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Parity vertex colouring of plane graphs

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Parity vertex colouring of plane graphs

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Abstract: A proper vertex colouring of a 2-connected plane graph G is a *parity vertex colouring* if for each face f and each colour c, no vertex or an odd number of vertices incident with f is coloured with c. The minimum number of colours used in such colouring of G is denoted by $\chi_p(G)$.

In this paper we prove that $\chi_p(G) \leq 118$ for every 2-connected plane graph G.

Keywords: discharging method, plane graph, vertex colouring, parity vertex colouring.

2010 Mathematics Subject Classification: 05C10, 05C15

1 Introduction

The Four Colour Problem [1, 14] has served as a starting point and motivation for many interesting problems. It is well known that to solve this problem it is sufficient to solve it for plane triangulations. The solution of the Four Colour Problem is known under the name the Four Colour Theorem.

It follows from the Four Colour Theorem that the vertices of any plane triangulation T can be coloured with at most four colours in such a way that vertices of the same face f receive different colours. This simple observation led Ore and Plummer [12] to introduce a *cyclic colouring*. A cyclic colouring of a plane graph G is a vertex colouring in which any two vertices incident with the same face receive different colours. The minimum number of colours in any cyclic colouring of a plane graph G is called the *cyclic chromatic number* of G and is denoted by $\chi_c(G)$. If a graph G is 2-connected, then any face f of G is incident with $deg(f)$ vertices. Hence, $\chi_c(G)$ is naturally lower bounded by $\Delta^*(G)$, the maximum face size of G. Sanders and Zhao [15] proved that $\chi_c(G) \leq \left\lceil \frac{5\Delta^*(G)}{3} \right\rceil$ for any 2-connected plane graph G. On the other hand for any $d \geq 4$ there is a 2-connected plane graph G_d satisfying $\Delta^*(G_d) = d$ and $\chi_c(G_d) = \left\lfloor \frac{3\Delta^*(G)}{2} \right\rfloor$. It is conjectured that $\chi_c(G) \leq \left\lfloor \frac{3\Delta^*(G)}{2} \right\rfloor$ for any 2-connected plane graph G.

Plummer and Toft [13] proposed the conjecture that if G is a 3-connected plane graph, then $\chi_c(G) \leq \Delta^*(G) + 2$. This conjecture is true for plane triangulations from the Four Colour Theorem, and for 3-connected plane graphs with $\Delta^*(G) \geq 18$, see Horňák and Jendroľ [10] for $\Delta^*(G) \geq 24$ and Horňák and Zlámalová [11] for the remaining cases. For $\Delta^*(G) = 4$ this conjecture is known to be true by Borodin [3]. Enomoto et al. [9] obtained for $\Delta^*(G) \geq 60$ even stronger result, namely $\chi_c(G) \leq \Delta^*(G) + 1$. The best known general result is the inequality $\chi_c(G) \leq \Delta^*(G) + 5$ of Enomoto and Horňák [8].

Another motivation for this paper comes from a parity colouring concept introduced in recent papers of Bunde, Milans, West, and Wu. In [4, 5] they introduced a *strong parity edge colouring* of graphs. It is an edge colouring of a graph G such that each open walk in G uses at least one colour an odd number of times. Let $p(G)$ be the minimum number of colours in a strong parity edge colouring of G. The exact value of $p(K_n)$ for complete graphs is determined in [4]. They mentioned that computing $p(G)$ is NP-hard even when G is a tree.

The authors of [7] focused on facial walks of plane graphs. They introduced a *facial parity edge colouring*, which is an edge colouring such that no two consecutive edges of a facial walk of any face f receive the same colour and for each face f and each colour c, no edge or an odd number of edges incident with f is coloured with c. The problem is to determine the minimum number of colours used in such a colouring. This number is called the *facial parity chromatic index*. Note that the facial parity chromatic index depends on the embedding of the graph. The authors of [7] proved that each 2-edge-connected plane multigraph has facial parity chromatic index at most 20. Moreover, the upper bound is at most 12 for any 3-edge-connected plane multigraph, and if it is a 4-edge-connected plane multigraph then the upper bound is at most 9.

