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Abstract

Let the cake be represented by the unit interval of reals, with two players having possibly different valuations. We propose a finite algorithm that produces contiguous pieces for both players such that their values differ by at most ε, where $\varepsilon > 0$ is a given small number. Players are not required to reveal their complete value functions, they only have to announce the bisection points of a sequence of intervals. If both utility functions are everywhere positive then the algorithm converges to the unique equitable point.

Keywords. Cake cutting, algorithm, approximation.

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1 Introduction

In this paper we deal with the problem of 'fairly' dividing a certain resource, from now on called the cake, between n people (players). The cake is represented by the interval $\langle 0, 1\rangle$ of reals. Players have different opinions about the values of different parts of the cake. We shall suppose that these valuations are private information of players.

We shall concentrate on equitable divisions, i.e. such that the values of pieces assigned to all players are equal (according to their valuations). In the literature, other concepts of fairness are considered, too. In a fair (sometimes called simple fair) division each player receives at least $1/n$ part of the cake according to his valuation, in an envy-free division no player thinks that he would be better off with somebody else's piece and an exact division assigns pieces such that everybody thinks that everybody's piece has value exactly $1/n$. It is known that in general, these properties are not equivalent, see e.g. [2] and [7], where also some other notions are defined and the relations between them explored.

People have long been concerned with dividing things 'fairly'. One of the oldest recorded fair-division problems is the bankruptcy problem from Talmud [1] and the famous algorithm 'I cut, you choose' goes back, according to [4], at least to the Hebrew Bible.

A rigorous mathematical theory of fair division was established soon after the second world war by a group of brilliant Polish mathematicians Steinhaus, Banach and Knaster [8]. In general, it is easier to prove the existence of a division fulfilling a certain property than to find such a division, see a nice review in [7], Chapter 7. In recent years, several papers studied what can be achieved by a finite algorithm. A finite algorithm, as specified by [7], [12] or [10], uses a finite number of requests of two types issued to players:

- 'Cut the given cake piece into two pieces whose values are in the given ratio!' (cutting query)
- 'What is your value of the given cake piece?' (evaluation query)

In particular, a finite algorithm does not require the knowledge of complete value functions of players and so also the famous moving-knife algorithms [9], [3] cannot be considered finite. Robertson and Webb [6] proved that there exists no finite algorithm that produces an exact division for two players and Stromquist [10] showed that neither an envy-free division among three players in which each player receives a single piece, can be obtained by a finite algorithm.

Hence algorithms that produce a 'nearly fair' division are called for. Robertson and Webb [6], [7] provided a near exact algorithm that, given a small $\varepsilon > 0$ and a set of real numbers $\alpha_1, \ldots, \alpha_n$ with $\sum_{i=1}^{n} \alpha_i = 1$, constructs a division such that for each player i, the value of his piece differs from α_i by at most ε. The main idea of the algorithm is the following. Player 1 cuts the cake into pieces which he considers to be smaller than k each, where k is a small number determined by ε and n; in the case of two players k may be set to $\varepsilon/2$. Then player 2 can reduce any of the pieces (if necessary) so that each new piece will be smaller than k according to him, etc. A near exact division is then produced by a suitable assignment of the obtained pieces to players. A disadvantage of this algorithm is that many small pieces arise and those assigned to one player can be scattered irregularly over the whole cake.

Another ε-exact division for two players can be obtained using the approach described by Simmons and Su in [11]. They considered the so-called consensus-halving, i.e. a division of an object into two portions so that each of n people believes the portions are equal. (If $n = 2$, we get exact division.) Simmons and Su showed, using methods from combinatorial topology, namely theorems of Borsuk-Ulam and Tucker, that such a division, needing at most n cuts exists, and this number of cuts is best possible. Moreover, they showed how a constructive proof of Tuckers lemma yields a finite algorithm for locating an ε-approximate solution that uses a minimal number of cuts.

