P. J. ŠAFÁRIK UNIVERSITY
FACULTY OF SCIENCE
INSTITUTE OF MATHEMATICS
Jesenná 5, 040 01 Košice, Slovakia

D. Jakubíková-Studenovská and M. Petrejčíková

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IM Preprint, series A, No. 7/2010
May 2010

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D. Jakubíková–Studenovská, M. Petrejčíková
Institute of Mathematics, P.J. Šafárik University,
Jesenná 5, 041 54 Košice, Slovak Republic
danica.studenovska@upjs.sk, maria.petrejcikova@upjs.sk

Abstract: For a monounary algebra $\mathcal{A} = (A, f)$ we study the lattice Quord $\mathcal{A}$ of all quasiorders of $\mathcal{A}$, i.e., of all reflexive and transitive relations compatible with f. In the present paper we find necessary and sufficient conditions for $\mathcal{A}$ under which Quord $\mathcal{A}$ is complementary. As a consequence, conditions under which Quord $\mathcal{A}$ is a Boolean lattice are described.

1 Introduction

If $\mathcal{A}$ is an algebra, then the set consisting of all reflexive and transitive relations on $\mathcal{A}$, which are compatible with all operations of $\mathcal{A}$ (i.e., quasiorders of $\mathcal{A}$), will be denoted Quord $\mathcal{A}$. Then Quord $\mathcal{A}$ is a lattice with respect to inclusion. It is easy to see that the latice Con $\mathcal{A}$ of all congruences of $\mathcal{A}$ is a sublattices of Quord $\mathcal{A}$.

According to the table in [2], Con $\mathcal{A}$ and especially Quord $\mathcal{A}$ have quite many elements. For example if $\mathcal{A}$ has no operation, i.e., $\mathcal{A} = A$, and if $|A| = 5$ then $|\operatorname{Con} A| = 52$ and $|\operatorname{Quord} A| = 6942$. Chajda and Czédli [2] showed that if $|A| \leq \kappa_n$ for some integer n, then Quord A has a three-element generating set, although the lattice Con A, $4 < |A| < \infty$, cannot be generated by three elements [11]. (Let us remark that according to the [2], $\kappa_0 = \aleph_0$ denote the smallest infinite cardinal, i.e., the cardinality of $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, and $\kappa_{n+1} = 2^{\kappa_n}$.)

Quasiorders and their lattices have been studied from different point of view by several autors (e.g., [2], [3], [6], [7], [8]).

The lattices of all quasiorders on a set have some special properties e.g.: they are atomistic, dually atomistic and complemented [6]. By [3], [8], every algebraic

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Mathematics Subject Classification (2000): 08A60, 08A02

Key words and phrases: monounary algebra, quasiorder, lattice, complement, complementary lattice

Acknowledgement. This paper was supported by Grant VEGA 2/0194/10.

lattice is isomorphic to the quasiorder lattice of a suitable algebra. Only few properties of the lattice of quasiorders are known in the case of concrete classes of algebras. For example, for the majority algebras it is known that their quasiorder lattice is always distributive [4], [9].

In [10] the question how endomorphisms of quasiorders behave, in particular, under which conditions $\mathrm{End}\ q \subseteq \mathrm{End}\ q^{'}$ for quasiorders $q, q^{'}$ on a set A ($\mathrm{End}\ q$ is the set of all mappings preserving q) is investigated. The authors describe the quasiorder lattice of the algebra $(A, \mathrm{End}\ q)$.

We will deal with the lattice $\mathrm{Quord}\,(A, f)$ of all quasiorders of (A, f), where (A, f) is a monounary algebra.

The aim of our paper is to find necessary and sufficient conditions for a monounary algebra (A, f) under which the lattice $\mathrm{Quord}\,(A, f)$ is complementary. For the lattice $\mathrm{Con}\,(A, f)$, an analogous problem was solved by Egorova and Skornyakov [5].

Further, we apply our results together with results of [7] and find necessary and sufficient conditions under which the lattice $\mathrm{Quord}\,(A, f)$ is Boolean.

2 Preliminaries

By a monounary algebra we will understand a pair $\mathcal{A} = (A, f)$ where A is a nonempty set and $f : A \to A$ is a mapping.

A monounary algebra $\mathcal{A}$ is called *connected* if for arbitrary $x, y \in A$ there are non-negative integers n, m such that $f^n(x) = f^m(y)$. A maximal connected subalgebra of a monounary algebra is called a *connected component*.

An element $x \in A$ is referred to as *cyclic*, if there exists a positive integer n such that $f^n(x) = x$. In this case the set $\{x, f^1(x), f^2(x), \dots, f^{n-1}(x)\}$ is said to be a *cycle*.

A *quasiorder* of an algebra $\mathcal{A} = (A, F)$ is a reflexive and transitive binary relation on A, which is compatible with all operations $f \in F$. A quasiorder is a congruence of $\mathcal{A}$, if it is symmetric. We will denote by $\mathrm{Quord}\ \mathcal{A}$ the lattice of all quasiorders ordered by inclusion and by $\mathrm{Con}\ \mathcal{A}$ its sublattice, the lattice of all congruences. The smallest and the greatest element of $\mathrm{Quord}\ \mathcal{A}$ and of $\mathrm{Con}\ \mathcal{A}$ are denoted as $I_A = \{(a, a) : a \in A\}$ and $A \times A$. If $\wedge_{\mathrm{Con}}$, $\vee_{\mathrm{Con}}$, $\wedge_{\mathrm{Quord}}$, $\vee_{\mathrm{Quord}}$ are the corresponding operations in the lattices $\mathrm{Con}\ \mathcal{A}$ and $\mathrm{Quord}\ \mathcal{A}$, then it is obvious, that $\wedge_{\mathrm{Con}} = \wedge_{\mathrm{Quord}} = \cap$ and $\vee_{\mathrm{Con}} = \vee_{\mathrm{Quord}}$ is the operation of the transitive hull. Therefore we will use the symbols $\wedge$ and $\vee$ for these operations.