In this paper we investigate a *parity vertex colouring* of 2-connected plane graphs which can be considered as a relaxation of the cyclic colouring. A proper vertex colouring of a 2-connected plane graph is a parity vertex colouring if for each face f and each colour c, no vertex or an odd number of vertices incident with f is coloured with c. The minimum number of colours in any parity vertex colouring of a 2-connected plane graph G is called the *parity chromatic number* of G and is denoted by $\chi_p(G)$.

If $\chi_0(G)$ denotes the (usual) chromatic number of a 2-connected plane graph G then immediately from the definitions we have

$$\chi_0(G) \leq \chi_p(G) \leq \chi_c(G).$$

Notice that for plane triangulations proper (usual) colourings, cyclic colourings, and parity vertex colourings coincide. Moreover, for 2-connected plane graphs with maximum face size at most 5 cyclic colourings and parity vertex colourings coincide too.

The parameter $\chi_p(G)$ has been introduced in Czap and Jendroľ [6], where the authors have conjectured an existence of a constant K such that $\chi_p(G) \leq K$ for every 2-connected plane graph G.

The main result of this paper is the proof of this conjecture, namely we show that $K \leq 118$.

2 Notation

Throughout this paper we use the standard terminology according to Bondy and Murty [2]. However, we recall some frequently used terms.

A *planar graph* is a graph which can be embedded in the plane. A *plane graph* is a fixed embedding of a planar graph.

In this paper we consider 2-connected plane graphs, they can have parallel edges but loops are not allowed.

Let $G = (V, E, F)$ be a connected plane graph with the vertex set V, the edge set E, and the face set F. The *degree* of a vertex v, denoted by $deg(v)$, is the number of edges incident with v. A k-vertex is a vertex of degree k. The *size* of a face f is defined to be the length of its *facial walk*, i.e. the shortest closed walk containing all edges from the boundary of f. The size of f is denoted by $deg(f)$. A k-face is a face of size k. Two faces are *adjacent* if they share an edge.

Given a graph G and one of its edges, say $e = uv$, the *contraction* of e, denoted by $G\%e$, consists of replacing u and v by a new vertex adjacent to all the former neighbours of u and v, and removing the loop corresponding to the edge e. (We keep multiple edges if they occur). Analogously we define the contraction of the set of edges $H = \{e_1, \ldots, e_k\}$ and we denote it by $G\%\{e_1, \ldots e_k\}$ or $G\%H$.

Let H be a subgraph of G. Then the graph $G \setminus H$ is defined as a graph obtained from G by deleting the edges in $E(H)$. (We delete isolated vertices if they occur).

A *vertex k-colouring* of a graph G is a mapping $\varphi : V(G) \rightarrow \{1, \ldots, k\}$. A vertex colouring is called *proper* if no two adjacent vertices have the same colour.

A proper vertex colouring φ is a *parity vertex (PV) colouring* of a 2-connected plane graph G if for each face f and each colour c, no vertex or an odd number of vertices incident with f is coloured with c.

3 Main result

The main result of this paper is the following theorem.

Theorem 1 *Let G be a 2-connected plane graph. Then*

$$\chi_p(G) \leq 118.$$

The remainder of this paper consists of a proof of this theorem. The proof uses a discharging method.

Suppose there is a counterexample to Theorem 1. Let G be a counterexample with minimum number of vertices, then minimum number of edges.

First, we prove several structural properties of G.

We say that a face f is *small* if $2 \leq deg(f) \leq 59$ otherwise it is called *big*.

3.1 Reducible configurations

We find such (forbidden) subgraphs H of G that the parity colouring of $G \setminus H$ or $G\%H$ using at most 118 colours can be extended to a required colouring of G using at most 118 colours, which is a contradiction to G being a counterexample. In the sequel, whenever we speak about a PV colouring, we always mean a PV colouring using at most 118 colours.

3.1.1 Small faces

Claim 1 *G does not contain any 2-face.*

Proof Let f be a 2-face incident two times with the edge uv. Let $G' = G \setminus \{uv\}$ be a graph obtained from G by deleting one uv edge. G' has less edges than G, hence, it has a required colouring. Clearly, the same colouring is a PV colouring of G. □

Claim 2 *G does not contain two triangles with a common edge.*

Proof Let f_1 and f_2 be two triangles with a common edge e, see Figure 1. The graph $G' = G \setminus \{e\}$ has less edges than G, therefore, it has a PV colouring. Let f' be the face of G' corresponding to the faces f_1 and f_2 in G.