Our algorithm `bisect` achieves a 'nearly equitable' division for two players, i.e. the values of the received pieces differ from each other by at most ε. It is less general than Robertson and Webb's algorithm, since the used approach cannot be generalized to a greater number of players. On the other hand, it is more intuitive, does not make use of any deep mathematical results, the values of the assigned pieces are in general greater than 1/2 and only one cut is used (although several 'marks' are made on the cake), so each player receives a contiguous interval. `bisect` uses only one type of elementary operation: players are asked to announce a bisection point (according to their values) of a certain subinterval.

The organization of the paper is as follows. In Section 2 we formally define the used notions and prove the basic property of equitable points. Section 3 is devoted to algorithm `bisect` and its properties. Finally, Section 4 concludes with some examples and open questions.

2 Definitions and basic properties of divisions

We will consider the set of *players* $N = \{1, 2, ..., n\}$; in the greatest part of this paper, $n = 2$. The *cake* is a single divisible good that is to be divided between players and is represented by the interval $\langle 0, 1\rangle$. In this work, the only *allowable portions – pieces* (see [5]) are intervals $\langle p, q\rangle$, $0 \le p \le q \le 1$. A cutpoint of two neighbouring pieces cannot belong to both of them, but since in our model the value of a piece is not influenced by a single point, we shall represent all pieces as closed intervals.

The model studied in this paper allows the players to value the same piece differently, as well as to assign different values to intervals with the same length but in different positions. The value that player i assigns to an interval $I = \langle p, q\rangle$ will be denoted by $U_i(I)$ or $U_i(p, q)$. We suppose that each player's valuation U_i is

(i) nonnegative, i.e. $U_i(I) \ge 0$ for each interval $I \subseteq \langle 0, 1\rangle$,

(ii) additive, i.e. $U_i(I \cup J) = U_i(I) + U_i(J)$ for any two disjoint intervals I, J,

(iii) divisible, i.e. for each $I = \langle p, q\rangle \subseteq \langle 0, 1\rangle$ and each $\lambda \in \langle 0, 1\rangle$ there exists $r \in \langle p, q\rangle$ such that $U_i(p, r) = \lambda\, U_i(p, q)$, and

(iv) normalized, i.e. $U_i(0, 1) = 1$.

Such a valuation can be represented by a nonnegative integrable utility function $u_i : \langle 0, 1\rangle \to \mathbb{R}$ such that $\int_0^1 u_i(x)\mathrm{d}x = 1$. So, $U_i(p, q) = \int_p^q u_i(x)\mathrm{d}x$ for $p \le q$.

We are interested in cake divisions where each player receives exactly one interval. Such cake divisions will be called *simple* and they are specified by their cutpoints and the player's order.

Definition 1 *A simple cake division is a pair $d_c = (d, \varphi)$, where d is an $(n-1)$ tuple $(x_1, x_2..., x_{n-1})$ of cutpoints with $0 < x_1 < x_2 < ... < x_{n-1} < 1$, and $\varphi : N \to N$ is a permutation of N.*

Sometimes we shall complement the cutpoints by points $x_0 = 0$ and $x_n = 1$. Further, we suppose that in a simple cake division $d_c = (d, \varphi)$ with $\varphi(j) = i$ player i is assigned the interval $\langle x_{j-1}, x_j \rangle$.

Now we formally define the notions of fairness used in this work (see also [2, 7] for other notions and relations between them).

Definition 2 *Let $d_c = (d, \varphi)$ be a simple cake division for the set of players N. Then d_c is called*

a) simple fair, if $U_{\varphi(j)}(x_{j-1}, x_j) \geq 1/n$ for each $j \in N$

b) exact, if $U_{\varphi(j)}(x_{k-1}, x_k) = 1/n$ for each $j, k \in N$

c) envy-free, if $U_{\varphi(j)}(x_{j-1}, x_j) \geq U_{\varphi(j)}(x_{k-1}, x_k)$ for each $j, k \in N$

d) equitable, if $U_{\varphi(j)}(x_{j-1}, x_j) = U_{\varphi(k)}(x_{k-1}, x_k)$ for each $j, k \in N$.