For $a, b \in A$ let $\alpha(a, b)$ and $\theta(a, b)$ be the smallest quasiorder and the smallest congruence, respectively, such that $(a, b) \in \alpha(a, b)$, $(a, b) \in \theta(a, b)$.

The symbol $\mathbb{N}$ is used for the set of all positive integers.

From the paper of Berman [1] concerning congruences it follows that, if $n \in \mathbb{N}$, then θ is a congruence relation of an n-element cycle (C, f) if and only if there is $d \in \mathbb{N}$ such that d divides n and for each $x \in C$, $[x]_\theta = \{x, f^d(x), \dots, f^{(\frac{n}{d}-1)\,d}(x)\} = \{f^k(x) : k \equiv 0(\mathrm{mod}\ d)\}$.

The congruence with this property will be denoted θ_d. It is easy to verify that for each $x \in C$, θ_d is the smallest congruence containing the pair $(x, f^d(x))$.

In [7] the following assertions was proved:

Lemma 2.1. *Let (A, f) be an n-element cycle, $n \in \mathbb{N}$. Then* $\mathrm{Quord}\,(A, f) = \mathrm{Con}\,(A, f) = \{\theta_d : d/n\}$.

Lemma 2.2. *Let (A, f) be an n-element cycle, $n \in \mathbb{N}$. If $a, b \in A$, $f^m(a) = b$, $d =$*g.c.d.(n, m), then $\alpha(a, b) = \theta_d$.

3 Necessary condition

The goal of this section is to find necessary conditions for a monounary algebra (A, f), under which the lattice $\mathrm{Quord}\,(A, f)$ is complementary.

In the following four lemmas we will assume that (A, f) is a monounary algebra and that the lattice $\mathrm{Quord}\,(A, f)$ is complementary.

Lemma 3.1. *Let (B, f) be a subalgebra of the algebra (A, f). Then the lattice* $\mathrm{Quord}\,(B, f)$ *is complementary.*

Proof. Let $\beta \in \mathrm{Quord}\,(B, f)$. Denote $\alpha = \beta \cup I_A$. It is easy to see that $\alpha \in \mathrm{Quord}\,(A, f)$. Then there exists a complement α' of α in $\mathrm{Quord}\,(A, f)$. Put $\beta' = \alpha' \cap B^2$. Obviously, $\beta' \in \mathrm{Quord}\,(B, f)$. We have $\alpha \wedge \alpha' = I_A$ and $\alpha \vee \alpha' = A^2$, which implies

$$I_B \subseteq \beta \wedge \beta' \subseteq \alpha \wedge \beta' = \alpha \cap \alpha' \cap B^2 = I_A \cap B^2 = I_B.$$

Next, $\beta \vee \beta' \subseteq B^2$. To prove that $\beta \vee \beta' \supseteq B^2$ suppose that $(a, b) \in B^2, a \neq b$. Then $(a, b) \in A^2 = \alpha \vee \alpha'$ and there exist $k \in \mathbb{N} \cup \{0\}$ and $y_0, \dots, y_k \in A$ such that

$$a = y_0\ \alpha\ y_1\ \alpha'\ y_2\ \alpha\ y_3 \dots y_{k-2}\ \alpha'\ y_{k-1}\ \alpha\ y_k = b.$$

We can assume that $y_i \neq y_{i+1}$ for each $0 < i < k-1$. Since $(a, b) \in B^2$, the definition of α yields $y_0\ \beta\ y_1 \in B$ and $y_{k-1}\ \beta\ y_k$, $y_{k-1} \in B$. For $0 < i < k-1$ such that $y_i \in B$ and $y_i\ \alpha\ y_{i+1}$ we get $y_{i+1} \in B$ and $y_i\ \beta\ y_{i+1}$. Let $0 < i < k-1$ be such that $y_i \in B$ and $y_i\ \alpha'\ y_{i+1}$. Then $y_{i+1}\ \alpha\ y_{i+2}$ and either $i = k-2$ or $y_{i+1} \neq y_{i+2}$, and it follows $y_{i+1} \in B$. Therefore $(y_i, y_{i+1}) \in \alpha' \cap B^2$, i.e., $(y_i, y_{i+1}) \in \beta'$. From this consideration we obtain the chain

$$a = y_0\ \beta\ y_1\ \beta'\ y_2\ \beta\ y_3 \cdots y_{k-2}\ \beta'\ y_{k-1}\ \beta\ y_k = b,$$

hence $(a, b) \in \beta \vee \beta'$. Thus β' is a complement of β in $\mathrm{Quord}\,(B, f)$. □

Lemma 3.2. *If $x \in A$, then there is $m \in \mathbb{N}$ such that $f^{m+1}(x) = f(x)$.*

Proof. Let $x \in A$. Denote

$$\alpha = \{(f^i(x), f^m(x)) : i, m \in \mathbb{N}\} \cup I_A.$$

Obviously, $\alpha \in \mathrm{Quord}\,(A, f)$. The lattice $\mathrm{Quord}\,(A, f)$ is complementary, hence there exists a complement β of α. There are $k \in \mathbb{N} \cup \{0\}$ and $y_0, \dots, y_k \in A$ such that

$$f(x) = y_0 \ \alpha \ y_1 \ \beta \ y_2 \ \alpha \ y_3 \dots y_{k-2} \ \beta \ y_{k-1} \ \alpha \ y_k = x.$$