Clearly, on the face f' there are four different colours, hence, this colouring induces a required colouring of G. □

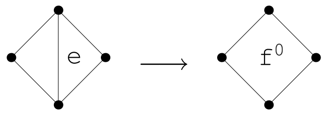

Figure 1: Two adjacent triangles form a reducible configuration: one can remove the edge they share.

Claim 3 *G does not contain a 3-face and a 4-face with a common edge.*

Proof We can use a similar reduction as above (remove the common edge or the common edges). □

3.1.2 Chains of 2-vertices

Claim 4 *There is no chain consisting of at least four consecutive 2-vertices in G.*

Proof Let $v_0e_0v_1e_1 \dots v_pe_pv_{p+1}$ be a chain consisting of p vertices of degree 2, $(v_1, \dots, v_p)$, where $p \geq 4$. The graph $G' = G\%\{e_0, e_1, e_2, e_3\}$ has a PV colouring φ'. Let v_0' and v_5' be the vertices in G' corresponding to the vertices v_0 and v_5 in G. Let $\varphi'(v_0') = c_0$ and $\varphi'(v_5') = c_1$. The PV colouring φ' of G' can be extended to a PV colouring φ of G by setting $\varphi(v_2) = \varphi(v_4) = c_0$ and $\varphi(v_1) = \varphi(v_3) = c_1$, see Figure 2. □

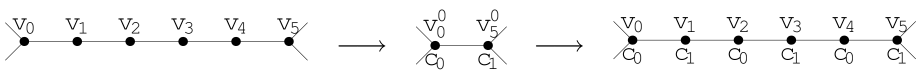

Figure 2: A chain of (at least) four 2-vertices is a reducible configuration.

Claim 5 *There are no adjacent faces f_1 and f_2 with two times at least two consecutive common 2-vertices.*

Proof We can use a similar reduction as above. □

3.1.3 Vertices and their neighbourhoods

For a vertex v and a face f let $F(v)$ be the set of faces incident with v and $V(f)$ be the set of vertices incident with f. Define for $v \in V(G)$

$$A(v) := \bigcup_{f \in F(v)} V(f) \setminus \{v\}.$$

Thus $A(v)$ is the set of all vertices w of G such that there is a facial walk in G containing v and w.

Claim 6 $|A(v)| \geq 118$ *for all* $v \in V$.

Proof Assume there is a vertex v with $|A(v)| \leq 117$. Let $v_1, \ldots, v_s$ be the neighbours of v in clockwise order. Let G' be a graph obtained from G by removing the vertex v and adding the edges $v_i v_{i+1}$ for $i = 1, \ldots, s-1$, and $v_s v_1$. Since the graph G' has less vertices than G it has a PV colouring with at most 118 colours. We can extend this colouring to a PV colouring of G. We colour the vertex v by a colour not used for the vertices of $A(v)$. □

Let v be a vertex and f_1 be a face incident with v. Define

$$A_{f_1}(v) := \bigcup_{f \in F(v) \setminus \{f_1\}} V(f) \setminus \{v\}.$$

Thus, we obtain $A_{f_1}(v)$ removing from $A(v)$ all vertices which are incident with f_1 only.

Claim 7 *Let* f *be a face of* G *and let* u, v *be two vertices incident with* f *such that* $u \notin A_f(v)$. *Then*

$$|A_f(u) \cup A_f(v)| \geq 117.$$

Proof Assume there is a face f with vertices u, v incident with it such that $u \notin A_f(v)$ and $|A_f(u) \cup A_f(v)| \leq 116$. Note that it follows $v \notin A_f(u)$ and the vertices u, v are not adjacent in G.

Let G' be a graph obtained from G in the following way. Remove u from G and join its neighbours in a clockwise order and remove v from G and join its neighbours in a clockwise order. The graph G' has less vertices than G, hence, it has a PV colouring.