Notice that when players have neither information about the valuations of other players nor about the pieces assigned to them, $a)$ is the only way they can use to judge the fairness of the division. If they know what the other players' pieces are, envy-freeness can also be considered. Further, if players are to be able to evaluate equitability of a division, they should know the values that other players received for their pieces.

It is easy to see that each envy-free cake division is simple fair. For $n = 2$, each simple fair cake division is envy-free (this is not necessarily true for more than two players), but not every simple fair division is equitable – an example is e.g. the result of the famous divide-and-choose procedure. Further, in an equitable division all players may receive equally little, but if the common value of the pieces is at least $1/n$, fairness is ensured. However, a fair equitable division may be neither envy-free nor exact, but for two players equitability with assigned values equal to $1/2$ is equivalent to exactness.

From now on we shall deal with simple equitable cake divisions for two players. A point $e \in (0, 1)$ is called an *equitable point* if $U_1(0, e) = U_2(e, 1)$ (or, equivalently, $U_1(e, 1) = U_2(0, e)$). The set of all equitable points will be denoted by $\mathcal{E}$.

Theorem 1 *For two players, the set $\mathcal{E}$ is always nonempty and connected. Moreover, $U_1(0, e)$ and $U_2(e, 1)$ are constant on $\mathcal{E}$ and either $U_1(0, e) = 1/2$ or exactly one of the two simple divisions generated by e is proportional for each $e \in \mathcal{E}$.*

Proof. Due to the assumptions in our model, the functions $U_1(0, x)$ and $U_2(0, x)$ of variable x are continuous and nondecreasing, whereas functions $U_2(x, 1) = 1 - U_2(0, x)$ and $U_1(x, 1) = 1 - U_1(0, x)$ continuous and nonincreasing. Then the function $f : \langle 0, 1 \rangle \to \mathbb{R}$ defined by $f(x) = U_1(0, x) - U_2(x, 1)$ is continuous, nondecreasing and $f(0) = -1, f(1) = 1$. According to the Intermediate Value Theorem there exists a point $e \in (0, 1)$ such that $0 = f(e)$.

Now let $e_1, e_2 \in \mathcal{E}$ and $e \in (e_1, e_2)$ be arbitrary. Again, since $U_1(0, x)$ is nondecreasing and $U_2(x, 1)$ nonincreasing we have that

$$U_1(0, e_1) \le U_1(0, e) \le U_1(0, e_2) = U_2(e_2, 1) \le U_2(e, 1) \le U_2(e_1, 1) = U_1(0, e_1)$$

hence there is equality throughout. So the second assertion follows and the rest is obvious. ■

Notice that if the utilities of both players are everywhere positive, then the equitable point is unique.

3 Bisection algorithm

So we know that an equitable point e for two players always exists. It could be computed from the equation

$$\int_0^e u_1(x)\mathrm{d}x = \int_e^1 u_2(x)\mathrm{d}x, \tag{1}$$

however, except in very trivial cases, such an equation usually cannot be solved by a closed formula. Further, to be at all able to perform such computations, the players must reveal their complete valuation functions to the algorithm. An algorithm for computing equitable division for any number of players, based on solving integral equations in $n-1$ unknowns and comparing their results for all the $n!$ possible players' orders was proposed in [2], but Hill and Morrison [5] pointed some weaknesses of this approach. Moreover, recall the result of Roberston and Webb ([6], see also Theorem 8.3 in [7]) stating that no finite algorithm can produce an exact fair division for two players in the ratio 1:1 (not even one with contiguous pieces).

