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An alternative description of Gabor spaces and Gabor-Toeplitz operators*

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Abstract. We present an alternative approach to study basic properties of Gabor-Toeplitz localization operators based on the general decomposition method of Vasilevski applied for particular case of spaces of Gabor transforms.

1 Introduction and preliminaries

In what follows $\langle \cdot, \cdot \rangle_d$ always means the inner product on $L_2(\mathbb{R}^d)$, whereas $\langle \cdot, \cdot \rangle_{2d}$ denotes the inner product on $L_2(\mathbb{R}^{2d})$. Let ϕ be normalized in $L_2(\mathbb{R}^d)$, i.e. $\|\phi\|_d = 1$. By $\phi_{q,p}$ we denote the *phase space shift* of a function ϕ by (q,p), i.e. $\phi_{q,p}(x) = \phi(x-q)e^{2\pi i p x}$, $q, p \in \mathbb{R}^d$. For a fixed $\phi \in L_2(\mathbb{R}^d)$, the functions of the form

$$(V_\phi f)(q,p) = \langle f, \phi_{q,p} \rangle_d, \quad f \in L_2(\mathbb{R}^d) \tag{1}$$

form a reproducing kernel Hilbert space $V_\phi(L_2(\mathbb{R}^d))$ which we call the *space of Gabor transforms* (these spaces are called "model spaces" in [3]). The transform (1) is known under a dozen names, such as short-time Fourier transform, Gabor transform, radar ambiguity function, coherent state transform, etc., each of which indicates a particular scientific application. Note that the transform (1) can also be defined on many pairs of distribution spaces and test functions, cf. [10]. The space $V_\phi(L_2(\mathbb{R}^d))$ is a closed subspace of $L_2(\mathbb{R}^{2d})$. Now, with $\|\phi\|_d = 1$, the functions $\phi_{q,p}$ give rise to the *Gabor reproducing formula*: for any $f \in L_2(\mathbb{R}^d)$ we have

$$f(x) = \int_{\mathbb{R}^{2d}} (V_\phi f)(q,p)\phi_{q,p}(x)\, dqdp, \tag{2}$$

Clearly, $\|V_\phi f\|_{2d} = \|f\|_d$, which means that the operator $V_\phi : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^{2d})$ given by (1) is an isometry and the integral operator

$$(P_\phi F)(q,p) = \int_{\mathbb{R}^{2d}} F(s,r) K_{s,r}(q,p)\, dsdr,$$

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is the orthogonal projection onto $V_\phi(L_2(\mathbb{R}^d))$, where $K_{q,p}(s,r) = \langle \phi_{q,p}, \phi_{s,r}\rangle_d$ is the reproducing kernel in $V_\phi(L_2(\mathbb{R}^d))$. Clearly, the kernel is hermitian symmetric $\overline{K_{q,p}(s,r)} = K_{s,r}(q,p)$ and has the reproducing property

$$F(q,p) = \langle F, K_{q,p}\rangle_{2d} = \int_{\mathbb{R}^{2d}} F(s,r) K_{s,r}(q,p)\, dsdr, \quad F \in V_\phi(L_2(\mathbb{R}^d)). \qquad (3)$$

For more information on time-frequency analysis, see e.g. [16].

In [19] Vasilevski introduced a method for decomposition of a Hilbert space onto L_2-type spaces (more precisely, the complete decomposition of $L_2(\Pi)$ onto Bergman and Bergman-type spaces of poly-analytic and poly-anti-analytic functions). The main advantage of this method is that it gives a description of the space of (analytic) functions under study in "real analysis terms" and (in some cases, e.g. the upper half-plane in cartesian coordinates, or the unit disk in polar coordinates) it permit a complete separation of variables in the Bergman space representation. The general scheme of such decomposition presented in [21] has been recently successfully used in the case of wavelets and Calderón-Toeplitz operators in [18]. Here, following the scheme of [21], we present similar results for Gabor-Toeplitz operators.

Firstly, we present a representation of $V_\phi(L_2(\mathbb{R}^d))$ which shows how much room do such spaces occupy inside $L_2(\mathbb{R}^{2d})$. We find a unitary operator $U : L_2(\mathbb{R}^{2d}, dqdp) \to L_2(\mathbb{R}^{2d}, dqdp)$ such that

$$U : V_\phi(L_2(\mathbb{R}^d)) \to L_0 \otimes L_2(\mathbb{R}^d, dp),$$

where L_0 is the one-dimensional subspace of $L_2(\mathbb{R}^d, dq)$ generated by function $\ell_0(q) = \overline{\phi(q)}$. Then we construct operators providing the following decompositions of the projection P_ϕ and of the identity operator on $L_2(\mathbb{R}^d)$

$$\begin{aligned} RR^* = I \ \ &: L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d), \\ R^*R = P_\phi &: L_2(\mathbb{R}^{2d}) \to V_\phi(L_2(\mathbb{R}^d)). \end{aligned}$$

Further, using the above mentioned representation and operators we investigate the Gabor-Toeplitz localization operators. The key tool in this study is the result of Theorem 3.2 claiming that for a measurable function $a = a(q)$ on $L_1(\mathbb{R}^d)$ the Gabor-Toeplitz operator T_a acting on $V_\phi(L_2(\mathbb{R}^d))$ is unitarily equivalent to the multiplication operator $\gamma_a I = RT_aR^*$ acting on $L_2(\mathbb{R}^d)$, where the function γ_a has the form (9). This provides an alternative tool and an easy access to study basic properties and the Wick calculus of such Gabor-Toeplitz operators in terms of function γ_a.