Now we extend the colouring of G' to a colouring of G. Clearly, there are at least two colours, say c_1, c_2, not used for the vertices of $A_f(u) \cup A_f(v)$. If one of these colours, say c_1, is used for $V(f)$ then colour both u and v by c_1. Otherwise colour u by c_1 and v by c_2. □

3.2 Bad vertices, faces, and structures

A vertex v incident with a big face f is called a *bad vertex with respect to* f if

1. $deg(v) = 2$ and v is incident with a small face f_1 or,

2. $deg(v) = 3$ and v is incident with a triangle f_1 and a face f_2 with $deg(f_2) \leq 11$ or,

3. $deg(v) = 3$ and v is incident with a 4-face f_1 and a face f_2 with $deg(f_2) \in \{4, 5\}$.

Let $B(f)$ be the set of bad vertices with respect to f. A face f_1 incident with a vertex from $B(f)$ and adjacent to f is called a *bad face with respect to f*. The set of bad faces with respect to f which are of degree $3, 4$, or 5 form a *bad structure with respect to f*.

Observation 1 *If f is a big face then there is a bad face f_1 with respect to f containing all bad vertices with respect to f.*

Otherwise there are vertices u and v fulfilling the assumption of Claim 7 with $|A_f(u) \cup A_f(v)| \leq 116$ contradicting the statement of that Claim.

By this observation we know that the number of bad vertices and bad faces with respect to a big face f is very limited. In the following we analyze these bad structures with respect to f. The motivation of this analysis is to show that by the discharging method a big face f has to give only a constant amount of its charge to the bad vertices and faces with respect to it. In fact we are going to prove that by Rule 6 defined in Section 4.1 f gives at most a charge 6 to these vertices and faces.

In the following let f be a big face. If we write bad vertices or bad faces we mean always bad vertices or bad faces with respect to f.

We distinguish cases considering a face f_1 containing all bad vertices. Note that it is possible that more than one face contains all bad vertices with respect to f. Thus some of the considered structures may occur more than once.

3.2.1 There is a 3-face f_1 containing all bad vertices

Note that the existence of such a bad triangle excludes the existence of a bad 4-face by Claim 3. We consider three types of bad triangles.

(a) f_1 shares a bad 2-vertex with f

Observation 2 *If $f_1 = v_1v_2v_3$ is a bad triangle with respect to f and $deg(v_1) = 2$ then there are at most three bad vertices, namely v_1, v_2, and v_3. The bad structure consists of f_1 only or of f_1 and a 5-face.*

Obviously v_1 belongs only to the faces f and f_1. Thus f_1 contains all bad vertices with respect to f. Note that $deg(v_2) \geq 3, deg(v_3) \geq 3$ since otherwise G is not 2-connected. A further bad face with respect to f must share the edge v_2v_3 with f_1 implying the statement of the above observation.

(b) f_1 shares an edge with f where both end vertices are bad 3-vertices with respect to f

Observation 3 *If $f_1 = v_1v_2v_3$ is a bad triangle with respect to f, v_1v_2 is an edge incident with both f and f_1, $deg(v_1) = deg(v_2) = 3$ and the vertices v_1 and v_2 are bad with respect to f then only v_1, v_2 are bad with respect to*

f. Moreover, in this case, the bad structure with respect to f consists of the triangle f_1 and at most two 5-faces.

Note that $deg(v_3) \geq 4$ because of Claim 6. Moreover, there cannot be a face f^* adjacent to f and f_1 containing v_1 and v_2 since otherwise v_3 is a cut vertex. Thus f_1 contains all bad vertices with respect to f.

(c) f_1 shares an edge with f where exactly one end vertex is a bad 3-vertex with respect to f.

If f_1 is a triangle of type (c) then we assume that f_1 is not incident with any vertex of degree 2 (if f_1 is incident with a 2-vertex then f_1 is a triangle of type (a)).

Observation 4 *If $f_1 = v_1v_2v_3$ is a bad triangle with respect to f and the 3-vertex v_1 is the only bad vertex with respect to f then the bad structure with respect to f consists of the triangle f_1 and at most one 5-face.*

It is worth to mention that if there is a triangle of type (a) or (b) then it is the only face which contains all bad vertices and cannot occur as part of a bad structure in the following cases.

3.2.2 There is a 4-face f_1 containing all bad vertices

Note first that if there is a bad 4-face containing all bad vertices then all other bad faces have to be 4- or 5-faces because of the definition of bad vertices and faces.

