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Rotations in the stable *b*-matching problem

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Rotations in the stable b-matching problem*

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Abstract

This paper deals with the stable b-matching problem on general multigraphs. We generalize the notion of singular and dual rotation and establish a one-one correspondence between stable b-matchings and certain sets of rotations. This correspondence is used to find all stable edges and a minimum regret stable b-matching in polynomial time. We also recall the NP-completeness of the egalitarian stable b-matching problem.

Keywords. Stable b-matching, rotation poset, minimum-regret, egalitarian

1 Introduction

The stable roommates problem (SR for short) was introduced in [10]. An instance of SR consists of n participants, who want to cooperate with one another, having preferences over possible partners. A matching is sought that is stable, i.e. it admits no pair of unmatched participants who prefer each other to their partners in the matching. However, unlike in the stable marriage problem (SM for short [10]), the bipartite version of SR, not all instances have a stable matching and the first polynomial algorithm to decide its existence and to find one was proposed by Irving [16]. A key notion that enabled that algorithm was the notion of the rotation.

Since then, SR was generalized in many ways, in particular to account for the possibility of agents having several partners in the matching: the stable crews problem [5], the stable fixtures problem [18], the stable multiple activities problem [6, 3, 7] or the stable roommates problem with choice function [12]. In all the above cases, the notion of the rotation was adjusted in a suitable way to enable

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an efficient algorithm. Notice that in the area of stable matchings, algorithms that do not use rotations were proposed too [19, 20, 2].

The structure of the set of all stable matchings turned out to be related to the structure of the set of rotations [13, 14, 17], for a complex treatment see [15]. In the bipartite versions of matching problems, rotations can be used to represent the set of all stable matchings (whose size can be exponential in the number of participants) in a compact way and also to efficiently find a stable matching that is optimal with respect to some objective function [4, 15]. However, in the non-bipartite case, the problem of finding an egalitarian stable matching is NP-complete [9].

The aim of this paper is to explore the structure of rotations and extend the one-one correspondence between stable matchings and complete closed sets of rotations to the case of nonbipartite multigraphs. We show how to use this correspondence to find all stable pairs, i. e. pairs that are in at least one stable matching and identify a minimum regret stable matching in polynomial time and also an egalitarian solution (however, the latter not in polynomial time).

The organization of the paper is as follows: in Section 2 we summarize the basic definitions and known results. In Section 3 we introduce dual rotations and derive the structure of rotation poset, in Section 4 we show how to find all stable pairs for a solvable instance and in Section 5 we deal with minimum regret and egalitarian stable matchings.

2 Definitions and known results

Let $G = (V, E)$, $|E| = m$, $|V| = n$ be a finite multigraph for which a function $b : V \to \mathbb{N}$ is given, called the *capacity function*. For each vertex $v \in V$, let $\prec_v$ be a linear order on the set of edges incident with v in G and $\mathcal{O} = \{\prec_v, v \in V\}$. If $e \prec_v f$, then e is for v better than f. The triple $I = (G, \mathcal{O}, b)$ is an instance of the Stable b-matching Problem (SbM for short).

Denote by $V(I)$ and $E(I)$ the vertex set and the edge set of graph G in $I = (G, \mathcal{O}, b)$ and by $E(v, I)$ the set of edges incident with vertex v in I. A subset F of $E(I)$ is said to b-*dominate* edge $e \in E(I)$ if there exists a vertex v with $e \in E(v, I)$ and distinct elements $f_1, f_2, \ldots, f_{b(v)}$ of $F \cap E(v, I)$ such that $f_j \prec_v e$ for $j = 1, 2, \ldots, b(v)$.

A subset M of $E(I)$ is a *stable b-matching* if each vertex $v \in V(I)$ is incident with at most $b(v)$ edges of M and each edge $e \in E(I) \setminus M$ is b-dominated by M. The set of all stable b-matchings for an SbM instance I will be denoted by $\mathcal{M}(I)$. An edge $e \in E(I)$ is a *stable* edge if e belongs to at least one stable b-matching, and e is a *fixed* edge if e belongs to each stable b-matching.

An instance $I' = (G', \mathcal{O}', b)$ is said to be a *subinstance* of an instance $I = (G, \mathcal{O}, b)$, written $I' \subseteq I$, if G' is a factor subgraph of G and $\prec'_v$ is the restriction of $\prec_v$ to $E(v, I')$ for each $v \in V(I)$. A subinstance I' is a *proper subinstance* of I, written $I' \subset I$, if G' is a proper factor subgraph of G. Notice that the capacity function is the same in both instances.

In what follows, we will denote by $s_I(v)$, $l_I(v)$ the edges that are $(b(v)+1)$st and last in $\prec_v$ in I, respectively. A vertex $v \in V(I)$ is called *good* in an instance I if $|E(v, I)| \leq b(v)$, otherwise it is called *bad* in I. Also, if a vertex v is incident with fewer than $b(v)$ edges in a matching M, it is said to be *undersubscribed* in M.

For a given SbM instance $I_0 = (G_0, \mathcal{O}_0, b)$ the aim is to determine whether a stable b-matching exists and if this is the case to find one. This can be performed by the two-phases SbM algorithm [6, 3] that systematically deletes edges in such a way that:

(i) edge deletion cannot establish any new stable b-matching, and

(ii) if there was some stable b-matching before edge deletion then there is also at least one stable b-matching after edge deletion.

Moreover, edges deleted during the first phase cannot belong to any stable b-matching (if one exists). At the termination of the algorithm either a stable b-matching $M \in \mathcal{M}(I_0)$ is obtained or a decision that $\mathcal{M}(I_0) = \emptyset$ is made.

Let $I = (G, \mathcal{O}, b)$ be an SbM instance and let $v \in V(I)$. Define

$$\begin{aligned} B(v, I) &:= \{f \in E(v, I) : |\{g \in E(v, I) : g \prec_v f\}| < b(v)\} \text{ and} \\ D(v, I) &:= \{f = vx \in E(v, I) : f \in B(x, I)\}. \end{aligned}$$

Each edge $f = vx \in B(v, I)$, called *a B-edge* at vertex v in I, can obviously be b-dominated only at its other end, thus at vertex x. An edge $f \in D(v, I)$ is called *a D-edge* at vertex v in I. During the first phase of the SbM algorithm edges are deleted to obtain an instance with the so-called *the first-last-property* (the FLP for short) when for each vertex $v \in V(I)$ and for each edge $e \in E(v, I)$ the inequality

$$|\{f \in D(v, I) : f \prec_v e\}| < b(v)$$

holds. Fundamental properties of SbM instances satisfying the FLP are summarized in the following lemma.

Lemma 1 ([3, 6]) *If an SbM instance I satisfies the FLP then*

(i) $|B(v, I)| = |D(v, I)|$ *for each vertex* $v \in V(I)$*;*

(ii) $l_I(v) \in D(v, I)$ *for all vertices* $v \in V(I)$*;*

From now on we will deal with SbM instances $I \subseteq I_0$. We shall also say that instance I is a *stable instance* if

$$e = uv \in E(I_0) \setminus E(I) \text{ if and only if } e \text{ is } b\text{-dominated by } D(u, I) \text{ or } D(v, I).$$

At the termination of the first phase of the SbM algorithm either the (unique) obtained subinstance $I^* = (G^*, \mathcal{O}^*, b)$, called *the Phase-1 subinstance*, already represents a stable b-matching or there exists at least one bad vertex in I^*. In the latter case, Phase 2 of the algorithm follows. The most important properties of the Phase-1 subinstance are:

Theorem 1 ([3]) *Let I_0 be a solvable SbM instance and I^* its Phase-1 subinstance, then*

(i) *I^* is a stable instance.*

(ii) *Each stable b-matching of I_0 is embedded in I^*.*

(iii) *each vertex v is assigned the same number of edges in all stable b-matchings, namely* $\min\{b(v), |E(v, I^*)|\}$.

(iv) *Each good vertex v in I^* is assigned to precisely the same set of edges $E(v, I^*)$ in all stable b-matchings. Moreover, if some vertex v is undersubscribed in one stable b-matching, then it is assigned the set $E(v, I^*)$ in all stable b-matchings.*

We can also summarize properties of stable instances:

Lemma 2 ([3]) *Let I be a stable instance. Then*

(i) $I \subset I^*$,

(ii) *if each vertex is good in I, then $E(I)$ determines a stable b-matching,*

(iii) *if I' is also a stable instance and $B(v, I) = B(v, I')$ for each vertex v, or equivalently $D(v, I) = D(v, I')$ for each v, then $I = I'$.*

After the termination of the first phase not resulting in a stable b-matching, the algorithm continues with Phase 2. In this phase, edges are deleted by eliminating the so-called rotations until each vertex is good or until the algorithm determines that $\mathcal{M}(I_0) = \emptyset$. A generalized definition of a rotation for the SbM problem was given in [6].

Definition 1 *A* rotation *exposed in stable instance $I = (G, \mathcal{O}, b)$ is a sequence of r ordered edge pairs $\varrho = (e_0^\varrho, f_0^\varrho)(e_1^\varrho, f_1^\varrho) \dots (e_{r-1}^\varrho, f_{r-1}^\varrho)$ such that*

$$e_j^\varrho = u_j^\varrho\, v_j^\varrho = l_I(v_j^\varrho) \quad \text{and} \quad f_j^\varrho = u_j^\varrho\, v_{j+1}^\varrho = s_I(u_j^\varrho),$$

where subscripts are taken modulo r.

The superscript ϱ may be omitted if the rotation is understood from the context. Edge sets $\{e_0^\varrho, e_1^\varrho, \dots, e_{r-1}^\varrho\}$ and $\{f_0^\varrho, f_1^\varrho, \dots, f_{r-1}^\varrho\}$ will be denoted by ϱ_E and ϱ_F, respectively. A vertex w that is incident with some edge of $\varrho_E \cup \varrho_F$ is said to be *covered* by rotation ϱ.

Lemma 3 ([6]) *If I is a stable instance with at least one bad vertex then I contains at least one exposed rotation and each vertex covered by this rotation is bad.*

Rotations for a stable instance I are sought by constructing an auxiliary digraph $H(I) = (V(I), A)$ where $\vec{a} = v\vec{w} \in A$ if $e = vu = l_I(v)$ and $f = uw = s_I(u)$. Each bad vertex has exactly one outgoing arc in H and it leads to another bad vertex. In such a digraph a cycle always exists, say $v_p, v_{p+1}, \dots, v_r$, $0 \le p \le r \le n$, and this cycle determines a rotation ϱ by taking as $e_j^\varrho = u_j v_j$ edge $l_I(v_j)$ and as $f_j^\varrho = u_j v_{j+1}$ edge $s_I(u_i)$. Each bad vertex w of H is either on a cycle representing some rotation ϱ or it is on a directed path leading to such a cycle. In the latter case, w is said to *lead to* rotation ϱ.