The algorithm that we propose finds a cutpoint of a cake division such that the difference between values of pieces assigned to players is not higher than a predetermined value ε. We call this property of a cake division ε-equitablity:

Definition 3 *Let $d_c = (d, \varphi)$ be a simple cake division and $\varepsilon > 0$ a real number. Then d_c is called ε-equitable if*

$$|U_{\varphi(j)}(x_{j-1}, x_j) - U_{\varphi(k)}(x_{k-1}, x_k)| \le \varepsilon \text{ for each } j, k \in N.$$

Our algorithm uses the assumption that players are willing to announce a bisection point for each piece of cake. There is no need for revealing players' valuations for pieces that are obtained during the algorithm.

In the beginning, the players are asked to announce their bisection points a_1, b_1 of interval $\langle 0,1\rangle$. If $a_1 = b_1$ then the algorithm cuts the cake at the point $a_1 = b_1$ and an equitable division (w.l.o.g with players' order (1,2)) is obtained with both players assigned pieces with value $\frac{1}{2}$. The algorithm terminates.

From now on suppose that $a_1 < b_1$ (otherwise the players are simply renamed). Player 1 is temporarily assigned the piece $\langle 0, a_1\rangle$, player 2 the piece $\langle b_1, 1\rangle$. Both pieces have the same value $\frac{1}{2}$ for their assignees. In the following iterations, the algorithm tries to enlarge the assignments by adding each player a piece with the value one half of the value considered in the previous iteration, using the information about the bisection points. Clearly, the right-end of the temporarily assigned piece of player 1 should move to the right and the left-end of the temporarily assigned piece of player 2 should move to the left.

To achieve this, in iteration j each player $i = 1,2$ is submitted a working interval $\langle p^i_j, q^i_j\rangle$ with value $1/2^{j-1}$ (disjoint from his temporarily assigned piece) and is asked to privately announce to the algorithm his bisection point (denoted by a_j and b_j for player 1 and 2, respectively). In the description of the algorithm, this operation is denoted by $\texttt{half}_1(p^1_j, q^1_j)$ and $\texttt{half}_2(p^2_j, q^2_j)$. Neither player has any information about the bisection point of the other player.

If $a_j < b_j$, then this enlargement is possible. In the following iteration, the left half of his current working interval is added to the temporary assignment of player 1 and he will bisect the right half of his current working interval. Similarly, the temporary assignment of player 2 will be increased by the right half of his current working interval and next he will be asked to bisect the left half of his current working interval.

If $a_i > b_i$, then increasing the assignments would lead to intersecting pieces, so the algorithm orders the players to retreat. This means that player 1 will next bisect the left half of his current working interval and player 2 the right half of his current working interval. If $a_i = b_i$, the referee can cut the cake in a_i. Player 1 is assigned the piece $\langle 0, a_i\rangle$, player 2 the piece $\langle a_i, 1\rangle$ and the algorithm terminates with an equitable division.

The choice of the left or the right half of an interval $\langle p, q\rangle$ determined by the bisection point m is concisely expressed by calls to functions $\texttt{left}(p, q, m, p\prime, q\prime)$ and $\texttt{right}(p, q, m, p\prime, q\prime)$. These functions output as the new working intervals $\langle p\prime, q\prime\rangle$ the intervals $\langle p, m\rangle$ or $\langle m, q\rangle$, respectively. The complete description of algorithm `bisect` is given in Figure 1.

In what follows we shall say that the j-th iteration is *successful*, if $a_j \leq b_j$. (Notice that the first iteration is always successful due to the renaming of players.) Otherwise the iteration is called *unsuccessful*.

Definition 4 *Temporary assignment in the j-th iteration is interval $\langle 0, A\rangle$ for player 1 and interval $\langle B, 1\rangle$ for player 2, where $A = p^1_{j+1}$, $B = q^2_{j+1}$.*