Observation 5 *Let $f_1 = v_1v_2v_3v_4$ be a bad 4-face containing all bad vertices with respect to f. Then one of the following holds.*

(i) No 2-vertex is bad with respect to f and the bad structure with respect to f consists of at most three faces.

(ii) The face f_1 is incident with one bad 2-vertex with respect to f. Assume that v_1, v_2, v_3 are consecutive vertices on f and f_1, $deg(v_2) = 2$. If $deg(v_4) = 3$ then the bad structure with respect to f consists of at most two faces by Claim 6. If $deg(v_4) \geq 4$ then the bad structure with respect to f consists of at most three faces, moreover, each of them is incident with a vertex of degree at least 4, see Figure 3.

(iii) The face f_1 is incident with two bad 2-vertices with respect to f and the bad structure with respect to f consists of at most two faces, see Figure 4.

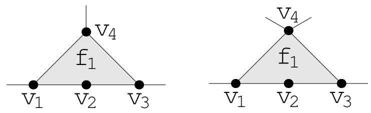

Figure 3: A bad 4-face f_1 with respect to f shares a 2-vertex with f.

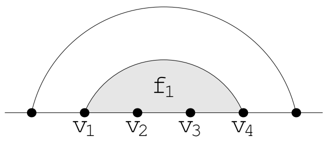

Figure 4: A 4-face f_1 shares two 2-vertices with a big face f.

3.2.3 There is a 5-face f_1 containing all bad vertices

Note first that if there is a bad 5-face containing all bad vertices with respect to f then all other bad faces have to be 3-faces of type (c) or 4-faces because of the definition of bad vertices and faces and the final remark in section 3.2.1.

Observation 6 *Let f be a big face and let $f_1 = v_1v_2v_3v_4v_5$ be a bad 5-face containing all bad vertices with respect to f. If there are bad faces with respect to f then they are 4-faces or 3-faces of type (c) and one of the following holds.*

(i) If f_1 contains at most one bad 2-vertex with respect to f then the bad structure consists of at most three faces, see Figure 5.

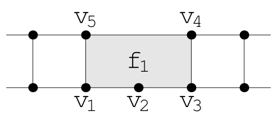

Figure 5: A bad structure with respect to f which consists of three faces.

(ii) Assume that f_1 contains two bad 2-vertices v_2, v_3 with respect to f, see Figure 6. If $deg(v_5) = 3$ then the bad structure with respect to f consists of f_1 and at most one more face by Claim 6. If $deg(u_5) \geq 4$ then the bad structure with respect to f consists of at most three faces, moreover, each of them is incident with a vertex of degree at least 4.

(iii) If f_1 contains three bad 2-vertices with respect to f then the bad structure with respect to f consists of f_1 and at most one more face, see Figure 7.

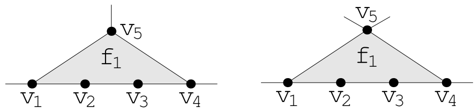

Figure 6: A bad 5-face f_1 with respect to f shares two 2-vertices with f.

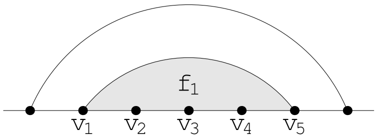

Figure 7: A bad structure with respect to f consist of two faces.

3.2.4 There is an ℓ-face f_1 with $6 \leq \ell \leq 11$ containing all bad vertices

Observation 7 *If $deg(f_1) = 11$ then it is incident with at most five bad 2-vertices with respect to f, if $deg(f_1) \leq 10$ then it is incident with at most four such 2-vertices.*

Let $v_1, \ldots, v_k$ be consecutive vertices on f_1 incident with both f_1 and f. Clearly $deg(v_1) \geq 3$ and $deg(v_k) \geq 3$. It follows from Claims 4 and 5 that if there are two or three consecutive 2-vertices then the other 2-vertices have to be separated from each other by vertices of degree at least three. Moreover, if between two 2-vertices there is a vertex of degree at least three then there are at least two such vertices since otherwise G is not 2-connected or the face f_1 does not contain all bad 2-vertices with respect to f.