If $k = 0$, then $f(x) = x$, $f^{1+1}(x) = f^1(x)$. If $k = 1$, then by the definition of α, $x = f^i(x)$ for some $i \in \mathbb{N}$, thus $f^{i+1}(x) = f(x)$. Let $k > 1$. Then $k \geq 3$. Again the definition of α yields that $y_{k-1} = x$, $y_{k-3} = f^i(x)$, $y_{k-2} = f^m(x)$ for some $i, m \in \mathbb{N}$, hence $f^m(x) = y_{k-2} \ \beta \ y_{k-1} = x$, from which we obtain

$$(f^{m+1}(x), f(x)) \in \beta. \tag{1}$$

Since also

$$(f^{m+1}(x), f(x)) \in \alpha, \tag{2}$$

the fact that $\alpha \wedge \beta = I_A$, in view of (1) and (2), implies that $f^{m+1}(x) = f(x)$. □

Lemma 3.3. *All cycles of (A, f) have the same number of elements.*

Proof. Assume that B and C are distinct cycles of (A, f) and they have n or m elements, respectively, $n > m$. Let $b \in B$, $c \in C$. Denote

$$\alpha = \{(f^i(b), f^j(b)) : i, j \in \mathbb{N}\} \cup I_A.$$

Then $\alpha \in \mathrm{Quord}\,(A, f)$, hence there exists a complement β of α. Since $(b, c) \in \alpha \vee \beta$, there are $k \in \mathbb{N}$ and $y_0, \dots, y_k \in A$ such that

$$b = y_0 \ \beta \ y_1 \ \alpha \ y_2 \ \beta \ y_3 \dots y_{k-2} \ \beta \ y_{k-1} \ \alpha \ y_k = c.$$

We can suppose that $y_i \neq y_{i+1}$ for each $0 < i < k-1$. Further, $y_{k-1} \ \alpha \ y_k = c \in C$ implies that $y_{k-1} = y_k = c$. Notice that k is even. If $k = 2$, then $b\beta c$. If $k > 3$, then $y_{k-3} \ \alpha \ y_{k-2}$ yields that there is $j \in \mathbb{N}$ such that $y_{k-2} = f^j(b)$. Hence in the both cases we get that there is $j \in \mathbb{N}$ such that $(f^j(b), c) \in \beta$. This implies that all pairs of the form $(f^{j+l}(b), f^l(c))$ belong to β. Since $n > m$, there are $l \in \mathbb{N}$, $0 \leq j_1 < j_2 < n$ such that

$$(f^{j_1+l}(b), f^l(c)) \in \beta, \ (f^{j_2+l}(b), f^l(c)) \in \beta. \tag{3}$$

(We note that for $j_1 + m < n$ there is $j_2 = j_1 + m$. In the case $j_1 + m \geq n$ there is $j_2 = j_1 + m - n$, so $0 \leq j_2 < j_1 < n$; for $j_1' = j_2, j_2' = j_1$ we get the required inequality $0 \leq j_1' < j_2' < n$.)

Further, $(c, b) \in \alpha \vee \beta$, thus there are $t \in \mathbb{N}$ and $v_0, \dots, v_t \in A$ such that

$$c = v_0 \ \beta \ v_1 \ \alpha \ v_2 \ \beta \ v_3 \dots v_{t-2} \ \beta \ v_{t-1} \ \alpha \ v_t = b,$$

$v_i \neq v_{i+1}$ for each $0 < i < t-1$. Then $v_1 = f^s(b)$ for some $s \in \mathbb{N}$, hence $c\beta f^s(b)$. Without loss of generality, j_1 and s are not congruent modulo n. In view of (3) we obtain

$$f^{j_1+l}(b)\ \beta\ f^l(c)\ \beta\ f^l(f^s(b)) = f^{s+l}(b).$$

Then $(f^{j_1+l}(b), f^{s+l}(b)) \in \alpha \wedge \beta = I_A$, which is a contradiction. □

Lemma 3.4. *If C is a cycle of (A, f) with n elements, then $n = 1$ or n is a product of mutually distinct primes.*

Proof. Let C satisfy the assumption, $c \in C$ and $n > 1$. By way of contradiction, let there be a prime p such that p^2/n. From Lemma 3.1 it follows that the lattice Quord (C, f) is complementary. Further,

$$\text{Quord}\,(C, f) = \text{Con}\,(C, f) = \{\theta_d : d/n\},$$

where $\theta_d = \theta(c, f^d(c))$. There exists a complement θ_k to θ_p in Con (C, f). Then $1 < k < n$. We have

$$A^2 = \theta_1 = \theta_p \vee \theta_k = \theta_{\text{g.c.d.}(p,k)},\ I_C = \theta_n = \theta_p \wedge \theta_k = \theta_{\text{l.c.m.}(p,k)}.$$

The first equality implies that p does not divide k, while the second one with the assumption p^2/n yields p/k, a contradiction. □

Theorem 3.5. *Let (A, f) be a monounary algebra. If the lattice* Quord (A, f) *is complementary, then*

1. *each connected component of (A, f) contains a cycle,*
2. *there is $n \in \mathbb{N}$ such that each cycle of (A, f) has n elements,*
3. *either $n = 1$ or n is a product of mutually distinct primes (n is square-free),*
4. *for each $a \in A$, the element $f(a)$ is cyclic.*

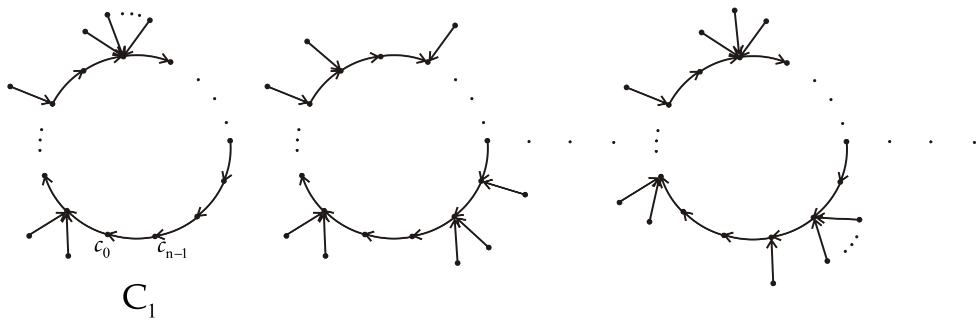

Figure 1: A monounary algebra $\mathcal{A} = (A, f)$ which satisfies above conditions.