Few comments about more general results obtained in connection with the time-frequency localization operators are given at the end of this note.

2 Representation of the space of Gabor transforms

Let ϕ be a fixed normalized window in $L_2(\mathbb{R}^d)$. It is well-known, cf. [16], that

$$(V_\phi f)(q,p) = \langle f, \phi_{q,p}\rangle_d = \mathcal{F}\{f(\cdot)\overline{\phi(\cdot - q)}\}(p),$$

where

$$\mathcal{F}\{g\}(\xi) = \hat{g}(\xi) = \int_{\mathbb{R}^d} g(x) e^{-2\pi i x\cdot\xi}\, dx$$

is the normalized Fourier transform. This enables to reformulate the Gabor transform in the terms of factorization, see Lemma 3.1.2 in [16]. Therefore we consider the unitary operator

$$U_1 = (I \otimes \mathcal{F}^{-1}) : L_2(\mathbb{R}^{2d}, dqdp) \to L_2(\mathbb{R}^d, dq) \otimes L_2(\mathbb{R}^d, dp),$$

and the unitary operator

$$U_2 : L_2(\mathbb{R}^d, dq) \otimes L_2(\mathbb{R}^d, dp) \to L_2(\mathbb{R}^d, dq) \otimes L_2(\mathbb{R}^d, dp)$$

given by $(U_2F)(q,p) = F(p-q,p)$ which provide the following description of the structure of the spaces of Gabor transforms $V_\phi(L_2(\mathbb{R}^d))$ inside $L_2(\mathbb{R}^{2d}, dqdp)$.

Theorem 2.1 *The unitary operator $U = U_2U_1$ gives an isometrical isomorphism of $L_2(\mathbb{R}^{2d}, dqdp)$ onto itself under which*

(i) the space of Gabor transforms $V_\phi(L_2(\mathbb{R}^d))$ is mapped onto $L_0 \otimes L_2(\mathbb{R}^d, dp)$, where L_0 is the one-dimensional subspace of $L_2(\mathbb{R}^d, dq)$ generated by function $\ell_0(q) = \overline{\phi(q)}$;

(ii) the projection P_ϕ is unitarily equivalent to $UP_\phi U^{-1} = P_0 \otimes I$, where P_0 is the one-dimensional projection of $L_2(\mathbb{R}^d, dq)$ onto L_0 given as

$$(P_0F)(q) = \overline{\phi(q)} \int_{\mathbb{R}^d} F(r)\phi(r)\, dr. \tag{4}$$

Proof. Denote by A_1 the image of the space $V_\phi(L_2(\mathbb{R}^d))$ under the mapping U_1. Obviously, A_1 is the space of all functions of the form

$$F(q,p) = f(p)\overline{\phi(p-q)}, \quad f \in L_2(\mathbb{R}^d),$$

and moreover, $\|F(q,p)\|_{A_1} = \|f(p)\|_{L_2(\mathbb{R}^d,dp)}$. The orthogonal projection $B_1 : L_2(\mathbb{R}^{2d}, dqdp) \to A_1$ has obviously the form

$$B_1 = U_1P_\phi U_1^{-1} = (I \otimes \mathcal{F}^{-1})P_\phi(I \otimes \mathcal{F}),$$

and is given by the formula

$$(B_1F)(q,p) = \overline{\phi(p-q)} \int_{\mathbb{R}^d} F(\tau,p)\,\phi(p-\tau)\,d\tau.$$

Then the image $A_2 = U_2(A_1)$ is the set of all functions of the form $F(q,p) = f(p)\overline{\phi(q)}$. Introducing $\ell_0(q) = \overline{\phi(q)}$ we obviously have that $\ell_0(q) \in L_2(\mathbb{R}^d, dq)$ and $\|\ell_0\|_d = 1$. If we denote by L_0 the one-dimensional subspace of $L_2(\mathbb{R}^d, dq)$ generated by $\ell_0(q)$, then the one-dimensional projection P_0 of $L_2(\mathbb{R}^d, dq)$ onto L_0 has the form

$$(P_0F)(q) = \langle F, \ell_0\rangle_d\, \ell_0 = \overline{\phi(q)} \int_{\mathbb{R}^d} F(r)\phi(r)\,dr,$$

which completes the proof. □

Remark 2.2 Observe that the obtained representation of the space of Gabor transforms is, actually, the image of L_2 under the action of the Gabor transforms with a fixed normalized window from L_2. For a more general situation see concluding remarks. In fact, this representation is also used in [17] in the context of Schwartz class functions.

Remark 2.3 Note that the result (i) of Theorem 2.1 was obtained independently in recent paper [1], Proposition 3, where author studies the structure of Gabor and super Gabor spaces, in particular those which are generated by vectors of Hermite functions. Also, there is presented an interesting connection between these spaces and the Fock spaces of poly-analytic functions using methods of time-frequency analysis. In what follows we underline this connection.