The key property of rotations is formulated as follows:

Lemma 4 ([6]) *Let I be a stable instance and let ϱ be a rotation exposed in I. Sets ϱ_E and ϱ_F are disjoint or identical. In the latter case, I has no stable b-matching.*

Suppose that I is a stable instance and $w \in V(I)$ is a bad vertex. If $D(w, I) \setminus l_I(w)$ is not an empty set, i. e. $b(w) \neq 1$, we denote by $k_I(w)$ the $\prec_w$-worst edge of $D(w, I) \setminus l_I(w)$. Otherwise, $k_I(w)$ is not defined.

Definition 2 *Let $\varrho = (e_0^\varrho, f_0^\varrho)(e_1^\varrho, f_1^\varrho) \dots (e_{r-1}^\varrho, f_{r-1}^\varrho)$ be a rotation exposed in a stable instance $I = (G, \mathcal{O}, b)$ with $e_j = u_j v_j = l_I(v_j)$, $f_j = u_j v_{j+1} = s_I(u_j)$ and $\varrho_E \cap \varrho_F = \emptyset$. The* elimination *of rotation ϱ is the deletion of all edges of the form $g = v_j w$, where $f_{j-1} \prec_{v_j} g$ as well as $k_I(v_j) \prec_{v_j} g$, for all $j = 0, 1, \dots, r-1$. The obtained instance will be denoted by $I \setminus \varrho = (G \setminus \varrho, \mathcal{O} \setminus \varrho, b)$.*

Lemma 5 ([3]) *Let I be a stable instance with an exposed rotation ϱ such that $\varrho_E \cap \varrho_F = \emptyset$. If $I \setminus \varrho$ is the instance obtained by the elimination of rotation ϱ then*

(i) $I \setminus \varrho$ is a stable instance, $I \setminus \varrho \subset I$ and $\mathcal{M}(I \setminus \varrho) \subseteq \mathcal{M}(I)$;
(ii) $B(w, I \setminus \varrho) = B(w, I) \setminus \{e_j\} \cup \{f_j\}$ for each w covered by ϱ as u_j;
(iii) $D(w, I \setminus \varrho) = D(w, I) \setminus \{e_j\} \cup \{f_{j-1}\}$ for each w covered by ϱ as v_j;
(iv) $B(w, I \setminus \varrho) = B(w, I)$ for each vertex w not covered by ϱ as u_j and $D(w, I \setminus \varrho) = D(w, I)$ for each vertex w not covered by ϱ as v_j, $j = 0, \dots, r-1$.

Lemma 6 ([3]) *Suppose that I and I' are stable instances of I_0 and $I' \subseteq I$.*

(i) If $\varrho = (e_0^\varrho, f_0^\varrho)(e_1^\varrho, f_1^\varrho) \dots (e_{r-1}^\varrho, f_{r-1}^\varrho)$ is a rotation exposed in I and if $B(w, I') \neq B(w, I)$ for at least one vertex w that leads to ϱ, then $I' \subseteq I \setminus \varrho$.
(ii) I' can be obtained from I by elimination of a rotation sequence.

Lemma 7 ([3]) *Each stable instance can be obtained from the Phase-1 subinstance I^* by elimination of an appropriate sequence of rotations. Moreover, each stable b-matching of the solvable SbM instance I_0 can be found by the SbM algorithm.*

Example 1 *In Fig. 1, SbM instance* $I_0 = (G_0, \mathcal{O}_0, b)$ *is given with* $|V(I_0)| = 7$ *and* $|E(I_0)| = 25$*. Capacities for vertices are displayed in brackets, e. g. vertex* w_1 *has capacity* 3*. For each vertex* w*, the preference order on* $E(w, I_0)$ *is given as an ordered list of edges from the most preferred one to the least preferred one. For convenience, the number in brackets gives the index of the other end vertex of the edge, i. e.* $\epsilon_2 = w_3w_7$*. D-edges are underlined and for each vertex, the number of D-edges is the same as the number of B-edges. Hence the instance already satisfies the FLP and* $I_0 = I^*$*. Therefore we can see that vertex* w_7 *is undersubscribed in each stable b-matching of* I_0 *if one exists.*

Since I_0 *is a stable instance with bad vertices, a rotation should be found and eliminated. Auxiliary digraph* $H(I_0)$ *for instance* I_0 *in Fig. 2 contains one cycle* (w_2, w_3) *corresponding to rotation* $\varrho^1 = (\epsilon_{19}, \epsilon_{10})(\epsilon_{20}, \epsilon_7)$*. Each bad vertex leads to this rotation. The stable instance* $I_0 \setminus \varrho^1$ *obtained by the elimination of* ϱ^1 *is in Fig. 3.*

A stable b-matching is obtained after eliminations of other six rotations, i. e. $M_1 = I_0 \setminus \varrho^1 \setminus \varrho^2 \setminus \varrho^5 \setminus \varrho^7 \setminus \varrho^3 \setminus \varrho^4 \setminus \varrho^6$ *in that order, see Fig. 4.*

$w_1(3)$	$\epsilon_{12}(5)$	$\epsilon_{18}(3)$	$\epsilon_{22}(6)$	$\epsilon_4(4)$	$\epsilon_3(6)$	$\underline{\epsilon_{15}(4)}$	$\epsilon_{11}(3)$	$\underline{\epsilon_5(3)}$	$\underline{\epsilon_{21}(4)}$
$w_2(2)$	$\underline{\epsilon_6(6)}$	$\epsilon_{13}(3)$	$\epsilon_8(5)$	$\epsilon_7(5)$	$\epsilon_{24}(5)$	$\epsilon_{25}(6)$	$\underline{\epsilon_{19}(6)}$		
$w_3(4)$	$\epsilon_5(1)$	$\underline{\epsilon_2(7)}$	$\epsilon_{10}(6)$	$\epsilon_9(4)$	$\epsilon_{11}(1)$	$\underline{\epsilon_{13}(2)}$	$\underline{\epsilon_{18}(1)}$	$\epsilon_{23}(4)$	$\underline{\epsilon_{20}(5)}$
$w_4(3)$	$\epsilon_{15}(1)$	$\underline{\epsilon_1(7)}$	$\epsilon_{21}(1)$	$\epsilon_{17}(6)$	$\underline{\epsilon_9(3)}$	$\epsilon_4(1)$	$\epsilon_{23}(4)$	$\underline{\epsilon_{14}(6)}$	
$w_5(1)$	$\epsilon_{20}(3)$	$\epsilon_7(2)$	$\epsilon_{24}(2)$	$\epsilon_8(2)$	$\epsilon_{16}(6)$	$\underline{\epsilon_{12}(1)}$			
$w_6(3)$	$\underline{\epsilon_6(2)}$	$\epsilon_{14}(4)$	$\epsilon_{19}(2)$	$\underline{\epsilon_{10}(3)}$	$\epsilon_3(1)$	$\epsilon_{16}(5)$	$\epsilon_{17}(4)$	$\epsilon_{25}(2)$	$\underline{\epsilon_{22}(1)}$
$w_7(3)$	$\underline{\epsilon_1(4)}$	$\underline{\epsilon_2(3)}$							

Fig. 1: The preference lists for SbM instance $I_0 = (G_0, \mathcal{O}_0, b)$

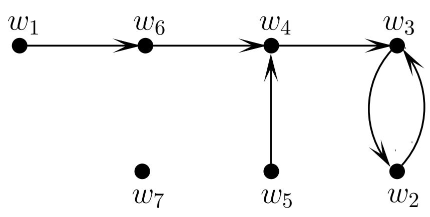

Fig. 2: Auxiliary digraph $H(I_0)$ for instance I_0.

3 Singular and dual rotations

For a given SbM instance I_0 we are now able to determine whether there is any stable b-matching in $\mathcal{M}(I_0)$ or not. In addition, by Lemma 7, each stable

$w_1(3)$	$\epsilon_{12}(5)$	$\epsilon_{18}(3)$	$\epsilon_{22}(6)$	$\epsilon_4(4)$	$\epsilon_3(6)$	$\underline{\epsilon_{15}(4)}$	$\epsilon_{11}(3)$	$\underline{\epsilon_5(3)}$	$\underline{\epsilon_{21}(4)}$
$w_2(2)$	$\underline{\epsilon_6(6)}$	$\epsilon_{13}(3)$	$\epsilon_8(5)$	$\underline{\epsilon_7(5)}$					
$w_3(4)$	$\epsilon_5(1)$	$\underline{\epsilon_2(7)}$	$\underline{\epsilon_{10}(6)}$	$\epsilon_9(4)$	$\epsilon_{11}(1)$	$\underline{\epsilon_{13}(2)}$	$\underline{\epsilon_{18}(1)}$		
$w_4(3)$	$\epsilon_{15}(1)$	$\underline{\epsilon_1(7)}$	$\epsilon_{21}(1)$	$\epsilon_{17}(6)$	$\underline{\epsilon_9(3)}$	$\epsilon_4(1)$	$\underline{\epsilon_{14}(6)}$		
$w_5(1)$	$\epsilon_7(2)$	$\epsilon_8(2)$	$\epsilon_{16}(6)$	$\underline{\epsilon_{12}(1)}$					
$w_6(3)$	$\underline{\epsilon_6(2)}$	$\epsilon_{14}(4)$	$\underline{\epsilon_{10}(3)}$	$\epsilon_3(1)$	$\epsilon_{16}(5)$	$\epsilon_{17}(4)$	$\underline{\epsilon_{22}(1)}$		
$w_7(3)$	$\underline{\epsilon_1(4)}$	$\underline{\epsilon_2(3)}$							

Fig. 3: The preference lists of $I_0 \setminus \varrho^1$ after elimination of ϱ^1.