Let us note that for a more clear visualization we write also the condition *1*, though it follows from the condition *4*.

4 Sufficient condition

In this section we will suppose that

- (A, f) is a monounary algebra,
- there is $n \in \mathbb{N}$ such that each cycle of (A, f) has n elements,
- either $n = 1$ or n is square-free,
- for each $a \in A$, the element $f(a)$ is cyclic.

Lemma 4.1. *Let (A, f) be a cycle, $\alpha = \theta_d$, d/n. If $e = \frac{n}{d}$, then $\beta = \theta_e$ is a complement to α in the lattice* Quord (A, f)*, and therefore the lattice* Quord (A, f) *is complementary.*

Proof. Let d, e be as above. Then g.c.d.$(d, e) = 1$, l.c.m.$(d, e) = n$, thus Lemma 2.2 implies $\theta_d \vee \theta_e = A \times A$, $\theta_d \wedge \theta_e = I_A$. Hence by Lemma 2.1 the lattice Quord (A, f) is complementary

□

Notation 4.2. *If $\alpha \in$ Quord (A, f), then $\bar{\alpha}$ denotes the relation (obviously, a quasiorder), such that, whenever $a, b \in A$,*

$$(a, b) \in \alpha \Longleftrightarrow (b, a) \in \bar{\alpha}.$$

For $a \in A$ we denote by $C(a)$ the cycle, which contains the element $f(a)$.

Let R be the binary relation defined on the set of all cycles of (A, f) as follows: If B, D are cycles of (A, f), then we put $B\ R\ D$, if there are $k \in \mathbb{N}$, cycles $B = C_0, C_1, \ldots, C_k = D$, elements $c_0 \in C_0, c_1 \in C_1, \ldots, c_k \in C_k$ such that for each $i \in \{0, 1, \ldots, k-1\}$, $(c_i, c_{i+1}) \in \alpha \cup \bar{\alpha}$.

If $a, b \in A$, then we set

$$a\ r\ b \Longleftrightarrow C(a)\ R\ C(b).$$

Lemma 4.3. *The relation r is an equivalence on A.*

Proof. It is easy to see, that r is reflexive: to prove that $a\ r\ a$, take $k = 1$, $c_0 = c_1 = f(a)$. Next, r is symmetric, since $\alpha \cup \bar{\alpha}$ is symmetric.

Now let us show transitivity. Assume that $c\ r\ d$ and $d\ r\ b$. Denote $C = C(c)$, $D = C(d)$, $B = C(b)$. There exist $m, l \in \mathbb{N}$, cycles $C = C_0, C_1, \ldots, C_m = D$, cycles $D = D_0, D_1, \ldots, D_l = B$, elements $c_0 \in C_0, c_1 \in C_1, \ldots, c_m \in C_m$, $d_0 \in D_0, d_1 \in D_1, \ldots, d_l \in D_l$ such that for each $i \in \{0, 1, \ldots, m-1\}$, $(c_i, c_{i+1}) \in \alpha \cup \bar{\alpha}$ and for each $j \in \{0, 1, \ldots, l-1\}$, $(d_j, d_{j+1}) \in \alpha \cup \bar{\alpha}$. Denote $k = m + l$ and for $j \in \{1, \ldots, l\}$ put

$$C_{m+j} = D_j.$$

Since $D = D_0 = C_m$ is a cycle and it contains the elements d_0, c_m, there is $t \in \{0, \ldots, n-1\}$ such that $d_0 = f^t(c_m)$. Further, the relation $(d_j, d_{j+1}) \in \alpha \cup \bar{\alpha}$ for $j \in \{0, 1, \ldots, l-1\}$ implies

$$(f^t(d_j), f^t(d_{j+1})) \in \alpha \cup \bar{\alpha}.$$

Now it suffices to denote $c_{m+j} = d_j$ for each $j \in \{1, \dots, l\}$ and the proof is complete.

□

Lemma 4.4. *If $a, b \in A$ belong to the same connected component, then $a\ r\ b$.*

Proof. Similarly as in the proof of reflexivity of the relation r, let us take $C_0 = C_1 = C(a) = C(b)$, $k = 1$, $c_0 = f(a) = c_1$. □

Let $A/r = \{A_j : j \in J\}$. Now we will work with the classes of the equivalence r.

The goal of the following construction is to define, for a given $j \in J$ and a given quasiorder $\alpha \in \mathrm{Quord}\,(A_j, f)$, some $\beta \in \mathrm{Quord}\,(A_j, f)$; later we show that β is a complement of α in $\mathrm{Quord}\,(A_j, f)$.

For simplification, we will write A instead of A_j, i.e., till the main result about complements in $\mathrm{Quord}\,(A_j, f)$ (Theorem 4.12) of this section, we assume that J is a one-element set.

Notation 4.5. *Let A' be the set of all noncyclic elements x of A such that*

$$(x, f^n(x)) \notin \alpha \text{ and } (f^n(x), x) \notin \alpha.$$

We define a binary relation ρ on A' as follows. Put $(a, b) \in \rho$ if $a, b \in A'$, $f(a) = f(b)$ and there are $k \in \mathbb{N}$ and $a = u_0, u_1, \dots, u_k = b$ elements of A' such that

$$(\forall i \in \{0, \dots, k-1\})(f(a) = f(u_i), (u_i, u_{i+1}) \in \alpha \cup \bar{\alpha}).$$

It is easy to verify that the relation ρ is an equivalence and that the following assertion is valid.