Following [21] introduce the isometrical imbedding $R_0 : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d) \otimes L_2(\mathbb{R}^d)$ by the rule

$$(R_0f)(q,p) = f(p)\ell_0(q).$$

Obviously, the image of R_0 coincides with the space A_2. The adjoint operator $R_0^* : L_2(\mathbb{R}^d) \otimes L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is given by

$$(R_0^*F)(r) = \int_{\mathbb{R}^d} F(s,r)\overline{\ell_0(s)}\,ds$$

and it is easy to check that

$$R_0^*R_0 = I \;: L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d),$$
$$R_0R_0^* = B_2 : L_2(\mathbb{R}^d) \otimes L_2(\mathbb{R}^d) \to A_2,$$

where $B_2 = U_2B_1U_2^{-1}$ is the orthogonal projection of $L_2(\mathbb{R}^{2d}, dqdp)$ onto A_2 and it has the form

$$(B_2F)(q,p) = \overline{\phi(q)} \int_{\mathbb{R}^d} F(\tau,p)\,\phi(\tau)\,d\tau.$$

Combining all the above we have the following result.

Theorem 2.4 *The operator $R = R_0^* U$ maps the space $L_2(\mathbb{R}^{2d},\, dqdp)$ onto $L_2(\mathbb{R}^d, dq)$, and the restriction*

$$R|_{V_\phi(L_2(\mathbb{R}^d))} : V_\phi(L_2(\mathbb{R}^d)) \to L_2(\mathbb{R}^d)$$

is an isometrical isomorphism. The adjoint operator

$$R^* = U^* R_0 : L_2(\mathbb{R}^d) \to V_\phi(L_2(\mathbb{R}^d))$$

is an isometrical isomorphism of the space $L_2(\mathbb{R}^d, dq)$ onto the space of Gabor transforms $V_\phi(L_2(\mathbb{R}^d)) \subset L_2(\mathbb{R}^{2d},\, dqdp)$.

Furthermore

$$\begin{aligned} RR^* &= I : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d), \\ R^*R &= P_\phi : L_2(\mathbb{R}^{2d},\, dqdp) \to V_\phi(L_2(\mathbb{R}^d)). \end{aligned}$$

Theorem 2.5 *The isometrical isomorphism*

$$R^* = U^* R_0 : L_2(\mathbb{R}^d) \to V_\phi(L_2(\mathbb{R}))$$

is given by

$$(R^* f)(q,p) = \int_{\mathbb{R}^d} f(\xi)\overline{\phi(\xi - q)} e^{-2\pi i p\xi}\, d\xi. \tag{5}$$

Proof. The direct calculation yields

$$\begin{aligned} (R^* f)(q,p) &= (U^* R_0 f)(q,p) = (U_1^* U_2^* R_0 f)(q,p) \\ &= (I \otimes \mathcal{F})(f(\xi)\ell_0(\xi - q))(p) \\ &= \int_{\mathbb{R}^d} f(\xi)\overline{\phi(\xi - q)} e^{-2\pi i p\xi}\, d\xi. \end{aligned}$$

□

Corollary 2.6 The inverse isomorphism

$$R = R_0^* U : V_\phi(L_2(\mathbb{R}^d)) \to L_2(\mathbb{R}^d)$$

is given by

$$(RF)(\xi) = \int_{\mathbb{R}^{2d}} F(q,p)\phi(\xi - q) e^{2\pi i p\xi}\, dqdp. \tag{6}$$

As we have mentioned in Remark 2.3, we describe the connection between the time-frequency analysis and complex analysis using the above stated operators. For this purpose let us consider the specific function

$$h_0(q) = \pi^{-d/4} e^{-q^2/2}.$$

Clearly, $\|h_0\|_d = 1$. Introduce the isomorphism

$$W_1 = W^* = W^{-1} : L_2(\mathbb{R}^{2d}, dqdp) \to L_2(\mathbb{R}^{2d}, dqdp),$$

where

$$(W_1 f)(q,p) = f\left(\frac{q+p}{\sqrt{2}}, \frac{q-p}{\sqrt{2}}\right),$$

the unitary operators $(W_2 f)(q,p) = (2\pi)^{-d/2}(U_1 f)(q, \frac{p}{2\pi})$ and

$$W_3 : L_2(\mathbb{R}^{2d}, dqdp) \to L_2(\mathbb{C}^d, d\mu)$$

given by

$$(W_3 f)(q,p) = \pi^{d/2} e^{(q^2+p^2)/2} f(q+ip),$$

where $L_2(\mathbb{C}^d, d\mu)$ is the Hilbert space of square-integrable functions on $\mathbb{C}^d$ with respect to the Gaussian measure $d\mu(z)$ over $\mathbb{C}^d$

$$d\mu(z) = \pi^{-d} e^{-z\cdot\overline{z}}\, dv(z)$$

and $dv(z) = dqdp$ is the usual Euclidean volume measure on $\mathbb{C}^d = \mathbb{R}^{2d}$. The closed subspace $F^2(\mathbb{C}^d)$ of $L_2(\mathbb{C}^d, d\mu)$ is usually called the Fock (or, the Segal-Bargmann) space. Then the unitary operator $W = W_3 W_2 W_1$ is an isometrical isomorphism