$\mathbf{M_1} = \mathbf{I_0} \setminus \varrho^{\mathbf{1}} \setminus \varrho^{\mathbf{2}} \setminus \varrho^{\mathbf{5}} \setminus \varrho^{\mathbf{7}} \setminus \varrho^{\mathbf{3}} \setminus \varrho^{\mathbf{4}} \setminus \varrho^{\mathbf{6}}$

$w_1(3)$	$\epsilon_{12}(5)$	$\epsilon_3(6)$	$\epsilon_{15}(4)$		$\varrho^1 = (\epsilon_{19}, \epsilon_{20})(\epsilon_{10}, \epsilon_7)$
$w_2(2)$	$\epsilon_6(6)$	$\epsilon_{13}(3)$			$\varrho^2 = (\epsilon_{21}, \epsilon_{17})(\epsilon_{22}, \epsilon_4)(\epsilon_{14}, \epsilon_3)$
$w_3(4)$	$\epsilon_2(7)$	$\epsilon_{10}(6)$	$\epsilon_9(4)$	$\epsilon_{13}(2)$	$\varrho^5 = (\epsilon_{18}, \epsilon_3)(\epsilon_{17}, \epsilon_9)$
$w_4(3)$	$\epsilon_{15}(1)$	$\epsilon_1(7)$	$\epsilon_9(3)$		$\varrho^7 = (\epsilon_4, \epsilon_{15})$
$w_5(1)$	$\epsilon_{12}(1)$				$\varrho^3 = (\epsilon_7, \epsilon_8)$
$w_6(3)$	$\epsilon_6(2)$	$\epsilon_{10}(3)$	$\epsilon_3(1)$		$\varrho^4 = (\epsilon_5, \epsilon_{11})$
$w_7(3)$	$\epsilon_1(4)$	$\epsilon_2(3)$			$\varrho^6 = (\epsilon_{11}, \epsilon_{13})(\epsilon_8, \epsilon_{12})$

Fig. 4: A stable b-matching M_1 for instance I_0 and the corresponding rotations.

b-matching can be found by the SbM algorithm. In order to obtain all stable b-matchings, we demonstrate in this section some properties of rotations that generalize the results for the stable roommates problem.

We stress that from now we restrict our work only to solvable SbM instances $I_0 = (G_0, \mathcal{O}_0, b)$, hence for each rotation ϱ we suppose that $\varrho_E \cap \varrho_F = \emptyset$. Recall that $I^* = (G^*, \mathcal{O}^*, b)$ is the Phase-1 subinstance of I_0 and $\mathcal{M}(I_0) = \mathcal{M}(I^*)$.

Definition 3 *Let $I = (G, \mathcal{O}, b)$ be a stable instance with an exposed rotation $\varrho = (e_0^\varrho, f_0^\varrho)(e_1^\varrho, f_1^\varrho) \dots (e_{r-1}^\varrho, f_{r-1}^\varrho)$, where $e_j^\varrho = u_j^\varrho v_j^\varrho = l_I(v_j^\varrho)$ and $f_j^\varrho = u_j^\varrho v_{j+1}^\varrho = s_I(u_j^\varrho)$ (subscripts taken modulo r).*

If there exists a stable instance $I' = (G', \mathcal{O}', b)$ such that the sequence of edge pairs $\bar{\varrho} = (f_0^\varrho, e_1^\varrho)(f_1^\varrho, e_2^\varrho) \dots (f_{r-2}^\varrho, e_{r-1}^\varrho)(f_{r-1}^\varrho, e_0^\varrho)$ is a rotation exposed in I', i. e. if

$$f_j^\varrho = v_{j+1}^\varrho u_j^\varrho = l_{I'}(u_j^\varrho) \quad \text{and} \quad e_{j+1}^\varrho = v_{j+1}^\varrho u_{j+1}^\varrho = s_{I'}(v_{j+1}^\varrho),$$

(subscripts taken modulo r), then ϱ is called nonsingular *and the rotation $\bar{\varrho}$ is called* the dual rotation *to rotation ϱ.*

If $\bar{\varrho}$ is not a rotation in any stable instance then ϱ is called singular.

If rotation ϱ is dual to rotation σ, then also σ is dual to rotation ϱ. Note that ϱ and $\bar{\varrho}$ cover the same set of vertices. However, in an auxiliary digraph $H(I)$ for some stable instance I, ϱ and $\bar{\varrho}$ correspond to two disjoint cycles.

Example 2 *Let us again use the instance from Example 1. Fig 5 represents a stable instance* $I' = I_0 \setminus \varrho^1 \setminus \varrho^2 \setminus \varrho^5 \setminus \varrho^7 \setminus \varrho^3 \setminus \varrho^4$ *with two exposed rotations* $\varrho^6 = (\epsilon_{11}, \epsilon_{13})(\epsilon_8, \epsilon_{12})$ *and* $\sigma = (\epsilon_{13}, \epsilon_{11})(\epsilon_{12}, \epsilon_8)$*. It is easy to see that* $\bar{\varrho}^6 = \sigma$*.*

$w_1(3)$	$\epsilon_{12}(5)$	$\epsilon_3(6)$	$\epsilon_{15}(4)$	$\epsilon_{11}(3)$	
$w_2(2)$	$\epsilon_6(6)$	$\epsilon_{13}(3)$	$\epsilon_8(5)$		
$w_3(4)$	$\epsilon_2(7)$	$\epsilon_{10}(6)$	$\epsilon_9(4)$	$\epsilon_{11}(1)$	$\epsilon_{13}(2)$
$w_4(3)$	$\epsilon_{15}(1)$	$\epsilon_1(7)$	$\epsilon_9(3)$		
$w_5(1)$	$\epsilon_8(2)$	$\epsilon_{12}(1)$			
$w_6(3)$	$\epsilon_6(2)$	$\epsilon_{10}(3)$	$\epsilon_3(1)$		
$w_7(3)$	$\epsilon_1(4)$	$\epsilon_2(3)$			

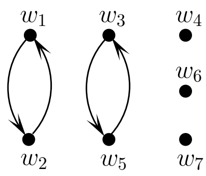

Fig. 5: Stable instance $I' = I_0 \setminus \varrho^1 \setminus \varrho^2 \setminus \varrho^5 \setminus \varrho^7 \setminus \varrho^3 \setminus \varrho^4$ with auxiliary digraph $H(I')$.

Lemma 8 *Let* ϱ *and* σ *be two rotations exposed in the same stable instance* I*. Then*

(i) $\varrho_E \cap \sigma_E \neq \emptyset$ *if and only if* $\varrho = \sigma$

(ii) $\varrho_E \cap \sigma_F \neq \emptyset$ *if and only if* $\varrho = \bar{\sigma}$

Proof. We will prove the 'only if' implications, as the 'if' direction is trivial. Let $\varrho = (e_0^\varrho, f_0^\varrho)(e_1^\varrho, f_1^\varrho)\ldots(e_{r-1}^\varrho, f_{r-1}^\varrho)$ and $\sigma = (e_0^\sigma, f_0^\sigma)(e_1^\sigma, f_1^\sigma)\ldots(e_{t-1}^\sigma, f_{t-1}^\sigma)$.

(i) Suppose that $g \in \varrho_E \cap \sigma_E$, thus there exist j and l, $0 \leq j \leq r-1$, $0 \leq l \leq t-1$, such that $g = e_j^\varrho = u_j^\varrho v_j^\varrho = e_l^\sigma = u_l^\sigma v_l^\sigma$.

If $u_j^\varrho = u_l^\sigma$ then $f_j^\varrho = f_l^\sigma$ and by induction we have $\varrho_E = \sigma_E$ and $\varrho_F = \sigma_F$, hence $\varrho = \sigma$.

Suppose that $u_j^\varrho = v_l^\sigma$. By Lemma 1(ii), $g = l_I(v_j^\varrho) \in D(v_j^\varrho, I)$, thus $e = l_I(v_l^\sigma) \in B(u_j^\varrho, I)$. But this is a contradiction, since only bad vertices are covered by a rotation.

(ii) Suppose that $g \in \varrho_E \cap \sigma_F$, thus there exist j and l, $0 \leq j \leq r-1$, $0 \leq l \leq t-1$, such that $g = e_j^\varrho = u_j^\varrho v_j^\varrho = f_l^\sigma = u_l^\sigma v_{l+1}^\sigma$.

By Lemma 1(ii), $g = s_I(u_l^\sigma) = l_I(v_j^\varrho) \in B(u_j^\varrho, I)$, so $u_j^\varrho \neq u_l^\sigma$ and thus $u_j^\varrho = v_{l+1}^\sigma$ and $v_j^\varrho = u_l^\sigma$. Hence, $|E(v_j^\varrho, I)| = |E(u_l^\sigma, I)| = b(u_l^\sigma) + 1 = b(v_j^\varrho) + 1$. By Lemma 1(i), $|B(u_l^\sigma, I)| = |D(u_l^\sigma, I)| = b(u_l^\sigma)$, so $|B(u_l^\sigma, I) \cap D(u_l^\sigma, I)| = b(u_l^\sigma) - 1$. Again by Lemma 1(ii), the worst edge in $\prec_{u_l^\sigma}$ is in $D(u_l^\sigma, I)$ hence

$$g = l_I(u_l^\sigma) \in D(u_l^\sigma, I) \setminus B(u_l^\sigma, I) = D(v_j^\varrho, I) \setminus B(v_j^\varrho, I)$$

and for a unique edge h:

$$h \in B(u_l^\sigma, I) \setminus D(u_l^\sigma, I) = B(v_j^\varrho, I) \setminus D(v_j^\varrho, I).$$

Edges $f_{j-1}^\varrho = u_{j-1}^\varrho v_j^\varrho$ and $e_l^\sigma = u_l^\sigma v_l^\sigma$ have vertex $v_j^\varrho = u_l^\sigma$ in common. $f_{j-1}^\varrho = s_I(u_{j-1}^\varrho)$, so $f_{j-1}^\varrho \notin D(v_j^\varrho, I)$ and therefore $u_{j-1}^\varrho v_j^\varrho = h$. Also $e_l^\sigma = l_I(v_l^\sigma)$ and by Lemma 1(ii), $e_l^\sigma \in D(v_l^\sigma, I)$, so $e_l^\sigma \in B(u_l^\sigma, I)$. Moreover, $|E(v_l^\sigma, I)| \geq b(v_l^\sigma) + 1$, so $e_l^\sigma \notin B(v_l^\sigma, I)$, hence $e_l^\sigma \notin D(u_l^\sigma, I)$ and therefore $e_l^\sigma = h$. Consequently $u_{j-1}^\varrho v_j^\varrho = h = u_l^\sigma v_l^\sigma$. So the assumption $g = e_j^\varrho = f_l^\sigma$ leads to $f_{j-1}^\varrho = e_l^\sigma = h$.

Now, if we take the edge $h = f_{j-1}^\varrho = e_l^\sigma$ and interchange the roles of rotations ϱ and σ in the proof above, we get $e_{j-1}^\varrho = f_{l-1}^\sigma$.

By induction we get that $\varrho_E = \sigma_F$ and $\varrho_F = \sigma_E$ in the same cyclic order and from the argument it is clear that $\varrho = \bar{\sigma}$. □

Although $\varrho_E \cap \sigma_E \neq \emptyset$ implies $\varrho = \sigma$ for arbitrary ϱ and σ exposed in the same instance, the implication $\varrho_F \cap \sigma_F \neq \emptyset \implies \varrho = \sigma$ is not valid in general. Example 3 illustrates such a situation.

We would like to stress, that assertion (i) from Lemma 8 holds not only for rotations σ and ϱ exposed in the same stable instance, but for an arbitrary pair of rotations. This can be proved by using several properties of singular and dual rotations that will be stated later (Lemma 15, Lemma 18). Notice that this is an analogy of the result for the two-sided many-to-many version of the stable marriage problem in [1], where it is proved that no (man, woman) pair can belong to two different meta-rotations.