Let $D \in A'/\rho$. We consider an equivalence relation $\alpha \cap \bar{\alpha}$ on D and choose one element from each class of this equivalence. The set of the chosen elements will be denoted by $P(D)$. Then the following lemma is valid.

Lemma 4.6. *1.* $(\forall x \in D \setminus P(D))(\exists y \in P(D))((x, y) \in \alpha, (y, x) \in \alpha)$;
2. $(\forall x, y \in P(D))((x, y) \in \alpha \Rightarrow (y, x) \notin \alpha)$.

Further we denote an arbitrary fixed element of the set $P(D)$ by the symbol $p(D)$.

Lemma 4.7. *Assume that x is a noncyclic element of A, $\alpha \restriction C(x) = \theta_d$, d/n. Next suppose that $k \in \mathbb{N}$ and either $(x, f^k(x)) \in \alpha$ or $(f^k(x), x) \in \alpha$. Then d/k.*

Proof. The assumption implies that either

$$(f(x), f^{k+1}(x)) \in \alpha \, or \, (f^{k+1}(x), f(x)) \in \alpha,$$

i.e., either $(f(x), f^{k+1}(x)) \in \theta_d$ or $(f^{k+1}(x), f(x)) \in \theta_d$. In both the cases we obtain that d/k. □

Now let us describe the relation β. Let $x, y \in A$. We put $(x, y) \in \beta$ if either $x = y$ or the pair (x, y) fulfils one of the steps of the construction. Let us remark that in (e) we use some previous steps.

Construction.

Step (a). Let x, y belong to the same cycle C, $y = f^k(x)$, $\alpha \upharpoonright C = \theta_d$, d/n and let $e = \frac{n}{d}$. We set $(x, y) \in \beta$ if and only if e/k.

Step (b). Let $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles. We put $(x, y) \in \beta$ if and only if there are $a \in C_1$ and $b \in C_2$ with $(b, a) \in \alpha, (a, b) \notin \alpha$.

Step (c). Suppose that $x, y \in P(D)$ for some $D \in A'/\rho$. Then $(x, y) \in \beta$ if and only if and $(y, x) \in \alpha$.

Step (d1). Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \upharpoonright C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \notin A'$, then $(x, y) \in \beta$ if and only if $(f^n(y), y) \notin \alpha, (y, f^n(y)) \in \alpha, x = f^k(y), e/k$.

Step (d'1). Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \upharpoonright C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \notin A'$, then $(x, y) \in \beta$ if and only if $(f^n(x), x) \in \alpha, (x, f^n(x)) \notin \alpha, y = f^k(x), e/k$.

Step (d2). Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \upharpoonright C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $y \in P(D), x = f^k(y), e/k$ and $(y, p(D)) \in \alpha$.

Step (d'2). Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \upharpoonright C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $x \in P(D), y = f^k(x), e/k$ and $(p(D), x) \in \alpha$.

Step (e). Suppose that x, y satisfy none of the assumptions of the previous steps. Then $(x, y) \in \beta$ if and only if $(x, f^n(x)) \in \beta$, $(f^n(x), f^n(y)) \in \beta$, $(f^n(y), y) \in \beta$.

Lemma 4.8. *Let $(x, y) \in \beta$. Then $(f(x), f(y)) \in \beta$.*

Proof. We can assume that $x \neq y$ and that the pair (x, y) is obtained according to the steps of the above construction.

(A) First x, y belong to the same cycle C, $y = f^k(x)$, $\alpha \upharpoonright C = \theta_d$, d/n, $e = \frac{n}{d}$ and e/k. Then $(f(x), f(y)) = (f(x), f^k(f(x)))$, thus $(f(x), f(y)) \in \beta$ by the step (a).

(B) Now $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles and there are $a \in C_1$ and $b \in C_2$ with $(b, a) \in \alpha, (a, b) \notin \alpha$. Since $f(x) \in C_1$ and $f(y) \in C_2$, the above step (b) yields that $(f(x), f(y)) \in \beta$.

(C) In the step (c) the assumption implies that $f(x) = f(y)$.

(D1) We will not repeat all assumptions of (d1). We have

$$y \notin A',\ (f^n(y), y) \notin \alpha,\ (y, f^n(y)) \in \alpha,\ x = f^k(y),\ e/k.$$

For verifying that $(f(x), f(y)) \in \beta$ we need to apply (a), because $f(x)$ and $f(y)$ belong to the same cycle. We have $f(y) = f^{n-k}(f(f^k(y))) = f^{n-k}(f(x))$ and $e/n-k$, therefore $(f(x), f(y)) \in \beta$.

(D'1) Analogously as (D1).

(D2) We suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further, $y \in A'$ and there is $D \in A'/\rho$ such that $y \in P(D), x = f^k(y), e/k$, $(y, p(D)) \in \alpha$. The elements $f(x)$ and $f(y)$ belong to the same cycle, $f(y) = f(p(D))$, thus $f(y) = f^{n-k}(f(f^k(y))) = f^{n-k}(f(x))$ and $e/n-k$, therefore $(f(x), f(y)) \in \beta$.

(D'2) Analogously as (D2).

(E) In this case we have $(x, f^n(x)) \in \beta$, $(f^n(x), f^n(y)) \in \beta$, $(f^n(y), y) \in \beta$. The elements $f^n(x), f^n(y)$ are cyclic, then (B), in view of $(f^n(x), f^n(y)) \in \beta$, implies $(f(f^n(x)), f(f^n(y))) \in \beta$, i.e., $(f(x), f(y)) \in \beta$.