$$W : L_2(\mathbb{R}^d, dq) \otimes L_2(\mathbb{R}^d, dp) \to L_2(\mathbb{C}^d, d\mu)$$

under which the space $L_0 \otimes L_2(\mathbb{R}^d, dp)$ is mapped onto $F^2(\mathbb{C}^d)$ and the operator

$$WR_0 : L_2(\mathbb{R}^d) \to F^2(\mathbb{C}^d) \subset L_2(\mathbb{C}^d, d\mu)$$

maps isomorphically and isometrically the space $L_2(\mathbb{R}^d, dp)$ onto the Fock space $F^2(\mathbb{C}^d)$. Moreover, $\widetilde{R} = WR_0$ and its adjoint

$$\widetilde{R}^* : L_2(\mathbb{C}^d, d\mu) \to L_2(\mathbb{R}^d)$$

satisfy the relations

$$\begin{aligned} \widetilde{R}^*\widetilde{R} = I \;\; &: L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d), \\ \widetilde{R}\widetilde{R}^* = B &: L_2(\mathbb{C}^d, d\mu) \to F^2(\mathbb{C}^d), \end{aligned}$$

where B is the Bargmann (orthogonal) projection of $L_2(\mathbb{C}^d, d\mu)$ onto $F^2(\mathbb{C}^d)$. It is easy to verify that the isomorphism $\widetilde{R}$ is exactly the Bargmann transform $\beta : L_2(\mathbb{R}^d) \to F^2(\mathbb{C}^d)$ in the form

$$(\beta f)(z) = \pi^{-d/4} \int_{\mathbb{R}^d} e^{-(z^2+\eta^2)/2 + \sqrt{2} z\cdot\eta} f(\eta)\, d\eta.$$

Clearly, $(\beta h_0)(z) = 1$ and in this case the Bargmann transform gives the direct connection between the Berezin-Toeplitz operators T_φ on $F^2(\mathbb{C}^d)$ with the differential operators $\mathcal{L}_\varphi^{h_0}$ (the so called Gabor-Daubechies localization operators) on $L_2(\mathbb{R}^d)$ as follows

$$\beta \mathcal{L}_\varphi^{h_0} \beta^{-1} = T_\varphi,$$

see e.g. [7] and [8]. Also, the function h_0 gives rise the Feichtinger algebra

$$S_0(\mathbb{R}^d) = \{f \in L_2(\mathbb{R}^d); V_{h_0} f(q,p) \in L_1(\mathbb{R}^{2d})\},$$

(the special case of modulation spaces), see [15]. By a simple computation we have

$$V_{h_0} f(q,-p) = e^{i\pi pq} e^{-|z|^2/2} (\beta f)(z), \quad z = \frac{\sqrt{2}}{2}(q + 2\pi i p).$$

This yields the following characterization of Feichtinger algebra in the spirit of Bargmann transform: $f \in S_0(\mathbb{R}^d)$ if and only if

$$\|f\|_{S_0(\mathbb{R}^d)} = \sqrt{2}\pi \int_{\mathbb{C}^d} |(\beta f)(z)| e^{-|z|^2} \, d\mu(z) < +\infty.$$

For more information on Fock and poly-analytic Fock spaces see [20] and for vector-valued version of Gabor analysis in connection with poly-analytic functions see [2].

Remark 2.7 In the above lines we recover the unitarity of the Bargmann transform, see also [16], Theorem 3.4.3. Similarly, when considering the general Hermite functions h_n we are able to provide an operator-theory proof of the result of paper [2], Theorem 1, i.e. we recover the unitarity of the true-polyanalytic Bargmann transform of order n.

3 Gabor-Toeplitz localization operators

In what follows we always consider $\phi \in S_0(\mathbb{R}^d)$. The transform (1) is the appropriate tool for defining localization operators and the Gabor reproducing formula (2) defines a natural context for time-frequency localization. By multiplying the amplitudes $(V_\phi f)(q,p)$ by some weight $a(q,p)$ we get the operator assigning to the original signal its time-frequency modification. This operator is known as the *Gabor-Toeplitz localization operator* and is defined to be the map of $L_2(\mathbb{R}^d)$ to $L_2(\mathbb{R}^d)$ given by

$$T_a f(x) = \int_{\mathbb{R}^{2d}} a(q,p)(V_\phi f)(q,p) \phi_{q,p}(x) \, dq dp,$$

where a is assumed to be non-negative, bounded and integrable function defined on the phase space. The weight function a is called the *symbol* of T_a, or its Gabor multiplier. Clearly, the formula $T_1 f = f$ is the Gabor reproducing formula (2). The name "Toeplitz" comes from the fact that T_a shares many features with Toeplitz operators. The name "localization operator" goes back to Daubechies [8] in 1988, while the case $\phi(t) = e^{-\pi t^2}$ comes back to Berezin in 1971, cf. [5], i.e. T_a boils down to the classical anti-Wick operator and the mapping $a \mapsto T_a$ is interpreted as a quantization rule [6, 22]. Besides, in other branches of mathematics, localization operators are also named short-time Fourier transform multipliers, cf. [14] and can be defined on more general spaces, the so-called modulation spaces, cf. [16].