begin a_1 :=$\texttt{half}_1(0,1)$; b_1 :=$\texttt{half}_2(0,1)$;
  if $a_1 = b_1$ **then** cut the cake in a_1 and halt;
  if $a_1 > b_1$ **then** rename the players;
  p^1_2 :=$\texttt{half}_1(0,1)$; $q^1_2 := 1$; $p^2_2 := 0$; q^2_2 :=$\texttt{half}_2(0,1)$; $j := 1$;
  repeat $j := j+1$; $a_j := \texttt{half}_1(p^1_j, q^1_j)$; $b_j := \texttt{half}_2(p^2_j, q^2_j)$;
    if $a_j \neq b_j$ **then**
    begin
      if $a_j < b_j$ **then**
        begin $\texttt{right}(p^1_j, q^1_j, a_j, p^1_{j+1}, q^1_{j+1})$; $\texttt{left}(p^2_j, q^2_j, b_j, p^2_{j+1}, q^2_{j+1})$ **end**;
      if $a_j > b_j$ **then**
        begin $\texttt{left}(p^1_j, q^1_j, a_j, p^1_{j+1}, q^1_{j+1})$; $\texttt{right}(p^2_j, q^2_j, b_j, p^2_{j+1}, q^2_{j+1})$ **end**;
    end
  until $a_j = b_j$ or $1/2^{j-1} < \varepsilon$;
  if $a_j \neq b_j$ **then** cut the cake in $c := \left(max\left\{p^1_j, p^2_j\right\} + min\left\{q^1_j, q^2_j\right\}\right)/2$
      else cut the cake in a_j;
end

Figure 1: Algorithm `bisect`

Players are certain to get their temporary assignments from any iteration, moreover, the values of their temporary assignments increase after every successful iteration and stay unchanged after an unsuccessful iteration (this follows from the definition of the algorithm). Further properties of the algorithm are summarized in the following lemma.

Lemma 1 *In each iteration $j \geq 2$ we have:*

(a) $\left\langle p^i_j, q^i_j\right\rangle \subset \left\langle p^i_{j-1}, q^i_{j-1}\right\rangle$ *for both players* $i = 1,2$

(b) $U_i(p^i_j, q^i_j) = 1/2^{j-1}$ *for both players* $i = 1,2$

(c) $p^1_j = a_\ell$ *and* $q^2_j = b_\ell$ *where* ℓ *was the last successful iteration before* j

(d) $U_1(0, a_j) = U_2(b_j, 1)$

(e) $U_1(0, p^1_j) = U_2(q^2_j, 1)$, *hence* $U_1(0, A) = U_2(B, 1)$

(f) $\left\langle p^1_j, q^1_j\right\rangle \cap \left\langle p^2_j, q^2_j\right\rangle \neq \emptyset$

(g) $|U_1(0,c) - U_2(c,1)| \leq 1/2^{j-1}$ *for each* $c \in \left\langle p^1_j, q^1_j\right\rangle \cap \left\langle p^2_j, q^2_j\right\rangle$

Proof.
$(a),(b),(c)$ Directly from the definition of algorithm `bisect`.

(d) Induction on j. Assume, that $U_1(0,a_k) = U_2(b_k,1)$ for every $k < j$. Then $U_1(0,a_j) = U_1(0,p^1_j) + U_1(p^1_j, a_j) = U_1(0,a_\ell) + U_1(p^1_j, q^1_j)/2 = U_2(b_\ell, 1) + U_2(p^2_j, q^2_j)/2 = U_2(b_j, q^2_j) + U_2(q^2_j, 1) = U_2(b_j, 1)$, hence the desired equality follows.

(e) Follows from (c) and (d).
(f) In the first iteration we have $\langle p_1^k, q_1^k \rangle = \langle 0, 1\rangle$ for both k; in the second one $\langle p_2^1, q_2^1\rangle = \langle a_1, 1\rangle$ and $\langle p_2^2, q_2^2\rangle = \langle 0, b_1\rangle$ and since after eventuall renaming of player we have $a_j < b_j$, the claim holds for $j = 1, 2$. Now let us suppose that $\langle p_j^1, q_j^1 \rangle \cap \langle p_j^2, q_j^2 \rangle \neq \emptyset$ for some $j \geq 2$. Those two intervals intersect if and only if $p_j^1 \leq q_j^2$ and simultaneously $p_j^2 \leq q_j^1$. The algorithm proceeds to the next iteration only if $a_j \neq b_j$ and we distinguish now the successful and unsuccessful iteration.