□

Lemma 4.9. *Let* $(x, y) \in \beta$, $(y, z) \in \beta$. *Then* $(x, z) \in \beta$.

Proof. We can assume that x, y, z are mutually distinct.

1) First assume that $C(x) \neq C(y)$. By (e) we have

$$(x, f^n(x)) \in \beta, \tag{1}$$

$$(f^n(x), f^n(y)) \in \beta, \tag{2}$$

$$(f^n(y), y) \in \beta. \tag{3}$$

Then (b) yields

$$\text{there are } a \in C(x), b \in C(y) \text{ with } (b, a) \in \alpha, (a, b) \notin \alpha \tag{4}$$

Similarly suppose that $C(z) \neq C(y)$. Then

$$(y, f^n(y)) \in \beta, \tag{5}$$

$$(f^n(y), f^n(z)) \in \beta, \tag{6}$$

$$(f^n(z), z) \in \beta, \tag{7}$$

$$\text{there are } b' \in C(y), c' \in C(z) \text{ with } (c', b') \in \alpha, (b', c') \notin \alpha. \tag{8}$$

From (4) and (8) it follows that there is $m \in \mathbb{N}$ with $b = f^m(b')$. Denote $c = f^m(c')$. Then

$$c = f^m(c')\ \alpha\ f^m(b') = b\ \alpha\ a.$$

Since $(a,b) \notin \alpha$, we get $(a,c) \notin \alpha$. Therefore

$$(c_1, c_2) \in \beta \text{ for each } c_1 \in C(x), c_2 \in C(z),$$

according to (b). Then $(f^n(x), f^n(z)) \in \beta$. Thus (1) and (7), in view of (e), imply $(x,z) \in \beta$.

2) Suppose that $C(x) \neq C(y) = C(z)$. If z is cyclic, then $(x,z) \in \beta$ by (4). Let z be noncyclic. If the elements y, z satisfy (e), then $(x,z) \in \beta$ analogously as in the first part of the proof. Hence y is cyclic.

Let $\alpha \restriction C(y) = \theta_{\frac{n}{e}}$. If $z \notin A'$, then by (d1), $(f^n(z), z) \notin \alpha, (z, f^n(z)) \in \alpha, y = f^k(z), e/k$. Thus again according to (d1), $(f^n(z), z) \in \beta$. If $z \in A'$, then by (d2) there is $D \in A'/\rho$ such that $z \in P(D), y = f^k(z), e/k$ and $(z, p(D)) \in \alpha$. Thus $(f^n(z), z) \in \beta$ in view of (d2). This in view of (1), (2) and (e) yields that $(x,z) \in \beta$.

3) The case when $C(x) = C(y) \neq C(z)$ is similar to 2).

4) Finally we suppose that $C(x) = C(y) = C(z)$, $\alpha \restriction C(x) = \theta_{\frac{n}{e}}$.

First we show the assertion for cyclic elements x, y, z. There are k, m with $y = f^k(x)$, $z = f^m(y)$, e/k, e/m. Then $z = f^{k+m}(x), e/k+m$, hence $(x,z) \in \beta$.

From the assumption $(x,y) \in \beta$, $(y,z) \in \beta$ it follows $(f^n(x), f^n(y)) \in \beta$, $(f^n(y), f^n(z)) \in \beta$, the elements $f^n(x), f^n(y), f^n(z)$ are cyclic, thus

$$(f^n(x), f^n(z)) \in \beta. \tag{9}$$

This implies that if $(x, f^n(x)) \in \beta$, $(f^n(z), z) \in \beta$ then the pair x, z satisfies (e) and then either $(x,z) \in \beta$ or x, z satisfy some of the assumptions of (a), (c), (d1), (d'1), (d2), (d'2). We will proceed according to this idea in the remaining part of the proof.

4.1) Let x, y be cyclic, z be noncyclic. By $(x,y) \in \beta$ we have $y = f^k(x), e/k$, thus also $x = f^n(x) = f^{k+i}(x) = f^i(f^k(x)) = f^i(y), e/i$. In view of (d1) or (d2), $y = f^m(z), e/m$. Then $x = f^{i+m}(z), e/i+m$ and $(x,z) \in \beta$ according to (d1) or (d2).

4.2) Let x, z be cyclic, y be noncyclic. For $y \notin A'$, then (d'1) by $(y,z) \in \beta$ implies that $(y, f^n(y)) \notin \alpha$ and (d1) by $(x,y) \in \beta$ implies that $(y, f^n(y)) \in \alpha$, a contradiction. If $y \in A'$, then (d'2) and $(y,z) \in \beta$ yield $y \in P(D)$ for some $D \in A'/\rho$ and $z = f^m(y), e/m$. Similarly, if $y \in A'$, then (d2) and $(x,y) \in \beta$ yield that $x = f^k(y), e/k$. There is $t \in \mathbb{N}$ with $m - k + tn \geq 0$ and then

$$z = f^{m+tn}(y) = f^{m-k+tn}(f^k(y)) = f^{m-k+tn}(x),\ e/m-k+tn.$$

Therefore $(x,z) \in \beta$ in view of (a).