In a weak sense, the definition of T_a can be expressed by

$$\langle T_a f, g\rangle_d = \langle aV_\phi f, V_\phi g\rangle_{2d} = \langle a, \overline{V_\phi f}\, V_\phi g\rangle_{2d}, \quad f, g \in L_2(\mathbb{R}^d)$$

according to the Gabor reproducing formula (2). Easily, the Gabor-Toeplitz operators T_a satisfy

$$0 \le \|T_a\| \le \|a\|_\infty,$$

they are trace class, and

$$\operatorname{tr} T_a = \int_{\mathbb{R}^{2d}} a(\eta)\, d\eta.$$

It is easy to check that $P_\phi M_a P_\phi$, where M_a is the operator of pointwise multiplication by a on $L_2(\mathbb{R}^{2d}, dqdp)$, has the matrix representation

$$\begin{bmatrix} V_\phi T_a V_\phi^* & 0 \\ 0 & 0 \end{bmatrix}$$

with respect to the decomposition $L_2(\mathbb{R}^{2d}, dqdp) = V_\phi(L_2(\mathbb{R}^d)) \oplus V_\phi(L_2(\mathbb{R}^d))^\perp$. The above representation shows that we may identify the operators T_a and $P_\phi M_a P_\phi$ on $V_\phi(L_2(\mathbb{R}^d))$. Thus for $F \in V_\phi(L_2(\mathbb{R}^d))$ we have

$$(T_a F)(q, p) = \langle aP_\phi F, K_{q,p}\rangle_{2d} = \langle aF, K_{q,p}\rangle_{2d}, \tag{7}$$

and the following easy result implies that the Gabor-Toeplitz operator acting on $V_\phi(L_2(\mathbb{R}^d))$ is the integral operator with kernel $(T_a K_{s,r})(q, p)$.

Theorem 3.1 *Let a be a bounded integrable function on $\mathbb{R}^{2d}$. Then the Gabor-Toeplitz operator acting on $V_\phi(L_2(\mathbb{R}^d))$ has the form*

$$(T_a F)(q, p) = \int_{\mathbb{R}^{2d}} F(s, r)(T_a K_{s,r})(q, p)\, dsdr, \quad F \in V_\phi(L_2(\mathbb{R}^d)). \tag{8}$$

Proof. Since $F \in V_\phi(L_2(\mathbb{R}^d))$, then according to (7) we may write

$$(T_a F)(q, p) = \int_{\mathbb{R}^{2d}} a(v, u)F(v, u)K_{v,u}(q, p)\, dvdu$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{2d}} a(v,u) \left(\int_{\mathbb{R}^{2d}} F(s,r) K_{s,r}(v,u)\, dsdr \right) K_{v,u}(q,p)\, dvdu \\
&= \int_{\mathbb{R}^{2d}} F(s,r) \left(\int_{\mathbb{R}^{2d}} a(v,u) K_{s,r}(v,u) K_{v,u}(q,p)\, dvdu \right) dsdr \\
&= \int_{\mathbb{R}^{2d}} F(s,r) (T_a K_{s,r})(q,p)\, dsdr,
\end{aligned}
$$

which completes the proof. □

It is easy to observe, that if T_a is bounded on $V_\phi(L_2(\mathbb{R}^d))$, then T_a^* is bounded on $V_\phi(L_2(\mathbb{R}^d))$ as well and $T_a^* = T_{\overline{a}}$. In particular, if a is real valued, then T_a is self-adjoint. Indeed, for $F \in V_\phi(L_2(\mathbb{R}^d))$ we have

$$
\begin{aligned}
(T_a^* F)(q,p) &= \langle T_a^* F, K_{q,p} \rangle_{2d} = \langle F, T_a K_{q,p} \rangle_{2d} = \langle F, a K_{q,p} \rangle_{2d} \\
&= (T_a F)(q,p).
\end{aligned}
$$

Now, using the representation of $V_\phi(L_2(\mathbb{R}^d))$ from the previous section we may state the following theorem describing an alternative tool for studying the properties of Gabor-Toeplitz localization operator T_a whose symbol a depends only on position q in phase space.

Theorem 3.2 *Let $a = a(q)$ be a measurable function on $L_1(\mathbb{R}^d)$. Then the Gabor-Toeplitz operator T_a acting on $V_\phi(L_2(\mathbb{R}^d))$ is unitarily equivalent to the multiplication operator $\gamma_a I = R T_a R^*$ acting on $L_2(\mathbb{R}^d)$, where R and R^* are given by (6) and (5), respectively, and the function γ_a has the form*

$$
\gamma_a(r) = \int_{\mathbb{R}^d} a(r-s)\, |\ell_0(s)|^2\, ds, \quad r \in \mathbb{R}^d. \tag{9}
$$

Proof. The operator T_a is obviously unitarily equivalent to the operator

$$
\begin{aligned}
R T_a R^* &= R P_\phi a P_\phi R^* = R(R^* R) a (R^* R) R^* \\
&= (R R^*) R a R^* (R R^*) = R a R^* \\
&= R_0^* U_2 (I \otimes \mathcal{F}^{-1}) a(q) (I \otimes \mathcal{F}) U_2^{-1} R_0 \\
&= R_0^* U_2 a(q) U_2^{-1} R_0 \\
&= R_0^* a(p-q) R_0.
\end{aligned}
$$

Finally, for $f \in L_2(\mathbb{R}^d)$ and $r \in \mathbb{R}^d$ we get

$$
\left(R_0^* a(p-q) R_0 f \right)(r) = f(r) \int_{\mathbb{R}^d} a(r-s) |\ell_0(s)|^2\, ds = f(r) \gamma_a(r).
$$

This completes the proof. □

Remark 3.3 On the strength of Theorem 3.2 it is easy to see that the function γ_a is nothing but the convolution of a symbol a with $|\ell_0|^2$. Also, it is independent on translation operator $(\tau_x f)(y) = f(y-x)$.