- (i) If the iteration is successful, which happens if $a_j < b_j$, the new working intervals are $\langle a_j, q_j^1 \rangle$ and $\langle p_j^2, b_j \rangle$. Now the inequality $p_j^2 \leq q_j^1$ folows from the induction hypothesis and $a_j < b_j$ from the definition of the successful iteration.
- (ii) In an unsuccessful iteration is the new working intervals are $\langle p_j^1, a_j \rangle$ and $\langle b_j, q_j^2 \rangle$. Again, the inequality $p_j^1 \leq q_j^2$ folows from the induction hypothesis and $b_j < a_j$ holds because the iteration was unsuccessful.

In both cases, the intersection of working intervals in nonempty also in the following iteration and by induction the claim is proved.
(g) Additivity of value functions and (b) imply that $0 \leq U_1(p_j^1, c) \leq 1/2^{j-1}$ and $0 \leq U_2(c, q_j^2) \leq 1/2^{j-1}$. Then, using (e)

$$\begin{aligned} |U_1(0,c) - U_2(c,1)| &= |U_1(0,p_j^1) + U_1(p_j^1,c) - (U_2(c,q_j^2) + U_2(q_j^2,1))| \\ &= |U_1(p_j^1,c) - U_2(c,q_j^2)| \leq 1/2^{j-1} \end{aligned}$$

and the claim is proved. ∎

Theorem 2 *For $\varepsilon > 0$, algorithm* `bisect` *outputs a fair ε-equitable division.*

Proof. If the procedure ends because $a_j = b_j$, the obtained simple division is equitable. If not, claim (f) of Lemma 1 ensures that ε-equitability for the given $\varepsilon > 0$ is achieved. Fairness is ensured by choosing the appropriate players' order: $(1, 2)$ if the condition in line 3 of the algorithm was not fulfilled, otherwise $(2, 1)$. ∎

For everywhere positive utility functions, the equitable point e is unique and also the bisection points of players are uniquely defined. This implies that there is exactly one possible execution of the algorithm and the bisection points converge to e. This is proved in the following Theorem.

Theorem 3 *If the utility functions of both players are everywhere positive then in each iteration $j \geq 2$ we have for the unique equitable point e:*

- (a) $min\{a_j, b_j\} \leq e \leq max\{a_j, b_j\}$
- (b) $e \in \langle p_j^1, q_j^1 \rangle \cap \langle p_j^2, q_j^2 \rangle$
- (c) $|U_1(0,e) - U_1(0,a_j)| < 1/2^j$ *and* $|U_2(e,1) - U_2(b_j,1)| < 1/2^j$

(*d*) $\lim_{j\to\infty} a_j = \lim_{j\to\infty} b_j = e$

Proof. (*a*) We distinguish three cases:

- $a_j = e$. Then, using Lemma 1(*d*), we have $U_2(b_j, 1) = U_1(0, a_j) = U_1(0, e) = U_2(e, 1)$. As function $U_2(x, 1)$ is strictly decreasing, this implies $b_j = e$.
- $a_j < e$. Since $U_1(0, x)$ is strictly increasing and $U_2(x, 1)$ is strictly decreasing in x, relations $U_2(e, 1) = U_1(0, e) > U_1(0, a_j) = U_2(b_j, 1)$ imply $e < b_j$.
- Similarly, $a_j > e$ implies $e > b_j$.

Summarizing: there are exactly three possibilities: $a_j = e = b_j$ or $a_j < e < b_j$ or $b_j > e > a_j$ and the claim is proved.

(*b*) Induction on j. Suppose that $e \in \langle p_k^1, q_k^1\rangle \cap \langle p_k^2, q_k^2\rangle$ for each $k < j$. According to (*a*), there are three different cases.