Example 3 *The following table represents an SbM instance I (with $b(u) = 1$ for all vertices for simplicity). The rotations exposed in I are written to the right of the preference lists.*

w_1	ϵ_1	ϵ_2	ϵ_3
w_2	ϵ_4	ϵ_2	ϵ_1
w_3	ϵ_3	ϵ_5	
w_4	ϵ_6	ϵ_7	
w_5	ϵ_5	ϵ_8	
w_6	ϵ_7	ϵ_9	ϵ_{10}
w_7	ϵ_{10}	ϵ_9	ϵ_4
w_8	ϵ_8	ϵ_6	

$\varrho = (\epsilon_1, \epsilon_2)$
$\sigma = (\epsilon_{10}, \epsilon_9)$
$\tau = (\epsilon_4, \epsilon_2)(\epsilon_3, \epsilon_5)(\epsilon_8, \epsilon_6)(\epsilon_7, \epsilon_9)$

It is easy to see that:
$\varrho_F \cap \tau_F = \{\epsilon_2\} \neq \emptyset$ *but* $\varrho \neq \tau$ *and also*
$\sigma_F \cap \tau_F = \{\epsilon_9\} \neq \emptyset$ *but* $\sigma \neq \tau$.

Lemma 9 *Let $\varrho^1, \varrho^2, \ldots, \varrho^p$, $p \geq 2$ be p different rotations exposed in the same stable instance I. Then each vertex w can be covered by at most two rotations. Moreover, if w is covered by ϱ^i and ϱ^j, then w is covered by ϱ^i only as some $u_l^{\varrho^i}$ and by ϱ^j only as some $v_r^{\varrho^j}$.*

Proof. Suppose that vertex w is covered by rotation ϱ^i. If $w = v_l^{\varrho^i}$, then w cannot be covered by any other rotation ϱ^j as some $v_r^{\varrho^j}$, as this would imply $e_l^{\varrho^i} = e_r^{\varrho^j} = l_I(w)$ and so $\varrho^i = \varrho^j$ by Lemma 8(i).

Suppose that $w = u_l^{\varrho^i}$. If w is covered by ϱ^j also as some $u_r^{\varrho^j}$, then $u_l^{\varrho^i} v_{l+1}^{\varrho^i} = s_I(u_l^{\varrho^i}) = s_I(u_r^{\varrho^j}) = u_r^{\varrho^j} v_{r+1}^{\varrho^j}$ and so $v_{l+1}^{\varrho^i} = v_{r+1}^{\varrho^j}$ which was proved at the beginning of this proof to be not possible. □

The following lemmas generalize Lemmas 4.3.1, 4.3.4 and 4.3.2 of [15] (in this order) and they explain the significance of the notion of dual rotation.

Lemma 10 *Let two rotations ϱ and σ, $\varrho \neq \sigma$ be exposed in the same stable instance I. Then either ϱ is exposed in $I \setminus \sigma$ or $\sigma = \overline{\varrho}$.*

Proof. Take any vertex w covered by $\varrho = (e_0^\varrho, f_0^\varrho)(e_1^\varrho, f_1^\varrho) \dots (e_{r-1}^\varrho, f_{r-1}^\varrho)$ as some v_j^ϱ, $0 \leq j \leq r-1$. By Lemma 9, w cannot be covered by σ as v_l^σ and thus by Lemma 5(iv) $D(v_j^\varrho, I \setminus \sigma) = D(v_j^\varrho, I)$. In particular, edge $e_j^\varrho = l_I(v_j^\varrho)$ is not deleted during the elimination of σ for any j, $0 \leq j \leq r-1$. So $\varrho_E \subseteq E(I \setminus \sigma)$. Now distinguish two cases.

(i) $\varrho_F \subseteq E(I \setminus \sigma)$. Take any vertex w covered by ϱ as some u_j^ϱ. By Lemma 9, w cannot be covered by σ as u_l^σ, so $B(u_j^\varrho, I \setminus \sigma) = B(u_j^\varrho, I)$ by Lemma 5(iv). Hence $f_j^\varrho = u_j^\varrho v_{j+1}^\varrho$ remains the $(b(u_j^\varrho)+1)$th best edge for u_j^ϱ in $I \setminus \sigma$ for each j, $j = 0, 1, \dots, r-1$ and so ϱ is exposed in $I \setminus \sigma$.

(ii) $\varrho_F \not\subseteq E(I \setminus \sigma)$. Without loss of generality, suppose that edge $f_0^\varrho = u_0^\varrho v_1^\varrho$ was deleted during the elimination of σ. Thus $f_0^\varrho = v_j^\sigma w$ for some j, $0 \leq j \leq k-1$ and some $w \in V(I)$ and $u_{j-1}^\sigma v_j^\sigma = f_{j-1}^\sigma \prec_{v_j^\sigma} f_0^\varrho$ as well as $k_I(v_j^\sigma) \prec_{v_j^\sigma} f_0^\varrho$. By Lemma 9, $u_0^\varrho = v_j^\sigma$, and so $f_0^\varrho = s_I(v_j^\sigma)$. But $k_I(v_j^\sigma) \prec_{v_j^\sigma} f_0^\varrho$, so $k_I(v_j^\sigma) \in B(v_j^\sigma, I)$ and hence $|B(v_j^\sigma, I) \cap D(v_j^\sigma, I)| = b(v_j^\sigma) - 1$. So there is a unique edge $h \in B(v_j^\sigma, I) \setminus D(v_j^\sigma, I)$. As $f_{j-1}^\sigma \prec_{v_j^\sigma} f_0^\varrho$, we have $f_{j-1}^\sigma \in B(v_j^\sigma, I)$. But also $f_{j-1}^\sigma = u_{j-1}^\sigma v_j^\sigma = s_I(u_{j-1}^\sigma) \notin D(v_j^\sigma, I)$, so $f_{j-1}^\sigma = h$. But as $v_j^\sigma = u_0^\varrho$ and $e_0^\varrho = u_0^\varrho v_0^\varrho$ is the last for v_0^ϱ, $f_0^\varrho \notin D(u_0^\varrho = v_j^\sigma, I)$ and so $e_0^\varrho = h = f_{j-1}^\sigma$. Hence $\varrho_E \cap \sigma_F \neq \emptyset$ and $\varrho = \bar{\sigma}$ by Lemma 8(ii). □

Lemma 11 *Let rotations ϱ and $\bar{\varrho}$ be both exposed in a stable instance I. Then*

(i) $|E(w,I)| = b(w)+1$ and $|E(w, I \setminus \varrho)| = |E(w, I \setminus \bar{\varrho})| = b(w)$ for each vertex w covered by ϱ (or equivalently, by $\overline{\varrho}$)

(ii) $E(w', I \setminus \varrho) = E(w', I) = E(w', I \setminus \bar{\varrho})$ holds for each vertex w' not covered by ϱ.

Proof. As both ϱ and $\bar{\varrho}$ are exposed in I, we have $f_j = s_I(u_j) = l_I(u_j)$ and $e_j = l_I(v_j) = s_I(v_j)$. Hence $|E(w,I)| = b(w)+1$ holds for each covered vertex w. Consequently, during the elimination of ϱ or $\bar{\varrho}$, only edges from ϱ_E or ϱ_F, respectively, are deleted. So the assertion follows. □

Lemma 12 *If ϱ and σ are two different rotations exposed in a stable instance I and $\varrho \neq \bar{\sigma}$, then $I \setminus \varrho \setminus \sigma = I \setminus \sigma \setminus \varrho$.*

Proof. As $\varrho \neq \bar{\sigma}$, Lemma 10 implies that ϱ is exposed in $I \setminus \sigma$ and σ is exposed in $I \setminus \varrho$. Therefore, instances $I' = I \setminus \varrho \setminus \sigma$ and $I'' = I \setminus \sigma \setminus \varrho$ are defined properly. We have three types of vertices:
If vertex w is not covered by any rotation ϱ, σ then by Lemma 5(iv)

$$B(w, I \setminus \varrho \setminus \sigma) = B(w, I \setminus \varrho) = B(w, I) = B(w, I \setminus \sigma) = B(w, I \setminus \sigma \setminus \varrho). \quad (1)$$

Suppose that vertex w is covered by exactly one rotation of ϱ and σ, without loss of generality say by ϱ. If w is not covered as any u_j^ϱ, Lemma 5(iv) implies that (1) holds. If $w = u_j^\varrho$ for some j, then

$$B(w, I \setminus \varrho \setminus \sigma) = B(w, I \setminus \varrho) = B(w, I) \setminus \{e_j^\varrho\} \cup \{f_j^\varrho\} \quad (2)$$
$$B(w, I \setminus \sigma \setminus \varrho) = B(w, I \setminus \sigma) \setminus \{e_j^\varrho\} \cup \{f_j^\varrho\} = B(w, I) \setminus \{e_j^\varrho\} \cup \{f_j^\varrho\} \quad .(3)$$

If vertex w is covered by both ϱ and σ then Lemma 9 implies, without loss of generality, $w = u_j^\varrho = v_l^\sigma$. So we have the same case as (2) and (3).

Therefore for each vertex w, $B(w, I \setminus \varrho \setminus \sigma) = B(w, I \setminus \sigma \setminus \varrho)$ and since both I' and I'' are stable instances, Lemma 2(iii) implies that they are equal. □

The previous lemma implies that if a stable instance I' was obtained by elimination of a sequence $\varrho^1, \varrho^2, \ldots, \varrho^r$ of mutually different and non-dual rotations from a stable instance I, then *the order of eliminated rotations is immaterial.* (However, note that a rotation cannot be eliminated until it has become exposed.) Because of that, we shall use the notation $I' = I \setminus R = (G \setminus R, \mathcal{O} \setminus R, b)$ where $R = \{\varrho^1, \varrho^2, \ldots, \varrho^r\}$, since the stable instance I' is completely determined by I and R. Lemma 14 is a generalization of Lemma 4.3.3 of [15] and it shows, that also the set of rotations R is completely determined by I and I'. At first, we state one more proposition that is a generalization of Lemma 8(i).

Lemma 13 *Let I and I' be stable instances and $I' \subseteq I$. Let rotation ϱ be exposed in I and rotation σ be exposed in I'. Then $\varrho_E \cap \sigma_E \neq \emptyset \iff \varrho = \sigma$.*

Proof. We shall prove only the "⇒" implication for the case $I' \subset I$, as the case $I' = I$ follows from Lemma 8(i). Let $\varrho_E \cap \sigma_E \neq \emptyset$. By Lemma 6(ii), I' can be obtained from I by elimination of a rotation sequence $\tau^1, \tau^2, \ldots, \tau^r$. Hence τ^1 and ϱ are both exposed in I. If $\varrho_E \cap \tau_E^1 \neq \emptyset$, then $\varrho = \tau^1$ (by Lemma 8) and no edge of ϱ_E is present in I', contradicting $\varrho_E \cap \sigma_E \neq \emptyset$.