Remark 3.4 Naturally we may ask: *How wide is a class of functions which are originated by Gabor-Toeplitz operator T_a with bounded measurable symbols $a(q)$ on $L_1(\mathbb{R}^d)$?* Respectively, *does for each function $\gamma \in L_2(\mathbb{R}^d)$ exist a bounded measurable symbol $a(q)$ on $L_1(\mathbb{R}^d)$ such that the Gabor-Toeplitz operator T_a is unitarily equivalent to the multiplication operator by this function, i.e., $\gamma_a = \gamma$?* Authors do not know about solution of this problem and they were unable to prove it.

Observe that the result of Theorem 3.2 suggests considering not only L_∞-symbols, but also unbounded ones. In this case we obviously have

Corollary 3.5 The Gabor-Toeplitz operator T_a with symbol $a = a(q)$ is bounded on $V_\phi(L_2(\mathbb{R}^d))$ if and only if the corresponding function γ_a is bounded.

Introduce the C^*-algebra $\mathcal{A}$ of bounded measurable symbols depending only on q, and consider the operator algebra $\mathcal{T}(\mathcal{A})$ generated by all the operators of the form

$$T_a : F \in V_\phi(L_2(\mathbb{R}^d)) \mapsto P_\phi M_a F \in V_\phi(L_2(\mathbb{R}^d)),$$

where $a \in \mathcal{A}$. According to Theorem 3.2 the algebra $\mathcal{T}(\mathcal{A})$ is obviously unitarily equivalent to the algebra $R\,\mathcal{T}(\mathcal{A})R^*$ and we have the following

Corollary 3.6 The algebra $\mathcal{T}(\mathcal{A})$ is commutative and is isometrically imbedded to $C_b(\mathbb{R}^d)$ (the bounded norm-continuous operator functions on $\mathbb{R}^d$). The isomorphic imbedding η is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A})$

$$\eta : T_a \mapsto \gamma_a(r) = \int_{\mathbb{R}^d} a(r-s)\, |\ell_0(s)|^2 \, ds, \quad r \in \mathbb{R}^d,$$

where $a = a(q) \in \mathcal{A}$, and $\ell_0(s) = \overline{\phi(s)}$.

We may also consider the algebra $\mathcal{R}(\mathcal{A}, P_\phi)$ generated by all the operators of the form

$$A = aI + bP_\phi,$$

where $a = a(q)$, $b = b(q) \in \mathcal{A}$ acting on $L_2(\mathbb{R}^{2d})$. Then the algebra $\mathcal{R}(\mathcal{A}, P_\phi)$ is naturally isomorphic to the algebra

$$\mathcal{R}_0 = U\mathcal{R}(\mathcal{A}, P_\phi)U^{-1} = U_2(I \otimes \mathcal{F}^{-1})\mathcal{R}(\mathcal{A}, P_\phi)(I \otimes \mathcal{F})U_2^{-1}$$

and under this isomorphism the generators of the algebra $\mathcal{R}(\mathcal{A}, P_\phi)$ are mapped to the following generators of the algebra $\mathcal{R}_0$

$$\begin{aligned} UP_\phi U^{-1} &= P_0 \otimes I, \\ Ua(q)U^{-1} &= a(p-q)I \end{aligned}$$

acting on the space $L_2(\mathbb{R}^{2d})$. Now observe that the case $a = a(q)$ admits the following representation of Gabor-Toeplitz localization operator.

Theorem 3.7 *If $a = a(q)$, then the Gabor-Toeplitz operator T_a has the form*

$$\langle T_a f, g\rangle_d = \int_{\mathbb{R}^d} \gamma_a(r) f(r)\overline{g(r)}\, dr, \quad f, g \in L_2(\mathbb{R}^d).$$

Proof. By Fourier representation of $V_\phi f$ we have

$$\begin{aligned} \langle T_a f, g\rangle_d &= \langle aV_\phi f, V_\phi g\rangle_{2d} = \int_{\mathbb{R}^{2d}} a(q)(V_\phi f)(q,p)\overline{(V_\phi g)(q,p)}\, dqdp \\ &= \int_{\mathbb{R}^{2d}} a(q) \left(\int_{\mathbb{R}^d} f(r)\overline{\phi(r-q)}e^{-2\pi ipr}\, dr\right) \cdot \overline{\left(\int_{\mathbb{R}^d} g(r)\overline{\phi(r-q)}e^{-2\pi ipr}\, dr\right)}\, dqdp \\ &= \int_{\mathbb{R}^{2d}} a(q) \left(\int_{\mathbb{R}^d} \overline{g(r)}\phi(r-q)e^{2\pi ipr}\, dr\right) \cdot \overline{\left(\int_{\mathbb{R}^d} \overline{f(r)}\phi(r-q)e^{2\pi ipr}\, dr\right)} dqdp \\ &= \int_{\mathbb{R}^d} a(q) \left(\int_{\mathbb{R}^d} f(r)\overline{g(r)}\, |\phi(r-q)|^2\, dr\right) dq. \end{aligned}$$

Applying Fubini theorem yields

$$\langle T_a f, g\rangle_d = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} a(q)|\phi(r-q)|^2\, dq\right) f(r)\overline{g(r)}\, dr.$$

Hence the result. □

Reverting the statement of Theorem 3.2 we come to the following spectral-type representation of a Gabor-Toeplitz operator which is easier than the representation (8). Its proof goes directly from Theorem 3.2, Theorem 2.5 and Corollary 2.6.