4.3) Let x be cyclic, y, z be noncyclic. First let $y, z \in P(D)$ for some $D \in A'/\rho$. Then $(z,y) \in \alpha$ in view of (c). Next, $x = f^m(y)$, e/m, $(y, p(D)) \in \alpha$, thus $(z, p(D)) \in \alpha$. Since $f^m(y) = f^m(p(D)) = f^m(z)$, we obtain by (d2) that $(x,z) \in \beta$. Now let $(y,z) \in \beta$ by (e). Then $(y, f^n(y)) \in \beta$, $(f^n(y), f^n(z)) \in \beta$,

$(f^n(z), z) \in \beta$. The second relation implies that $y = f^k(z), e/k$. From (d1), (d2) for the elements x, y we get that $x = f^m(y), e/m$, thus $x = f^{m+k}(z), e/m + k$. If $z \notin A'$, then by (d1), $(f^n(z), z) \notin \alpha, (z, f^n(z)) \in \alpha$ and then $(x, z) \in \beta$. If $z \in A'$, then according to $(f^n(z), z) \in \beta$ by (d2) we obtain $z \in P(D)$ for some $D \in A'/\rho$ and $(z, p(D)) \in \alpha$, therefore $(x, z) \in \beta$.
4.4) The case when x, y be noncyclic, z be cyclic is dual to 4.3).
4.5) Let x, z be noncyclic, y be cyclic. From $(x, y) \in \beta$ and (d'1), (d'2) it follows that either $x \notin A', (f^n(x), x) \in \alpha, (x, f^n(x)) \notin \alpha, y = f^k(x), e/k$, or $x \in A'$, there is $D \in A'/\rho$ such that $x \in P(D), y = f^k(x), e/k$ and $(p(D), x) \in \alpha$. Next, (d'1), (d'2) yield $(x, f^n(x)) \in \beta$. It can be shown analogously that $(f^n(z), z) \in \beta$. Therefore we either obtain that $(x, z) \in \beta$ according to (e) or x, z satisfy the assumption of (c). Then $z \in P(D)$. Since $(y, z) \in \beta$, (d2) implies that $y = f^m(z), e/m$ and $(z, p(D)) \in \alpha$. Therefore

$$z \;\alpha\; p(D) \;\alpha\; x,$$

hence $(x, z) \in \beta$ by (c).
4.6) Finally suppose that x, y, z are noncyclic. Then either x, y satisfy the assumption of (c) and

$$x, y \in P(D), D \in A'/\rho, (y, x) \in \alpha$$

or x, y satisfy the assumption of (e) and

$$(x, f^n(x)) \in \beta, (f^n(x), f^n(y)) \in \beta, (f^n(y), y) \in \beta.$$

Similarly, either y, z satisfy the assumption of (c) and

$$y, z \in P(D_1), D_1 \in A'/\rho, (z, y) \in \alpha$$

or y, z satisfy the assumption of (e) and

$$(y, f^n(y)) \in \beta, (f^n(y), f^n(z)) \in \beta, (f^n(z), z) \in \beta.$$

Let x, y satisfy the assumption of (c) and y, z satisfy the assumption of (c). Then $D_1 = D$, $z \;\alpha\; y \;\alpha\; x$, thus $(x, z) \in \beta$ by (c).

Let x, y satisfy the assumption of (c) and y, z satisfy the assumption of (e) (the case when x, y satisfy the assumption of (e) and y, z satisfy the assumption of (c) is analogous). We have $(y, f^n(y)) \in \beta$, thus by (d'2), $(p(D), y) \in \alpha$, which yields $p(D) \;\alpha\; y \;\alpha\; x$. Then (d'2) implies that $(x, f^n(x)) \in \beta$, therefore (e) according to (9) yields $(x, z) \in \beta$.

Let x, y satisfy the assumption of (e) and y, z satisfy the assumption of (e). In view of (9), if $(x, z) \notin \beta$, then $x, z \in P(D_2), D_2 \in A'/\rho, (z, x) \notin \alpha$. Since $(f^n(z), z) \in \beta$, by (d2) we obtain $(z, p(D_2)) \in \alpha$, and from (d'2) and $(x, f^n(x)) \in \beta$ it follows that $(p(D_2), x) \in \alpha$. Therefore $(x, z) \in \beta$, a contradiction.

□

Lemma 4.10. *If $(x, y) \in \alpha \wedge \beta$, then $x = y$.*

Proof. Let $(x, y) \in \alpha \wedge \beta$, $x \neq y$.

(A) Assume that x, y belong to the same cycle C. There is $d \in \mathbb{N}$ such that $\alpha \restriction C = \theta_d$, d/n. Step (a) implies that $\beta \restriction C = \theta_e$, where $e = \frac{n}{d}$. We have $(x, y) \in \alpha \restriction C \cap \beta \restriction C = \theta_d \cap \theta_e$. Then according to Lemma 4.1, $x = y$.

(B) Suppose that $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles. There is $d \in \mathbb{N}$ such that $\alpha \restriction C_2 = \theta_d$, d/n. Then $(x, y) \in \beta$ if and only if there are $a \in C_1$ and $b \in C_2$ with $(b, a) \in \alpha, (a, b) \notin \alpha$. There are $k, m \in \mathbb{N}$ such that $a = f^k(x), b = f^m(y)$. Since $(x, y) \in \alpha$, also $(f^k(x), f^k(y)) \in \alpha$, hence

$$f^m(y) = b \ \alpha \ a = f^k(x) \ \alpha \ f^k(y).$$

The elements $f^m(y), f^k(y)$ belong to C_2 and $(f^m(y), f^k(y)) \in \theta_d$, which yields that $d/m - k$. Then also $(f^k(y), f^m(y)) \in \alpha$ and

$$a = f^k(x) \ \alpha \ f^k(y) \ \alpha \ f^m(y) = b,$$

which is a contradiction.

(C) Let $x, y \in P(D)$ for some $D \in A'/\rho$. Then $(x, y) \in \beta$ if and only if and $(y, x) \in \alpha$. We assumed that $(x, y) \in \alpha$, but this is a contradiction, because $x, y \in P(D)$.

(D1) Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$ and let $y \notin A'$. Then $(f^n(y), y) \notin \alpha$, $(y, f^n(y)) \in \alpha$, $x = f^k(y)$, e/k. Next, $(f^{k+1}(y), f(y)) = (f(x), f(y)) \in \alpha$, which implies that d/k. The assumption about n at the beginning of the section yields ed/k, i.e., n/k and $x = f^n(y) = y$.

