaf w_i such that $\varphi(w_i) = i$. Since $H \in \mathcal{C}_n$ we have $ls(G) \leq ls(H) = t_n$.

In a graph H, there is a face α_i such that $w_i \in \alpha_i$ for every $i \in \{2, \dots, t_n - 1\}$. Since each face is incident with at least one vertex coloured with the colour 1, the faces α_i are pairwise distinct. Therefore, $|F(H)| \geq t_n - 2$.

Let W be a graph obtained from H by deleting all the leaves. Then

$$|V(W)| \leq n - (t_n - 2).$$

The graph W is a simple connected plane graph, therefore,

$$|F(W)| \leq 2|V(W)| - 4 \leq 2(n - t_n + 2)) - 4 = 2n - 2t_n.$$

On the other hand, $|F(W)| = |F(H)| \geq t_n - 2$, hence

$$2n - 2t_n \geq |F(W)| \geq t_n - 2,$$
$$t_n \leq \frac{2n+2}{3}.$$

The sharpness of this bound follows from the next theorem. ■

Theorem 4.2 *For any integer $t \geq 1$ and any $k \in \{1, 2, 3\}$ there exists a simple k-connected plane graph $G = (V, E, F)$ on n vertices, $n \geq t$, such that*

$$ls(G) = \frac{2n+2}{3}.$$

Proof
Let $k \in \{1, 2, 3\}$. Let T be an arbitrary (3-connected) triangulation on t vertices. We insert a new vertex into each face of T. We join each new vertex with k vertices of a corresponding face. We obtain a k-connected plane graph G_k.

The graph G_k has $t + (2t - 4) = 3t - 4$ vertices. If we colour the vertices of T with the same colour 1 and the other vertices with colours $2, \dots, 2t - 3$ we obtain a nonloose $(2t - 3)$-colouring of G_k for $k \in \{1, 2, 3\}$. Hence $ls(G_k) \geq 2t - 2$.

$$ls(G_k) \geq 2t - 2 = \frac{2}{3}(3t - 4) + \frac{2}{3} = \frac{2}{3}|V(G_k)| + \frac{2}{3} = \frac{2|V(G_k)| + 2}{3}.$$

The reverse inequality follows from Theorem 4.1. ■

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