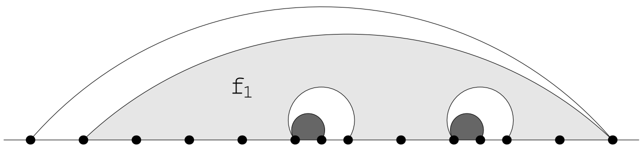

Figure 8: An 11-face can be incident with five bad 2-vertices.

Now we consider the case that there are more bad faces with respect to f adjacent to f_1. Because of the definition of bad faces we do not have bad 4- or 5-faces. We cannot have triangles of type (a) or (b) because of the final remark in Section 3.2.1. Thus we have to look for triangles of type (c) only. We ask for the maximum number of such triangles with respect to f.

Observation 8 *Let f_1 be a bad ℓ-face with $6 \leq \ell \leq 11$ containing all bad vertices with respect to f. If $t \geq 0$ is the number of 2-vertices shared by f and f_1 then there are at most $\frac{2}{3} \cdot (deg(f_1) - t)$ bad triangles of type (c) with respect to f.*

Let $f_2 = v_1v_2v_3$ be a bad triangle of type (c) with respect to f where v_1v_2 is the edge incident with f and v_1 is the bad 3-vertex with respect to f. Observe that v_1v_3 is incident with f_1 because of $deg(v_1) = 3$. If v_3 belongs to f then $deg(v_3) \geq 4$ since otherwise $deg(v_2) = 2$. Thus v_3 belongs to f_1 but in any case it is not a bad vertex with respect to f.

By these arguments we obtain that any s consecutive vertices of f_1 with degree at least 3 are incident with at most $\frac{2}{3} \cdot s$ bad triangles with respect to f. Hence, the number of these bad triangles with respect to f is at most $\frac{2}{3} \cdot (deg(f_1) - t)$.

4 Discharging

4.1 Discharging rules

Because G is the minimum counterexample it contains no reducible configuration. Let the initial charge of each vertex v be $\psi(v) = 2\deg(v) - 6$ and the initial charge of each face f be $\psi(f) = \deg(f) - 6$. We can easily derive from Euler's formula that

$$\sum_{f \in F} (\deg(f) - 6) + \sum_{v \in V} (2\deg(v) - 6) = -12.$$

It is obvious that all the negative charge is in the faces of size 3, 4, and 5 and in the vertices of degree 2.

Rule 1: Let v be a vertex of degree at least 4.

- It sends charge 1 to every incident triangle and charge $\frac{1}{2}$ to every incident 4- or 5-face.

Rule 2: Let f be a face of size at least 12.

- If f is adjacent to f_1 with $deg(f_1) = 3$ and $e = v_1v_2$ is a common edge of f and f_1 then:
  - if $deg(v_1) = deg(v_2) = 3$ then f sends charge 1 to f_1.
  - if $deg(v_1) = 3$ and $deg(v_2) \geq 4$ then f sends charge $\frac{1}{2}$ to f_1.

Rule 3: Let f be a big face.

- If a vertex v is incident with f and $deg(v) = 4$ and v is incident with two triangles and a 5-face then f sends charge $\frac{1}{2}$ to v.
- If a vertex v is incident with f and $deg(v) = 2$ then f sends charge 1 to v.

Rule 4: These rules apply only if the corresponding face does not belong to a bad structure with respect to f. Let f be a big face adjacent to f_1 with $deg(f_1) \in \{4, 5\}$.

- If $deg(f_1) = 4$ and $e = v_1v_2$ is an edge incident with both f and f_1 then:
  - if $deg(v_1) = deg(v_2) = 3$ then f sends charge 1 to f_1 through e.
  - if $deg(v_1) = 3$ and $deg(v_2) \geq 4$ then f sends charge $\frac{1}{2}$ to f_1 through e.
- If $deg(f_1) = 5$ and $e = v_1v_2$ is an edge incident with f and f_1 then:
  - if $deg(v_1) = 3$ and $deg(v_2) \geq 3$ then f sends charge $\frac{1}{3}$ to f_1 through e.

Rule 5: Let f be an 11-face.

- If f is incident with exactly five 2-vertices, then it sends charge 1 to each of them.

Rule 6: Extra charges. Let f be a big face.