Hence $\varrho_E \cap \tau_E^1 = \emptyset$. Suppose that $\tau^1 = \bar{\varrho}$. Then $|E(w, I \setminus \tau^1)| = b(w)$ for all vertices w covered by τ^1 by Lemma 11. Therefore no vertex w covered by ϱ can be covered by any other rotation after elimination of τ^1, so necessarily $\sigma_E \cap \varrho_E = \emptyset$, again a contradiction.

It follows that ϱ is exposed in $I \setminus \tau^1$. We can now repeat the arguments for $\tau^2, \tau^3, \ldots, \tau^r$ until we get a contradiction with $\sigma_E \cap \varrho_E \neq \emptyset$ or until ϱ is exposed in I'. Then by Lemma 8(i), $\varrho = \sigma$. □

Lemma 14 *If I and I' are stable instances, $I' \subset I$ and $I' = I \setminus R = I \setminus R'$ then $R = R'$.*

Proof. Suppose that $R = \{\varrho^1, \varrho^2, \ldots, \varrho^t\}$ and $R' = \{\sigma^1, \sigma^2, \ldots, \sigma^r\}$ and that these rotations were eliminated in this order to get I' from I.

Rotations ϱ^1 and σ^1 are both exposed in I. Suppose that $\varrho^1 \neq \sigma^1$. If ϱ^1 is dual to σ^1, then by Lemma 11(i), no vertex covered by ϱ^1 and σ^1 can be covered by any other rotation in any following stable instance. So $E(v, I \setminus \varrho^1) = E(v, I \setminus R)$ and $E(v, I \setminus \sigma^1) = E(v, I \setminus R')$ for each vertex v covered by ϱ^1 and σ^1. Hence, together with Lemma 5(ii-iv) this implies that $\varrho_E^1 \subseteq E(G \setminus R')$ and $\varrho_F^1 \subseteq E(G \setminus R)$. Since we have supposed that $\varrho_E \cap \varrho_F = \emptyset$, Lemma 2(iii) implies $I \setminus R \neq I \setminus R'$ – a contradiction. So ϱ^1 is exposed in $I \setminus \sigma^1$. We can now repeat this discussion for rotations $\sigma^2, \sigma^3, \ldots, \sigma^t$. As $I \setminus R' = I \setminus R$ and $\varrho^1_E \cap I \setminus R = \emptyset$, it follows that ϱ^1 has to be eliminated to get I', so $\varrho^1 \in R'$ and $\varrho^1 = \sigma^j$ for some j, $0 \leq j \leq r$. As the order of elimination of exposed rotations is immaterial, we can suppose that $\varrho^1 = \sigma^1$ and by repeating the arguments above for instance $I \setminus \varrho^1$ and sets $R \setminus \{\varrho^1\}$ and $R' \setminus \{\varrho^1\}$ we get $R' = R$. □

Lemma 15 *Let I and I', $I' \subseteq I$ be stable instances. If a rotation ϱ is exposed in I and a rotation σ is exposed in I', then either*

(i) $\sigma = \varrho$,

(ii) $\sigma = \bar{\varrho}$, or

(iii) there is stable subinstance of $I \setminus \varrho$, in which σ is exposed.

Proof. If σ is exposed in I then the lemma follows from Lemma 10. Suppose now, that σ is not exposed in I and let I' be a maximal stable subinstance of I in which σ is exposed, that is, such that there is no stable subinstance I'' of I in which σ is exposed and $I' \subset I'' \subset I$. By Lemma 6(ii) I' can be obtained from I by rotation eliminations, so suppose that $I' = I \setminus R$ for a set R of rotations.

If $\varrho \in R$, then clearly I' is a stable subinstance of $I \setminus \varrho$ and (iii) holds.

If $\bar{\varrho} \in R$, then clearly $\varrho \notin R$. Denote by I'' a stable subinstance of I obtained by the elimination of a rotation subset of R such that $\bar{\varrho}$ becomes exposed. By Lemma 10, ϱ remains exposed. So in I'', both ϱ and $\bar{\varrho}$ are exposed and also $I' \subseteq I''$. By Lemma 11, elimination of $\bar{\varrho}$ does not affect the preference list for any vertex w not covered by $\bar{\varrho}$ (i. e. neither by ϱ). It follows, that σ is exposed in $I \setminus (R \setminus \{\bar{\varrho}\})$, which is a contradiction with our assumption that I' is maximal.

If neither ϱ nor $\bar{\varrho}$ belong to R, Lemma 10 implies that ϱ must be exposed in I'. So if $\sigma \neq \varrho, \bar{\varrho}$, then σ must be exposed in $I' \setminus \varrho$, which is a subinstance of $I \setminus \varrho$. □

The first main result of this section is a generalization of Theorem 4.3.1 of [15] and it says that each stable b-matching of the given solvable SbM instance $I_0 = (G_0, \mathcal{O}_0, b)$ is associated with a unique set of rotations.

Theorem 2 *For a given solvable SbM instance I_0 let I^* be its Phase-1 subinstance and let $M = I^* \setminus R$ be any stable b-matching of I_0. Then R contains every singular rotation and exactly one rotation of each dual pair.*

Proof. Suppose that $R = \{\varrho^0, \varrho^1, \ldots, \varrho^{t-1}\}$, ϱ^0 is exposed in I^*, and ϱ^j is exposed in $I_j = I^* \setminus \{\varrho^0, \ldots, \varrho^{j-1}\}$ $(1 \leq j \leq t-1)$, so that $I_t = I^* \setminus R = M$. Let σ be some rotation, and I' a stable instance in which σ is exposed.

Suppose that σ is a singular rotation and $\sigma \notin R$. We suppose, that $\sigma \neq \varrho^0$ and since σ is singular and $I' \subseteq I^*$, $\bar{\sigma} \neq \varrho^0$, so by Lemma 15 there exists a stable subinstance of $I_1 = I^* \setminus \varrho^0$ in which σ is exposed. Likewise, there is a stable subinstance of I_2, a stable subinstance of I_3, ..., and a stable subinstance of

I_t in which σ is exposed. But this is a contradiction since $I_t = M$ is a stable b-matching and hence there is no exposed rotation.

Suppose that σ is nonsingular. The definition of the dual rotation and Lemma 11 imply that R cannot contain both σ and $\bar{\sigma}$ since elimination of one prevents the possibility of the elimination of the other. So suppose that neither σ nor $\bar{\sigma}$ belong to R. Rotation ϱ is exposed in a stable instance $I' \subseteq I^*$ and since $\varrho^0 \neq \sigma, \bar{\sigma}$, by Lemma 15 there exists a subinstance of $I_1 = I^* \setminus \varrho^0$ in which σ is exposed. Likewise, for $I_2, \dots, I_t$, giving a contradiction as above. □

Theorem 2 describes a mapping from $\mathcal{M}(I_0)$ to the family of sets of rotations that contain every singular rotation and exactly one rotation of each dual pair. By Lemma 14, this mapping is injective. However, since a rotation cannot be eliminated until it has become exposed, not all such sets of rotations necessarily represent stable b-matchings. Therefore, this mapping might be not onto. However there exists a one-one correspondence between the stable b-matchings and certain sets of rotations. First we state one more lemma that is a generalization of Lemma 4.3.6 of [15].

Lemma 16 *A rotation ϱ exposed in a stable instance I is singular if and only if there is a stable subinstance $I' \subseteq I$ in which ϱ is the only exposed rotation.*

Proof. Suppose first that ϱ is singular. If in I some other rotation is exposed, say σ, then as it cannot be dual to ϱ, Lemma 10 implies that ϱ is exposed also in $I \setminus \sigma$. This argument can be repeated and a sequence of stable instances with ϱ exposed is produced. As it has to be finite, the sequence ends with a stable subinstance, in which ϱ is the only exposed rotation.

Now suppose that ϱ is nonsingular, i. e. $\bar{\varrho}$ exists. Suppose that ϱ is the only exposed rotation in some stable instance $I' = I^* \setminus R$ for a set of rotations $R = \{\sigma^1, \dots, \sigma^r\}$ with $\bar{\varrho} \notin R$ and suppose that rotations from R were eliminated in this order to get I' from I^*. Since $\bar{\varrho} \neq \sigma^1, \bar{\sigma}^1$, Lemma 15 implies that $\bar{\varrho}$ is exposed in some subinstance of $I^* \setminus \sigma^1$. By the same argument, $\bar{\varrho}$ is exposed in some subinstance of $I^* \setminus \sigma^1 \setminus \sigma^2$. Repeated applications of Lemma 15 imply that there exists a stable instance $I'' \subseteq I' = I^* \setminus R$ in which $\bar{\varrho}$ is exposed. If ϱ was the only rotation exposed in I', after its elimination rotation $\bar{\varrho}$ would disappear - a contradiction. □

Definition 4 *Rotation σ is said to be a* predecessor *of rotation ϱ, $\sigma \neq \varrho$, written $\sigma \prec \varrho$, if for each stable instance $I = I^* \setminus R \subseteq I^*$ in which ϱ is exposed, σ belongs to R.*

If σ is a predecessor of ϱ, rotation σ has to be eliminated for ϱ to become exposed. The reflexive closure $\preceq$ of the predecessor relation is clearly a partial order on the set of rotations of a given SbM instance. The set of rotations with relation $\preceq$ will be referred to as *the SbM rotation poset* and denoted by $\Pi(I_0)$. A set $R \subseteq \Pi(I_0)$ will be called *closed*, if for each $\varrho \in R$ we have $\tau \in R$ whenever $\tau \prec \varrho$. A subset of the rotation poset containing all singular rotations and exactly one of each dual pair will be called *complete*.

The following lemmas generalize Lemmas 4.3.7 and 4.3.8 of [15].

Lemma 17 *If ϱ, σ are nonsingular rotations and π a singular one, then*

(i) $\varrho \not\prec \bar{\varrho}$;

(ii) $\varrho \prec \sigma \Longleftrightarrow \bar{\sigma} \prec \bar{\varrho}$;

(iii) $\tau \prec \pi \Longrightarrow \tau$ *is singular; i. e. a predecessor of a singular rotation is also singular.*

Proof. (i) Let $\varrho = (\varrho_E, \varrho_F)$. During the elimination of ϱ, all edges of ϱ_E are deleted. But as $\bar{\varrho} = (\varrho_F, \varrho_E)$, clearly $\varrho \not\prec \bar{\varrho}$.