- $a_{j-1} = e = b_{j-1}$. Algorithm **bisect** terminates and there is nothing to be proved for j.
- $a_{j-1} < e < b_{j-1}$. According to the definition of the algorithm, iteration $j-1$ is successful, $p_j^1 = a_{j-1} < e < q_{j-1}^1 = q_j^1$ and $p_j^2 = p_{j-1}^2 < e < b_{j-1} = q_j^2$, the claim is proved.
- $a_{j-1} > e > b_{j-1}$. In this case iteration $j-1$ is unsuccessful, $p_j^1 = p_{j-1}^1 < e < a_{j-1} = q_j^1$ and $p_j^2 = b_{j-1} < e < q_{j-1}^2 = q_j^2$, so the claim follows.

(*c*) From the definition of bisection points and Lemma 1(*b*) we have for each j:

$$U_1(p_j^1, a_j) = U_1(a_j, q_j^1) = 1/2^j \text{ and } U_2(p_j^2, b_j) = U_2(b_j, q_j^2) = 1/2^j.$$

We again distinguish three cases:

- $a_j = b_j = e$ then the assertion is trivial.
- $a_j > e > b_j$. Then, using (*b*), $|U_1(0, e) - U_1(0, a_j)| = U_1(e, a_j) < U_1(p_j^1, a_j) = 1/2^j$. Similarly, $|U_2(e, 1) - U_2(b_j, 1)| = U_2(b_j, e) < U_2(b_j, q_j^2) = 1/2^j$.
- Case $a_j < e < b_j$ is proved similarly.

(*d*) Follows from (*c*). ■

4 Discussion and conclusions

It is relatively straightforward to modify the presented algorithm to achieve a division which assigns to two players contiguous pieces such that the ratio of their values is approximately $r : s$ for some predetermined values $r \neq s$. (An easy modification of Theorem 1 shows that such a division always exists under relatively simple assumptions about the players' valuations.) In the first iteration player one is asked to tell the two points a_1^L, a_1^R such that $U_1(0, a_1^L) = U_1(a_1^R, 1) = \frac{r}{r+s}$ and similarly player 2 tells the two points b_1^L, b_1^R such that $U_2(0, b_1^L) = U_1(b_1^R, 1) = \frac{s}{r+s}$. (It may again be necessary to change the players' order, but let us further suppose that this was not the case.) Then, in iterations j player 1 demands a piece with value $\frac{r}{2^j(r+s)}$ neighbouring his temporary assignment from the right and player 2 demands a piece with value $\frac{s}{2^j(r+s)}$ neighbouring his temporary assignment from the left. Similarly, as in Figure 1, iteration is either successful and both players receive the demanded piece, or it is unsuccessful and neither does. Inequality $|sU_1(0, c) - rU_2(c, 1)| < \varepsilon$ will be proved similarly as in the case $r = s$. Further, the values of the temporarily assigned pieces converge to $r \sum_{j \in J} 2^j$ and $s \sum_{j \in J} 2^j$, respectively, with J denoting the set of all successful iterations, so the approximate ratio $r : s$ will be achieved.

Now we illustrate some problems that may occur if the utility functions of players admit nontrivial intervals with zero value.

Example 1. Let the utility functions be

$$u_1(x) = \begin{cases} 2 & \text{if } x \in \langle 0, 1/4 \rangle \\ 0 & \text{if } x \in (1/4, 3/4 \rangle \\ 2 & \text{if } x \in (3/4, 1 \rangle \end{cases} \qquad u_2(x) = \begin{cases} 3/2 & \text{if } x \in \langle 0, 1/3 \rangle \\ 0 & \text{if } x \in (1/3, 2/3 \rangle \\ 3/2 & \text{if } x \in (2/3, 1 \rangle \end{cases}$$