- If v is a vertex of degree 2 incident with f and with a small face f_1 then f sends extra charge 1 to v except of the case when $deg(f_1) = 11$ and f_1 is incident with exactly five 2-vertices.
- If f_1 is a triangle of type (a) then f sends extra charge 3 to f_1.
- If f_1 is a triangle of type (b) then f sends extra charge 1 to f_1.
- If f_1 is a triangle of type (c) then f sends extra charge $\frac{1}{2}$ to f_1.
- If f_1 is a bad 4-face with respect to f then:
  - if f_1 is incident with a vertex of degree at least 4 then f sends extra charge $\frac{3}{2}$ to f_1.
  - if f_1 is incident only with vertices of degree at most 3 then f sends extra charge 2 to f_1.
- If f_1 is a bad 5-face with respect to f then:
  - if f_1 is incident with a vertex of degree at least 4 then f sends extra charge $\frac{1}{2}$ to f_1.
  - if f_1 is incident only with vertices of degree at most 3 then f sends extra charge 1 to f_1.

4.2 Analysis of the graph

The aim of the analysis in this section is to show that after redistributing original charges according to the above mentioned Rules 1–6 the new charge of each vertex and each face is nonnegative. This leads to a contradiction with the fact that the sum of charges of all vertices and faces equals to -12.

4.2.1 Vertices

1. Let v be a vertex of degree 2.

   Let f_1 and f_2 be the faces incident with v. It follows from Claim 6 that v is incident with at least one big face. If both faces are big then v receives charge 1 from each of them (Rule 3). If v is incident with a small face then it receives charge $1+1$ from the big face (Rules 3 and 6) or charge 1 from the big face and charge 1 from the incident 11-face (Rules 3 and 5). Thus the new charge is $-2+1+1=0$.

2. Let v be a vertex of degree 3. Then it does not send and does not get any charge, hence it charge is 0.

3. Let v be a vertex of degree 4. Its initial charge is $\psi(v)=2$.

   There is only one situation where the charge $\psi(v)=2$ is not enough for the incident faces $f_1,\dots,f_4$: f_1 and f_3 are triangles and f_2 is a 5-face. Then, by Claim 6 f_4 is a big face. By Rule 3 the vertex v receives an additional charge $\frac{1}{2}$ from f and its new charge is nonnegative.

4. Let v be a vertex of degree at least 5. It is incident with at most $\lfloor\frac{deg(v)}{2}\rfloor$ triangles and at most $\lceil\frac{deg(v)}{2}\rceil$ 5-faces (see Claims 2 and 3). Therefore the charge of v is at least $2deg(v)-6-(\lfloor\frac{deg(v)}{2}\rfloor\cdot 1+\lceil\frac{deg(v)}{2}\rceil\cdot\frac{1}{2})\geq 2deg(v)-6-(deg(v)-1)=deg(v)-5\geq 0$ (Rule 1).

4.2.2 Small faces

1. Let f be a triangle incident with vertices v_1,v_2,v_3.

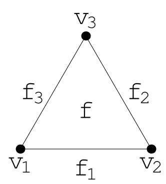

Figure 9: A triangle $v_1v_2v_3$ is incident with faces f_1,f_2,f_3.

   (a) If one of the vertices incident with f is a 2-vertex, then this 2-vertex is incident with a big face and the triangle is of type (a). Hence, it receives charge 3 from the big face (Rule 6).

   (b) If $deg(v_i)=3$ for $i=1,2,3$, then because of Claim 6 at least two of the faces f_1,f_2,f_3 are big. If all of them are faces with degree at least 12, then the charge of the triangle is $-3+1+1+1=0$ (Rule 2).

Otherwise assume that f_1 is a face with $deg(f_1) \leq 11$. Then f_2 and f_3 are big faces and f is a bad triangles of type (c) with respect to both big faces f_2 and f_3. In this case f receives the charge $1 + \frac{1}{2} + 1 + \frac{1}{2}$ (Rules 2 and 6).

(c) If $deg(v_1) = deg(v_2) = 3$ and $deg(v_3) \geq 4$, then there are two possibilities. The triangle f is adjacent to one big face or at least two big faces. In the first case it receives charge 1 from v_3 (Rule 1), charge 1 from the adjacent big face (Rule 2), and extra charge 1 from the big face (Rule 6) since it is a bad triangle of type (b). In the second case it receives charge 1 from v_3 (Rule 1) and charge at least 2 from the big faces.