(ii) Suppose that $\varrho \prec \sigma$ and $\bar{\sigma} \not\prec \bar{\varrho}$, so there exists a stable instance $I = I^* \setminus R$, where $\bar{\varrho} \in R$ and $\bar{\sigma} \notin R$. It follows that $\varrho \notin R$, and so $\sigma \notin R$ either.

As $\sigma \notin R$ and $\bar{\sigma} \notin R$, repeated applications of Lemma 15 imply that there exists a stable instance $I' \subseteq I$ in which σ is exposed (a similar approach as in the proof of Lemma 16). By Lemma 7, I' can be obtained from I by elimination of a set of rotations. So σ is exposed in $I' = I^* \setminus R'$, where $\varrho \notin R'$, contradicting $\varrho \prec \sigma$.

(iii) Suppose that $\tau \prec \pi$ and that τ is nonsingular. So $\bar{\tau}$ is also a rotation and it is exposed in some subinstance $I = I^* \setminus R$, where $\tau \notin R$. From Lemmas 5 and 7 and the fact that the given SbM instance is solvable it follows, that there is at least one stable b-matching embedded in $I \setminus \bar{\tau}$. This stable b-matching can be clearly obtained also from the Phase-1 subinstance I^* by eliminating a set of rotations that does not include τ. But as $\tau \prec \pi$, this set can include neither the singular rotation π, contradicting Theorem 2. □

Lemma 18 *Let R_0 denote the set of all singular rotations. R_0 is closed and every stable b-matching is embedded in the stable instance $I^* \setminus R_0$.*

Proof. First we will show that R_0 is closed. Suppose that $\varrho \in R_0$ and $\sigma \prec \varrho$ for some rotation σ. As ϱ is singular, σ is also singular (Lemma 17(iii)), i. e. $\sigma \in R_0$ and so R_0 is closed.

By Theorem 2, for finding any stable b-matching of a given SbM instance, each singular rotation needs to be eliminated. Notice, that the order of rotation eliminations is immaterial, subject to the predecessor relation, so each stable b-matching is embedded in stable instance $I^* \setminus R_0$. □

Following the notation for the SR, we will call stable instance $I^* \setminus R_0 = \widehat{I} = (\widehat{G}, \widehat{\mathcal{O}}, b)$ *the reduced Phase-1 subinstance.* The poset $\widehat{\Pi}(I_0)$ obtained from the poset $\Pi(I_0)$ by omitting singular rotations will be called *the reduced rotation poset.* Now a complete subset of $\widehat{\Pi}(I_0)$ is a set containing exactly one rotation of each dual pair.

$w_1(3)$	$\epsilon_{12}(5)$	$\epsilon_4(4)$	$\underline{\epsilon_3(6)}$	$\underline{\epsilon_{15}(4)}$	$\epsilon_{11}(3)$	$\underline{\epsilon_5(3)}$
$w_2(2)$	$\underline{\epsilon_6(6)}$	$\epsilon_{13}(3)$	$\epsilon_8(5)$	$\underline{\epsilon_7(5)}$		
$w_3(4)$	$\epsilon_5(1)$	$\underline{\epsilon_2(7)}$	$\underline{\epsilon_{10}(6)}$	$\underline{\epsilon_9(4)}$	$\epsilon_{11}(1)$	$\underline{\epsilon_{13}(2)}$
$w_4(3)$	$\epsilon_{15}(1)$	$\underline{\epsilon_1(7)}$	$\underline{\epsilon_9(3)}$	$\underline{\epsilon_4(1)}$		
$w_5(1)$	$\epsilon_7(2)$	$\epsilon_8(2)$	$\underline{\epsilon_{12}(1)}$			
$w_6(3)$	$\underline{\epsilon_6(2)}$	$\underline{\epsilon_{10}(3)}$	$\underline{\epsilon_3(1)}$			
$w_7(3)$	$\underline{\epsilon_1(4)}$	$\underline{\epsilon_2(3)}$				

Fig. 6: The reduced Phase-1 subinstance $\widehat{I} = I^* \setminus \varrho^1 \setminus \varrho^2 \setminus \varrho^5$ obtained by the elimination of all singular rotations.

Example 4 *For the instance from Example 1, the only rotation exposed in the Phase-1 instance is rotation* $\varrho^1 = (\epsilon_{19}, \epsilon_{10})(\epsilon_{20}, \epsilon_7)$*, thus it is a singular rotation. There are two more singular rotations for this instance, namely* $\varrho^2 = (\epsilon_{21}, \epsilon_{17})(\epsilon_{22}, \epsilon_4)(\epsilon_{14}, \epsilon_3)$ *and* $\varrho^5 = (\epsilon_{18}, \epsilon_3)(\epsilon_{17}, \epsilon_9)$*. The reduced Phase-1 subinstance* $\widehat{I} = I^* \setminus \varrho^1 \setminus \varrho^2 \setminus \varrho^5$ *can be found on Fig. 6. According to Lemma 18, each stable b-matching of the example instance is embedded in* $\widehat{I}$*.*

The following theorem gives a one-one correspondence between $\mathcal{M}(I_0)$ and particular closed subsets of $\widehat{\Pi}(I_0)$. It is a generalization of Theorem 4.3.2 of [15].

Theorem 3 *For a solvable SbM instance* $I_0 = (G_0, \mathcal{O}_0, b)$*, there is a one-one correspondence between* $\mathcal{M}(I_0)$ *and the complete closed subsets of* $\widehat{\Pi}(I_0)$*.*

Proof.

Take any stable b-matching $M \in \mathcal{M}(I_0)$. Lemma 18 implies $M = \widehat{I} \setminus R$ with R containing only dual rotations and this set is complete by Theorem 2. Since a rotation cannot be eliminated before it becomes exposed, set R is closed.

Suppose that R is a complete closed subset of $\widehat{\Pi}(I_0)$. So R can be eliminated from $\widehat{I}$. If there is a bad vertex w in $\widehat{I}$ then there is a rotation ϱ exposed in $\widehat{I} \setminus R$. But then $\varrho \notin R$ and also $\bar{\varrho} \notin R$ contradicting R being complete. Therefore each vertex is good and by Lemma 2(ii) $\widehat{I} \setminus R$ is a stable b-matching. □

Example 5 *Fig. 4 on page 7 shows stable b-matching* $M_1 = I_0 \setminus \varrho^1 \setminus \varrho^2 \setminus \varrho^5 \setminus \varrho^7 \setminus \varrho^3 \setminus \varrho^4 \setminus \varrho^6$*. Rotations* ϱ^1*,* ϱ^2 *and* ϱ^5 *are singular, hence* $M_1 = \widehat{I} \setminus \varrho^7 \setminus \varrho^3 \setminus \varrho^4 \setminus \varrho^6$ *and rotations* ϱ^3*,* ϱ^4*,* ϱ^6 *and* ϱ^7 *are nonsingular. Altogether* 11 *rotations become exposed in all stable instances for this example, namely*

Singular rotations:

$\varrho^1 = (\epsilon_{19}, \epsilon_{20})(\epsilon_{10}, \epsilon_7)$ $\quad \varrho^2 = (\epsilon_{21}, \epsilon_{17})(\epsilon_{22}, \epsilon_4)(\epsilon_{14}, \epsilon_3)$ $\quad \varrho^5 = (\epsilon_{18}, \epsilon_3)(\epsilon_{17}, \epsilon_9)$

Nonsingular rotations:

$\varrho^3 = (\epsilon_7, \epsilon_8)$ $\quad \varrho^4 = (\epsilon_5, \epsilon_{11})$ $\quad \varrho^6 = (\epsilon_{11}, \epsilon_{13})(\epsilon_8, \epsilon_{12})$ $\quad \varrho^7 = (\epsilon_4, \epsilon_{15})$

$\bar{\varrho}^3 = (\epsilon_8, \epsilon_7)$ $\quad \bar{\varrho}^4 = (\epsilon_{11}, \epsilon_5)$ $\quad \bar{\varrho}^6 = (\epsilon_{11}, \epsilon_{13})(\epsilon_8, \epsilon_{12})$ $\quad \bar{\varrho}^7 = (\epsilon_4, \epsilon_{15})$

and Fig. 7 illustrates how they become exposed.
For this instance $\mathcal{M}(I_0) = \{M_1, M_2, M_3, M_4, M_5, M_6\}$ *and stable b-matchings are shown in Fig. 8*

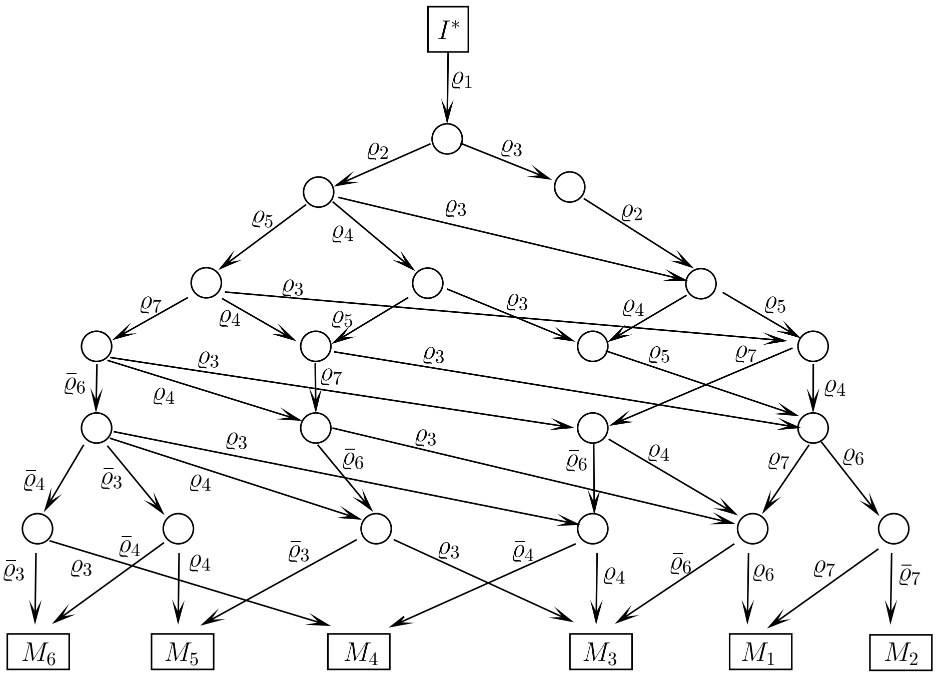

Fig. 7: The diagram illustrating how rotations are exposed.

4 Finding all stable edges

Suppose that $I_0 = (G_0, \mathcal{O}_0, b)$ is a solvable SbM instance. Recall, that an edge $e \in E(G_0)$ is called *stable* if it belongs to some stable b-matching and it is called *fixed* if e belongs to each stable b-matching.

In the stable marriage problem, stable pairs can be found in $O(n^2)$ time by algorithm minimal-differences (n is the size of an instance) [15]. Stable pairs are precisely pairs from the woman-optimal matching and pairs that are in a rotation. An algorithm to find all stable pairs in the many-to-many (bipartite) stable matchings was proposed in [8].