Theorem 3.8 *If $a = a(q)$ is a measurable function on $L_1(\mathbb{R}^d)$, the Gabor-Toeplitz operator T_a acting on $V_\phi(L_2(\mathbb{R}^d))$ has the following representation*

$$(T_a F)(q,p) = \int_{\mathbb{R}^d} \gamma_a(\xi)\overline{\phi(\xi - q)} f(\xi) e^{-2\pi ip\xi}\, d\xi, \tag{10}$$

where $f(\xi) = (RF)(\xi)$.

Remark 3.9 At the same time it is instructive to give a direct proof of the theorem which does not use the results of the previous section.

Indeed, for a symbol $a = a(q)$ consider the Gabor-Toeplitz operator, see (7),

$$(T_aF)(q,p) = \int_{\mathbb{R}^{2d}} a(s)F(s,r)K_{s,r}(q,p)\,dsdr,$$

and represent the function $F(s,r)$ as follows

$$F(s,r) = \int_{\mathbb{R}^d} f(\xi)\overline{\phi(\xi - s)}e^{-2\pi ir\xi}\,d\xi,$$

where $f \in L_2(\mathbb{R}^d)$. Thus,

$$\begin{aligned}(T_aF)(q,p) &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} a(s)\left(\int_{\mathbb{R}^d} f(\xi)\overline{\phi(\xi - s)}e^{-2\pi ir\xi}\,d\xi\right)K_{s,r}(q,p)\,ds\right)dr\\ &= \int_{\mathbb{R}^d} f(\xi)\left(\int_{\mathbb{R}^d} a(s)\overline{\phi(\xi - s)}\left(\int_{\mathbb{R}^d} K_{s,r}(q,p)e^{-2\pi ir\xi}\,dr\right)ds\right)d\xi\\ &= \int_{\mathbb{R}^d} f(\xi)\overline{\phi(\xi - q)}e^{-2\pi ip\xi}\left(\int_{\mathbb{R}^d} a(s)|\phi(\xi - s)|^2\,ds\right)d\xi.\end{aligned}$$

The following result gives the form of the Wick symbol of Gabor-Toeplitz operator T_a depending on q. For the proof it is sufficient to use Theorem 3.7.

Theorem 3.10 *Let $a = a(q)$ be a measurable function on $L_1(\mathbb{R}^d)$. Then the Wick symbol $\widetilde{a}(q,p)$ of the Gabor-Toeplitz operator T_a depends only on q as well, and has the form*

$$\widetilde{a}(q) = \int_{\mathbb{R}^d} \gamma_a(\xi)|\phi(\xi - q)|^2\,d\xi. \tag{11}$$

The corresponding Wick function is given by the formula

$$\widetilde{a}(q,p,s,r) = \frac{1}{K_{q,p}(s,r)}\int_{\mathbb{R}^d} \gamma_a(\xi)\phi(\xi - q)\overline{\phi(\xi - s)}e^{-2\pi i\xi(r-p)}\,d\xi, \tag{12}$$

where $(q,p),(s,r) \in \mathbb{R}^{2d}$.

Remark 3.11 By virtue of Corollary 3.5 we may ask the following natural question which is open for authors: *Is it true that for a measurable function $a = a(q)$ on $L_1(\mathbb{R}^d)$ the boundedness of Gabor-Toeplitz operator (and hence the boundedness of function γ_a) is equivalent to the boundedness of its Wick symbol?*

Remark 3.12 Let $(q,p),(s,r) \in \mathbb{R}^{2d}$, $a = a(q)$, and $F \in V_\phi(L_2(\mathbb{R}^d))$. Applying the general approach to coherent states, cf. [6], to our particular case it is immediate that

$$\widetilde{a}(s,r,q,p) = \frac{\langle T_a\phi_{s,r},\phi_{q,p}\rangle_d}{\langle\phi_{s,r},\phi_{q,p}\rangle_d} = \frac{\langle aK_{s,r},K_{q,p}\rangle_{2d}}{K_{s,r}(q,p)} = \frac{(T_aK_{s,r})(q,p)}{K_{s,r}(q,p)},$$

and therefore by using representation (8) and explicit form for the Wick function (12) we have