(d) If $deg(v_1) = 3, deg(v_2) \geq 4, deg(v_3) \geq 4$ then f receives charge at least 1 from the adjacent big face (or big faces) (Rules 2 and 6) and charge 2 from the vertices v_2, v_3 (Rule 1).

(e) If all vertices of f have degree at least 4 then f gets charge 3 from its vertices.

Thus the new charge of the triangle f is nonnegative in all above cases.

2. Let f be a face of size 4. The original charge of f is -2.

   (a) If f is bad with respect to some big face then it receives charge at least 2 from the adjacent big face and from the incident vertices (Rules 1 and 6).

   In the following we assume that f is not bad with respect to any big face.

   (b) If all four vertices of f have degree 3 then at least two of the adjacent faces are big faces because of Claim 6. These faces send charge at least 2 to f (Rule 4).

   (c) If three vertices of f have degree 3 then again two of the adjacent faces are big faces and at least one of them sends 1, the other one $\frac{1}{2}$ and the vertex of degree at least 4 sends also charge $\frac{1}{2}$ to f.

   (d) Analogous arguments work for two vertices or one vertex of degree 3.

   (e) If all vertices of f have degree 4 then they send charge $4 \cdot \frac{1}{2}$ to f.

3. Let f be a face of degree 5.

   (a) If f is bad with respect to some big face then it receives charge at least 1 from the adjacent big face and from the incident vertices (Rules 1 and 6).

   In the following we assume that f is not bad with respect to any big face.

(b) If a face f has at least 4 vertices of degree three, then it is adjacent to at least 2 big faces, and they send to f charge at least $\frac{1}{3} \cdot 2$ (Rule 4) and f receives charge $\frac{1}{2}$ from the incident vertex (Rule 1) or charge $\frac{1}{3}$ from the third adjacent big face (Rule 4).

(c) If f has at most 3 vertices of degree 3 then it receives charge at least 1 from the incident vertices (Rule 1).

4. Let f be a face of degree $6 \leq deg(f) \leq 10$. Then it does not send and does not get any charge, hence its charge is nonnegative.

5. Let f be a face of degree 11. Then its initial charge is 5 and it sends charge at most 5 to the incident 2-vertices (Rule 5).

6. Let f be a face of degree $12 \leq deg(f) \leq 59$.

   Because of Claim 2 f sends at most $\lfloor \frac{deg(f)}{2} \rfloor$ to adjacent triangles (Rule 2). For the new charge we obtain $\psi(f) - \lfloor \frac{deg(f)}{2} \rfloor = deg(f) - 6 - \lfloor \frac{deg(f)}{2} \rfloor = \lceil \frac{deg(f)}{2} \rceil - 6 \geq 0$.

4.2.3 Big faces

Let f be a big k-face, i.e. a face of degree at least 60. Then its initial charge is $k-6$. The face f sends a part of its charge to some adjacent faces and incident vertices in two ways.

The first way is according Rules 1–5. In this way f sends charge at most $\frac{4}{5} \cdot k$ that can be easily seen by careful checking of the possible situations when traversing along the facial cycle of f. Note that the average $\frac{4}{5}$ for an edge we can get when we have six consecutive vertices v_i, $i = 1, \ldots, 6$ (and hence five edges) on the boundary of f such that $deg(v_1) \geq 3, deg(v_2) = deg(v_3) = deg(v_4) = 2, deg(v_5) = deg(v_6) = 3$, and the edge v_5v_6 is incident with a triangle.

In the second way (according Rule 6) the face f sends charge at most 6 to complete the missing positive charge in some bad 2-vertices and/or bad faces with respect to f (Observations 1–8).

The new charge of f is therefore at least

$$k - 6 - \left(\frac{4}{5} \cdot k + 6\right) = \frac{1}{5} \cdot k - 12 = \frac{k-60}{5} \geq 0.$$

Thus the charge of all elements of the graph is nonnegative, but the sum of all charges is -12. This contradiction implies that the minimum counterexample does not exist.

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