In the stable roommates problem, stable pairs can be found in $O(n^3 \log n)$ time. Stable pairs that are not fixed are precisely those covered by nonsingular rotations [15].

Lemma 19 generalizes the result for the SbM problem (Lemma 4.4.1 in [15]).

$\mathbf{M_1} = \widehat{I} \setminus \{\varrho^3, \varrho^4, \varrho^6, \varrho^7\}$

$w_1(3)$	$\epsilon_{12}(5)$	$\epsilon_3(6)$	$\epsilon_{15}(4)$	
$w_2(2)$	$\epsilon_6(6)$	$\epsilon_{13}(3)$		
$w_3(4)$	$\epsilon_2(7)$	$\epsilon_{10}(6)$	$\epsilon_9(4)$	$\epsilon_{13}(2)$
$w_4(3)$	$\epsilon_{15}(1)$	$\epsilon_1(7)$	$\epsilon_9(3)$	
$w_5(1)$	$\epsilon_{12}(1)$			
$w_6(3)$	$\epsilon_6(2)$	$\epsilon_{10}(3)$	$\epsilon_3(1)$	
$w_7(3)$	$\epsilon_1(4)$	$\epsilon_2(3)$		

$\mathbf{M_2} = \widehat{I} \setminus \{\varrho^3, \varrho^4, \varrho^6, \bar{\varrho}^7\}$

$w_1(3)$	$\epsilon_{12}(5)$	$\epsilon_4(4)$	$\epsilon_3(6)$	
$w_2(2)$	$\epsilon_6(6)$	$\epsilon_{13}(3)$		
$w_3(4)$	$\epsilon_2(7)$	$\epsilon_{10}(6)$	$\epsilon_9(4)$	$\epsilon_{13}(2)$
$w_4(3)$	$\epsilon_1(7)$	$\epsilon_9(3)$	$\epsilon_4(1)$	
$w_5(1)$	$\epsilon_{12}(1)$			
$w_6(3)$	$\epsilon_6(2)$	$\epsilon_{10}(3)$	$\epsilon_3(1)$	
$w_7(3)$	$\epsilon_1(4)$	$\epsilon_2(3)$		

$\mathbf{M_3} = \widehat{I} \setminus \{\varrho^3, \varrho^4, \bar{\varrho}^6, \bar{\varrho}^7\}$

$w_1(3)$	$\epsilon_3(6)$	$\epsilon_{15}(4)$	$\epsilon_{11}(3)$	
$w_2(2)$	$\epsilon_6(6)$	$\epsilon_8(5)$		
$w_3(4)$	$\epsilon_2(7)$	$\epsilon_{10}(6)$	$\epsilon_9(4)$	$\epsilon_{11}(1)$
$w_4(3)$	$\epsilon_{15}(1)$	$\epsilon_1(7)$	$\epsilon_9(3)$	
$w_5(1)$	$\epsilon_8(2)$			
$w_6(3)$	$\epsilon_6(2)$	$\epsilon_{10}(3)$	$\epsilon_3(1)$	
$w_7(3)$	$\epsilon_1(4)$	$\epsilon_2(3)$		

$\mathbf{M_4} = \widehat{I} \setminus \{\varrho^3, \bar{\varrho}^4, \bar{\varrho}^6, \varrho^7\}$

$w_1(3)$	$\epsilon_3(6)$	$\epsilon_{15}(4)$	$\epsilon_5(3)$	
$w_2(2)$	$\epsilon_6(6)$	$\epsilon_8(5)$		
$w_3(4)$	$\epsilon_5(1)$	$\epsilon_2(7)$	$\epsilon_{10}(6)$	$\epsilon_9(4)$
$w_4(3)$	$\epsilon_{15}(1)$	$\epsilon_1(7)$	$\epsilon_9(3)$	
$w_5(1)$	$\epsilon_8(2)$			
$w_6(3)$	$\epsilon_6(2)$	$\epsilon_{10}(3)$	$\epsilon_3(1)$	
$w_7(3)$	$\epsilon_1(4)$	$\epsilon_2(3)$		

$\mathbf{M_5} = \widehat{I} \setminus \{\bar{\varrho}^3, \varrho^4, \bar{\varrho}^6, \varrho^7\}$

$w_1(3)$	$\epsilon_3(6)$	$\epsilon_{15}(4)$	$\epsilon_{11}(3)$	
$w_2(2)$	$\epsilon_6(6)$	$\epsilon_7(5)$		
$w_3(4)$	$\epsilon_2(7)$	$\epsilon_{10}(6)$	$\epsilon_9(4)$	$\epsilon_{11}(1)$
$w_4(3)$	$\epsilon_{15}(1)$	$\epsilon_1(7)$	$\epsilon_9(3)$	
$w_5(1)$	$\epsilon_7(2)$			
$w_6(3)$	$\epsilon_6(2)$	$\epsilon_{10}(3)$	$\epsilon_3(1)$	
$w_7(3)$	$\epsilon_1(4)$	$\epsilon_2(3)$		

$\mathbf{M_6} = \widehat{I} \setminus \{\bar{\varrho}^3, \bar{\varrho}^4, \bar{\varrho}^6, \varrho^7\}$

$w_1(3)$	$\epsilon_3(6)$	$\epsilon_{15}(4)$	$\epsilon_5(3)$	
$w_2(2)$	$\epsilon_6(6)$	$\epsilon_7(5)$		
$w_3(4)$	$\epsilon_5(1)$	$\epsilon_2(7)$	$\epsilon_{10}(6)$	$\epsilon_9(4)$
$w_4(3)$	$\epsilon_{15}(1)$	$\epsilon_1(7)$	$\epsilon_9(3)$	
$w_5(1)$	$\epsilon_7(2)$			
$w_6(3)$	$\epsilon_6(2)$	$\epsilon_{10}(3)$	$\epsilon_3(1)$	
$w_7(3)$	$\epsilon_1(4)$	$\epsilon_2(3)$		

Fig. 8: Stable b-matchings of the example instance.

Lemma 19 *Let I_0 be a solvable SbM instance, let $\widehat{I}$ be its reduced Phase-1 subinstance and $e \in E(\widehat{I})$. Then*

(i) $e = uv$ is a fixed edge if and only if $e \in B(u, \widehat{I}) \cap B(v, \widehat{I})$;

(ii) $e = uv$ is a stable nonfixed edge if and only if e is covered by a nonsingular rotation.

Proof. (i) Suppose that $e \in B(u, \widehat{I}) \cap B(v, \widehat{I})$. Then it cannot be deleted via any rotation elimination in the SbM algorithm and so it is in each stable b-matching obtained by the algorithm. As the algorithm can find each stable b-matching and each stable b-matching is embedded in $\widehat{I}$, e is a fixed edge.

Now suppose that $e = uv$ is a fixed edge and $e \notin B(u, \widehat{I})$. Then u is a bad vertex in $\widehat{I}$ and it leads to some rotation. Since $\widehat{I}$ is the reduced Phase-1 subinstance, all singular rotations were eliminated. Hence by Lemma 16 there is always at least one rotation that u does not lead to. An execution of Phase 2 that in each subsequent step eliminates only rotations to which u does not lead to will never delete any edge of $B(u, \widehat{I})$ (Lemma 5(ii-iv)). Therefore $B(u, \widehat{I}) \subseteq M$ where M is the obtained stable b-matching and so e does not belong to M, a contradiction.

(ii) Suppose that $e = uv$ is a stable edge, $e \notin B(u, \widehat{I}) \cap B(v, \widehat{I})$ and $e \in M = \widehat{I}_r = \widehat{I} \setminus \{\varrho_0, \varrho_1, \ldots, \varrho_r\}$ where $M \in \mathcal{M}(I_0)$. Since $e \in B(u, \widehat{I}_r) \cap B(v, \widehat{I}_r)$, by Lemma 5(iv) e had to be covered by some nonsingular rotation $\varrho \in \{\varrho_0, \varrho_1, \ldots, \varrho_r\}$.

On the other hand, suppose that e is covered by a nonsingular rotation ϱ and $e \in \varrho_E = \bar{\varrho}_F$. Suppose that ϱ is exposed in a stable instance I', $I' \subseteq \widehat{I}$. We will show that there is a stable subinstance I'' of I' in which both $\varrho, \bar{\varrho}$ are exposed. Then the result follows from Lemma 11.

If $\bar{\varrho}$ is exposed in I', then $I'' = I'$. Suppose that $\bar{\varrho}$ is not exposed in I'. As each singular rotation was eliminated, by Lemma 16 there is at least one rotation $\sigma \neq \varrho$ exposed in I'. Repeated applications of Lemma 15 and Lemma 16 assure that there exists a subinstance I'' of I' with $\varrho, \bar{\varrho}$ embedded. □

Based on results from Section 3 together with Lemma 19 we can show how to find all fixed and stable edges.

Suppose that $I_0 = (G_0, \mathcal{O}_0, b)$ is a solvable SbM instance. After one run of the SbM algorithm, some stable b-matching $M = I^* \setminus R$ is obtained where I^* is the Phase-1 subinstance and R is the set of eliminated rotations containing each singular rotation and one of each dual pair (Theorem 2). If we were able to distinguish singular rotations from nonsingular ones, we would be able to identify each stable edge and each fixed edge.

Singularity of a rotation $\varrho \in R$ can be verified exactly as in SR [15]. It is sufficient to run the SbM algorithm once more but avoiding elimination of ϱ. If the algorithm yields some stable b-matching, the ϱ is clearly nonsingular otherwise it is singular. Hence we are able to test singularity of any particular rotation in $O(m)$ time. We need to run this test at most $|R|$-times, i. e. $O(m)$ times since each rotation elimination deletes at least one edge.

To summarize, we can identify all singular rotations in $O(m^2)$ time and so by Lemma 19, we can find all stable edges in $O(m^2)$ time. (A more subtle approach derived from the cutoff-point approach for the SR stated in [15] results in an $O(m^{\frac{3}{2}} \log m^{\frac{1}{2}})$ algorithm.)

Corollary 1 *For a solvable SbM instance $I_0 = (G_0, \mathcal{O}_0, b)$, all stable and fixed edges can be found in $O(m^2)$ time.*

Example 6 *Reduced Phase-1 subinstance $\widehat{I}$ for our example instance is on Fig. 6. The set of edges that are B-edges for both endvertices contains six elements: $\epsilon_1 = \{w_4, w_7\}$, $\epsilon_2 = \{w_3, w_7\}$, $\epsilon_3 = \{w_1, w_6\}$, $\epsilon_6 = \{w_2, w_6\}$, $\epsilon_9 = \{w_3, w_4\}$ and $\epsilon_{10} = \{w_3, w_6\}$. By Lemma 19 those edges are fixed edges (see Fig. 8). All nonsingular rotations for the example instance are listed on page 15, hence beside fixed edges there are another 8 stable nonfixed edges: $\epsilon_4, \epsilon_5, \epsilon_7, \epsilon_8, \epsilon_{11}, \epsilon_{12}, \epsilon_{13}$ and ϵ_{15}.*

5 Optimal stable b-matching

Given a solvable SbM instance $I_0 = (G_0, \mathcal{O}_0, b)$, an optimality criterion of a stable b-matching can be defined in several ways. We will deal with minimum regret and egalitarian stable b-matchings.