$$\begin{aligned}
(T_aF)(q,p) &= \int_{\mathbb{R}^{2d}} F(s,r)(T_aK_{s,r})(q,p)\,dsdr \\
&= \int_{\mathbb{R}^{2d}} \widetilde{a}(s,r,q,p)F(s,r)K_{s,r}(q,p)\,dsdr \\
&= \int_{\mathbb{R}^{2d}} F(s,r)\left(\int_{\mathbb{R}^d} \gamma_a(\xi)\phi(\xi-s)\overline{\phi(\xi-q)}e^{-2\pi i\xi(p-r)}\,d\xi\right) dsdr \\
&= \int_{\mathbb{R}^d} \gamma_a(\xi)\overline{\phi(\xi-q)}e^{-2\pi ip\xi}\left(\int_{\mathbb{R}^{2d}} F(s,r)\phi(\xi-s)e^{2\pi ir\xi}\,dsdr\right) d\xi \\
&= \int_{\mathbb{R}^d} \gamma_a(\xi)\overline{\phi(\xi-q)}(RF)(\xi)e^{-2\pi ip\xi}\,d\xi,
\end{aligned}$$

i.e. writing the Gabor-Toeplitz localization operator T_a in terms of its Wick symbol yields the spectral-type decomposition (10) of the operator T_a.

Corollary 3.13 Let T_{a_1} and T_{a_2} be two Gabor-Toeplitz operators with symbols $a_1(q)$ and $a_2(q)$, where $a_1(q)$, $a_2(q)$ are bounded measurable functions on $L_1(\mathbb{R}^d)$, and let $\widetilde{a_1}(q)$ and $\widetilde{a_2}(q)$ be their Wick symbols, respectively. Then the Wick symbol $\widetilde{a}(q)$ of the composition $T_{a_1}T_{a_2}$ is given by

$$\widetilde{a}(q) = (\widetilde{a_1} \star \widetilde{a_2})(q) = \int_{\mathbb{R}^d} \gamma_{a_1}(\xi)\gamma_{a_2}(\xi)|\phi(\xi-q)|^2\,d\xi. \tag{13}$$

Proof. The result follows immediately from Theorem 3.2 and Theorem 3.10. □

Observe that the set $\mathcal{W}(\mathcal{A})$ of all Wick symbols for Gabor-Toeplitz operators T_a with (anti-Wick) symbols $a(q) \in \mathcal{A}$ coincides with the set of all functions of the form (11)

$$\widetilde{a}(q) = \int_{\mathbb{R}^d} \gamma_a(\xi)|\phi(\xi-q)|^2\,d\xi,$$

where $\gamma_a \in L_2(\mathbb{R}^d)$. Obviously, the linear space $\mathcal{W}(\mathcal{A})$ is a commutative algebra with respect to multiplication given by (13), and the operator algebra $\mathcal{T}(\mathcal{A})$ (besides the isomorphism of Corollary 3.6) is isomorphic to $\mathcal{W}(\mathcal{A})$ via the mapping

$$\omega : T_a \in \mathcal{T}(\mathcal{A}) \mapsto \widetilde{a}(q) = \int_{\mathbb{R}^d} \gamma_a(\xi)|\phi(\xi-q)|^2\,d\xi \in \mathcal{W}(\mathcal{A}).$$

4 Concluding remarks

In this paper we provide an alternative description to study some properties of Gabor-Toeplitz localization operators T_a. This tool is an application of Vasilevski method used in this particular case of Gabor spaces and operators acting on them with restriction to the operators whose symbols depend only on space variable which is consequence of decomposition method we used. This enables us to obtain representation formulas for T_a, Wick calculus and proofs particularly simple.

At the same time we have to mention that the Vasilevski method might be applied in more general setting to study the general case of localization operators, cf. [9],

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega)(V_{\varphi_1} f)(x,\omega) M_\omega T_x \varphi_2(t)\, dx d\omega,$$

where M_ω and T_x are the operators of modulation and translation, respectively, φ_1, φ_2 are the analysis and synthesis windows, respectively, and a is the symbol of the operator. For a nice survey on time-frequency methods in the study of localization operators, cf. [13]. Clearly, for $\varphi_1 = \varphi_2$ the operator $A_a^{\varphi_1,\varphi_2}$ reduces to T_a. However, general case includes the following issues:

- apart from the L_2 space, more general families of function spaces and (ultra)distributions may be used;
- instead of a single function ϕ used for both analysis and synthesis of T_a the different functions φ_1, φ_2 may be used for analysis and synthesis of $A_a^{\varphi_1,\varphi_2}$;
- the symbol a is defined in phase space, i.e. depend both on frequency and space variable.

From this point of view it would be interesting and useful to investigate this general case in the context of Vasilevski method (at least in L_2-space case) and construct the analogues of the classical Bargmann transform, cf. [4] and its inverse (i.e. the operators R and R^* restricted onto the analytic space), which will be used then as the unitary multiples in the representation

$$R A_a^{\varphi_1,\varphi_2} R^* = \gamma_a^{\varphi_1,\varphi_2} I,$$

see our particular Theorem 3.2 for $\varphi_1 = \varphi_2 = \phi \in L_2(\mathbb{R}^d)$ and $a = a(q)$ being a measurable function on $L_1(\mathbb{R}^d)$. As far as we know such a method has never been used in the context of localization operators and surely brings new results. Naturally, the following question arise: *Will be these results significantly different when comparing with the well-known results obtained by time-frequency (phase-space) methods in connection with symbolic calculus, spectra, compactness, fredholmness, cf. [10, 11, 12], etc.*? Or, *do they provide an alternative description of already known results*?

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