Clearly, $\mathcal{E} = \langle 1/3, 2/3 \rangle$ and $U_1(0, e) = U_2(e, 1) = 1/2$. Since the bisection points of players are not defined uniquelly, let us suppose that the players state at the beginning of the algorithm $a_1 = 1/4$ and $b_1 = 1/3$. It is easy to see that all the subsequent iterations will be unsuccessful, since $a_j > 3/4$ and $b_j < 1/3$. Further, in each iteration $p_j^1 = 1/4$, $q_j^1 = 3/4 + 1/2^{j+1}$ and $q_j^2 = 1/3$, $p_j^2 = 1/3(1 - 1/2^{j-1})$, so if the algorithms stops in iteration j, we get $c = 1/3(1 - 1/2^j)$. This is not an equitable point, but ε-equitability will be achieved. Further, the sequence of bisection points of player 2 converges to the equitable point $1/3$, while $\lim_{j \to \infty} a_j = 3/4$, which is not an equitable point.

Example 2. In the second example the utility functions of players are

$$u_1(x) = \begin{cases} 2 & \text{if } x \in \langle 0, 1/4) \\ 3 & \text{if } x \in \langle 1/4, 1/3) \\ 0 & \text{if } x \in \langle 1/3, 2/3) \\ 3/4 & \text{if } x \in \langle 2/3, 1 \rangle \end{cases} \qquad u_2(x) = \begin{cases} 1/2 & \text{if } x \in \langle 0, 1/2) \\ 0 & \text{if } x \in \langle 1/2, 3/4) \\ 2 & \text{if } x \in \langle 3/4, 7/8) \\ 4 & \text{if } x \in \langle 7/8, 1 \rangle \end{cases}$$

Here it is easy to see that $\mathcal{E} = \langle 1/2, 2/3 \rangle$ with the common fair values equal to 3/4. The bisection points in the first iteration of the algorithm are uniquely determined, namely $a_1 = 1/4$ and $b_1 = 7/8$. However, in the second iteration there are already several choices, $a_2 \in \langle 1/3, 2/3 \rangle$ and $b_2 \in \langle 1/2, 3/4 \rangle$. The rest of the algorithm depends on this step:

(i) If the second iteration is successful, then $A \geq 1/3$ and $B \leq 3/4$ and all the subsequent iterations are unsuccessful, since $a_j > 2/3$ and $b_j < 1/2$. Both players receive pieces with value equal to 3/4 and their bisection points a_j and b_j converge to 2/3 and 1/2 respectively, hence to two different equitable points.

(ii) If the second iteration is unsuccessful, then the following bisection points of player 1 will be in the open interval $(1/4, 1/3)$ and those of player 2 will be in $(3/4, 7/8)$. All the following iterations will be sucessful, $\lim_{j\to\infty} a_j = 1/3$, $\lim_{j\to\infty} b_j = 3/4$, so neither sequences will converge to an equitable point.

Finally, we would like to formulate an open problem.

We know that for two players an equitable division with contiguous pieces exists, giving everybody the value at least 1/2. If $n \geq 3$, then it may happen that while an equitable division giving everybody a piece with value 1 exists, the maximum value when restricted to contiguous pieces is much less. Consider n players with the following utility functions:

$$u_i(x) = \begin{cases} 2n-1 & \text{if } x \in \left\langle \frac{2i-1}{2n-1}; \frac{2i}{2n-1} \right\rangle \\ 0 & \text{otherwise} \end{cases} \quad \text{for } i = 1, 2, \ldots, n-1$$

$$u_n(x) = \begin{cases} \frac{2n-1}{n} & \text{if } x \in \bigcup_{k=0}^{n-1} \left\langle \frac{2k}{2n-1}, \frac{2k+1}{2n-1} \right\rangle \\ 0 & \text{otherwise} \end{cases}$$

A division with pieces of value equal 1 for each player is achieved if everybody receives the portion of the interval where his utility function is positive. On the other hand, if neither player is to receive a piece with value 0, player n cannot receive more than $1/n$ (it is easy to see that a simple equitable division with such values exists). When the number of players increases, the common value of a simple equitable division diminishes to 0.

So we have the following question: Is it possible to determine the maximum number v such that there exists a simple equitable division assigning each player a piece with value at least v?

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