Suppose that for each vertex $v \in V(I_0)$ a *weight function* $c_v : E(v, I_0) \to R$ is given, i. e. a function that assigns a numerical value to each edge incident to v. We suppose that this function is strictly increasing with respect to $\preceq_v$, i. e. $e \prec_v f$ if and only if $c_v(e) < c_v(f)$.

The weight function provides a unified language for both the minimum regret as well as the egalitarian criterion. A stable b-matching is a minimum regret stable b-matching if the weight of the worst matched edge is minimized over all vertices. In an egalitarian stable b-matching, the sum of weights of all matched edges is minimum over all stable b-matchings.

Minimum regret stable b-matching

Given a stable instance $I = (G, \mathcal{O}, b)$, the *regret* $\tilde{r}_I(v)$ of some vertex v is the weight of $l_I(v)$. The *regret* $\tilde{r}(I)$ of stable instance I is the maximum regret of vertices. A *minimum regret* stable b-matching M^R is a stable b-matching for which the regret is of a minimum value, i. e. $M^R = \arg\{\min_{M \in \mathcal{M}(I_0)} \tilde{r}(M)\}$.

A minimum regret stable b-matching can be found by a method similar to that for the SR [15]. Consider the reduced Phase-1 subinstance $\widehat{I} \subseteq I_0$ and suppose that $w \in V(\widehat{I})$ is the vertex with maximum regret $\tilde{r}_{\widehat{I}}(w) = \tilde{r}(\widehat{I})$. If w is a good vertex, than clearly each stable b-matching is of the minimum regret $\tilde{r}(\widehat{I})$. If w is bad, then in subsequent steps of the algorithm, there is always at least one rotation to which w does not lead. Hence if only such rotations were eliminated, the regret of each obtained stable b-matching would be $\tilde{r}(\widehat{I})$. It follows that in order to obtain a stable b-matching with a smaller regret, one has to eliminate rotations to which w leads, where w is a bad vertex with maximum regret. Whenever a vertex with maximum regret happens to be a good vertex, any sequence of rotations eliminated from that point leads to a minimum regret stable b-matching. Notice that identifying the vertex with maximum regret and choosing an appropriate rotation can be done in $O(m)$ time.

Theorem 4 *For a solvable SbM instance $I_0 = (G_0, \mathcal{O}_0, b)$, a minimum regret stable b-matching can be found in $O(m)$ time.*

Egalitarian stable b-matching

Define the *weight* of a stable b-matching M as $w(M) = \sum_{e=uv \in M}(c_v(e) + c_u(e))$. Among all stable b-matchings, an *egalitarian stable b-matching* M^E is a b-matching with minimum weight $w(M)$, i. e. $M^E = \arg\{\min_{M \in \mathcal{M}(\mathcal{I})} w(M)\}$.

For a stable instance I that is not a stable b-matching, let us define the *weight* of a vertex v as $w_I(v) = \sum_{e \in B(v,I)} c_v(e)$ and the *weight* of an instance I as number $w(I) = \sum_{v \in V(I)} w_I(v)$.

To find an egalitarian stable b-matching, one can use Theorem 3 giving the one-one correspondence between $\mathcal{M}(I_0)$ and the complete closed subsets of $\widehat{\Pi}(I_0)$. Suppose that a rotation ϱ is exposed in a stable instance I. According to Lemma 5(ii,iv), the weight of stable instance $I \setminus \varrho$ differs from the weight of instance I by the weight of rotation $w(\varrho) = \sum_{j=0}^{r} \left(c_{u_j}(f_j) - c_{u_j}(e_j)\right)$, i. e. $w(I \setminus \varrho) = w(I) + w(\varrho)$. Moreover if $I' = I \setminus R$ where R is the set of rotations eliminated from I in order to get I', then $w(I') = w(I) + \sum_{\varrho \in R} w(\varrho) = w(I) + w(R)$. Hence the weight of each stable b-matching $M \in \mathcal{M}(I_0)$ can be easily computed as

$$w(M) = w(\widehat{I} \setminus R) = w(\widehat{I}) + w(R),$$

where $M = \widehat{I} \setminus R$ and $R = \{\varrho^1, \varrho^2, \dots, \varrho^k\}$ is the complete closed set of rotations eliminated from reduced Phase-1 subinstance $\widehat{I}$.

An egalitarian stable b-matching therefore corresponds to a minimum weight complete closed subset of the (reduced) rotation poset. As was discussed in Section 4 it is possible to identify all singular and dual rotations in $O(m^2)$ time. For each rotation one can calculate its weight with which it contributes to the overall weight in a constant time. However, to find a minimum weight complete closed subset is not an easy problem. In fact, the problem of finding an egalitarian stable matching is NP-complete already for the stable roommates problem [9]. On the other hand, the problem is polynomially solvable for two-sided matching problems [1, 4, 15]. We state here a short proof of NP-completeness for the SR from [9]. Since SR is a special case of SbM, the result holds also for SbM.

Theorem 5 (Theorem 8.3 [9]) *The egalitarian stable roommates problem is NP-complete.*

Proof. To prove the NP-hardness, a polynomial transformation from the NP-complete problem Vertex cover [11] is used. In Vertex cover a graph $G = (V, E)$ and a positive number $K \leq |V|$ are given. The question is whether there exists a vertex cover of size at most K for G, that is, a subset $V' \subseteq V$ such that for each edge $ij \in E$, at least one of i and j belongs to V' and $|V'| \leq K$.

For each instance (G, K) of Vertex cover, we construct a SR instance $I = (N, \mathcal{P})$, where N is the set of vertices and $\mathcal{P} = \{\mathcal{P}_i, i \in N\}$ the set of preference profiles for each vertex $i \in N$. The construction will be such that G contains a vertex cover of size at most K if and only if I admits a stable matching of weight at most $2.|E| + 6.|V| + K$ and moreover each minimum vertex cover of G corresponds to some egalitarian stable matching of I.

For each vertex $i \in V$ introduce four vertices p_i, $\bar{p}_i$, q_i and $\bar{q}_i$, hence $|N| = 4.|V|$. Preference list $\mathcal{P}_i$ for $i \in V$ is as follows (notice, that since G is simple, instead of edges we write the other endvertices):

p_i	$\bar{p}_i$	$p_{j_1} \dots p_{j_k}$	$\bar{q}_i$	
$\bar{q}_i$	p_i		q_i	$\bar{p}_i\ \bar{p}_{j_1} \dots \bar{p}_{j_k}$
$\bar{p}_i$	q_i	$\bar{q}_i\ \bar{q}_{j_1} \dots \bar{q}_{j_k}$	p_i	
q_i	$\bar{q}_i$		$\bar{p}_i$	

where $\{ij_1, ij_2, \dots ij_k\} \in E(i,G)$.

Further suppose that $c_v(u)$ is the rank of vertex u on v's preference list.

We will show, that stable edges can be only of the form $(p_i, \bar{p}_i)$, $(p_i, \bar{q}_i)$, $(q_i, \bar{q}_i)$ or $(q_i, \bar{p}_i)$ for some i. This is straightforward for each $\bar{q}_i$, $\bar{p}_i$ and q_i. For vertex p_i, assume for the sake of contradiction, that (p_i, p_j) belongs to some stable matching M. Then necessarily $(\bar{p}_i, q_i) \in M$ and so $\bar{q}_i$ is without a partner in M. But then $(q_i, \bar{q}_i)$ is a blocking edge, contradicting a stability of M.

Hence either $\{(p_i, \bar{p}_i), (q_i, \bar{q}_i)\} \subseteq M$ or $\{(p_i, \bar{q}_i), (\bar{p}_i, q_i)\} \subseteq M$ for any stable matching M. Set $i \in V'$ if the first case occurs and otherwise $i \notin V'$. Such a set V' must cover all edges of G. Suppose not, then for some edge $e = ij \in E$ neither $i \in V'$ nor $j \in V'$. But then $(p_i, \bar{q}_i) \in M$ and also $(p_j, \bar{q}_j) \in M$, so (p_i, p_j) is a blocking edge - again a contradiction. Thus there is a one-one correspondence between stable matchings of I and vertex covers of G.

Take now some stable matching M. The sum of weights of edges corresponding to vertex $i \in V'$ is $w_M(i) = w_M(p_i) + w_M(\bar{q}_i) + w_M(\bar{p}_i) + w_M(q_i) = 7 + |E(i,G)|$ and for vertex $i \notin V'$ it is $w_M(i) = 6 + |E(i,G)|$. The total weight of a stable matching is $w_M = \sum_{i \in V} (|E(i,G)| + 6) + |V'| = 2.|E| + 6.|V| + |V'|$. An egalitarian stable matching M^E therefore corresponds to a minimum vertex cover V'.

For a given stable matching M its weight can be calculated in polynomial time. This proves the NP-completeness of the egalitarian stable roommates problem. □

Example 7 *For the instance from Example 1, suppose that the weight function* $c_v : E(v, I_0) \to R$ *is given for each vertex* $v \in V(I_0)$*. Suppose that this function represents ranks of incident edges, e. g.* $c_{w_1}(\epsilon_{12}) = 1$, $c_{w_1}(\epsilon_{18}) = 2$, $c_{w_1}(\epsilon_{21}) = 9$*. All stable b-matchings for this instance can be found in Fig. 8. There are two minimum regret stable b-matchings* M_1 *and* M_2 *with the regret* $\tilde{r}(M^R) = 6$*. There are two egalitarian stable b-matchings* M_1 *and* M_6 *both of a weight* $w(M^E) = 57$*.*

6 Conclusion

In this paper we dealt with the stable b-matching problem and presented a one-one correspondence between the set of all stable b-matchings and the set of closed complete sets of the (reduced) rotation poset. This correspondence was used to find a minimum-regret stable b-matching and all stable (fixed) edges in polynomial time. On the other hand, in the non-bipartite case, the egalitarian stable matching problems is NP-hard and for completeness, we provided a brief proof of this property taken from [9].

Recently, Fleiner generalized Irving's algorithm to the stable roommates problem with choice functions [12], generalizing in an appropriate way the notion of rotation. However, detail description of a structure of rotations and its use for finding stable partnerships fulfilling some optimality criteria remains open.

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