(D2) Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$ and $y \in P(D)$ for $D \in A'/\rho$. Then $x = f^k(y), e/k$ and $(y, p(D)) \in \alpha$. Similarly as in (D1), $(f^{k+1}(y), f(y)) = (f(x), f(y)) \in \alpha$, therefore we obtain $x = y$.

(D'1), (D'2) Analogously as (D1), (D2).

(E) Now x, y satisfy none of the assumptions of the previous steps and

$$(x, f^n(x)) \in \beta, (f^n(x), f^n(y)) \in \beta, (f^n(y), y) \in \beta.$$

From of the assumption of the lemma it follows that $(f^n(x), f^n(y)) \in \alpha$. For the cyclic elements $f^n(x), f^n(y)$ we can apply (A) or (B), thus $f^n(x) = f^n(y)$. If y is cyclic, then $y = f^n(x)$, hence $(x, y) = (x, f^n(x)) \in \beta, (x, y) \in \alpha$ and $x = y$. Therefore we can assume that x and y are noncyclic. If $x \notin A'$, then $(x, f^n(x)) \in \beta$ by (d'1) implies $(f^n(x), x) \in \alpha$, thus

$$f^n(y) = f^n(x) \ \alpha \ x \ \alpha \ y,$$

a contradiction to $(f^n(y), y) \in \beta$. Similarly for y; therefore let $x, y \in A'$. From $f(x) = f^{n+1}(x) = f^{n+1}(y) = f(y)$ it follows that $x, y \in P(D)$ for some $D \in A'/\rho$. This completes the proof according to (C).

□

Lemma 4.11. $\alpha \vee \beta = A \times A$.

Proof. Let $x, y \in A$, $x \neq y$.

1) If x, y belong to the same cycle, then the assertion follows from Lemma 4.1.

2) Let x, y belong to distinct cycles. First let us prove that if C, D are distinct cycles, $c \in C$, $d \in D$ and $(c, d) \in \alpha \cup \bar{\alpha}$, then $(c', d') \in \alpha \vee \beta$ for each $c' \in C, d' \in D$. Let $c' \in C, d' \in D$. If $(c, d) \in \bar{\alpha}$ and $(c, d) \notin \alpha$, then $(d, c) \in \alpha$ and (b) implies $(c', d') \in \beta$. If $(c, d) \in \alpha$, then using the proved case 1) we get

$$c' \ (\alpha \vee \beta) \ c \ \alpha \ d \ (\alpha \vee \beta) \ d'.$$

By the assumption, $x \ r \ y$. Then $C(x) \ R \ C(y)$ and there are $k \in \mathbb{N}$, cycles $C(x) = C_0, \ C_1, \ \dots, \ C_k = C(y)$ and elements $c_0 \in C_0, \ c_1 \in C_1, \ \dots, \ c_k \in C_k$ such that for each $i \in \{0, 1, \dots, k-1\}$, $(c_i, c_{i+1}) \in \alpha \cup \bar{\alpha}$. Then by induction, $(x, y) \in \alpha \vee \beta$.

3) Let $C(x) = C(y)$ and either x is noncyclic, $x \notin A'$, y is cyclic, or x is cyclic, y is noncyclic, $y \notin A'$. We prove only the first case; the second one is analogous. Since $x \notin A'$, thus either $(x, f^n(x)) \in \alpha$ or $(f^n(x), x) \in \alpha$, $(x, f^n(x)) \notin \alpha$, which by (d'1) implies $(x, f^n(x)) \in \beta$. Then $(x, y) \in \alpha \vee \beta$ by 1).

4) Assume that x, y belong to the same connected component, $x, y \notin A'$. Then $(x, y) \in \alpha \vee \beta$ in view of 3). From this and from 1) it follows, that the condition that x, y belong to the same connected component, can be omitted.

5) Let $x, y \in D$, $D \in A'/\rho$. Then there are $k \in \mathbb{N}$ and $x = u_0, u_1, \dots, u_k = y$ elements of $P(D) \subseteq D$ such that $f(x) = f(y) = f(u_i), (u_i, u_{i+1}) \in \alpha \cup \bar{\alpha}$ for each $i \in \{0, \dots, k-1\}$. It can be shown analogously as in 2) that $(x, y) \in \alpha \vee \beta$.

6) Let $D \in A'/\rho$. In view of (d'2) we obtain $(p(D), f^n(p(D))) \in \beta$. This, together with the previous steps, implies that if $x \in A'$, then $x \in D$ for some $D \in A'/\rho$, thus $(x, p(D)) \in \alpha \vee \beta$ and $(f^n(p(D)), y) \in \alpha \vee \beta$ for each $y \notin A'$. So then $(x, y) \in \alpha \vee \beta$.

7) Let $D \in A'/\rho$. Then $(f^n(p(D)), p(D)) \in \beta$ by (d2). Thus if x is cyclic, $y \in A'$, then $y \in D$ for some $D \in A'/\rho$ and we get by (2) that $(x, f^n(p(D))) \in \alpha \vee \beta$, $(f^n(p(D)), p(D)) \in \beta$ and by (5) that $(p(D), y) \in \alpha \vee \beta$.

It follows from the previous steps that $(x, y) \in \alpha \vee \beta$ for arbitrary $x, y \in A$, so the claim is proved.

□

In view of construction and Lemmas 4.8 - 4.11 we obtain:

Theorem 4.12. *Let* $\alpha \in \mathrm{Quord}\,(A, f)$, *$r$ and J be as above. Let* $j \in J$. *Then there exists a complement* β_j *of* $\alpha_j = \alpha \restriction A_j$ *in the lattice* $\mathrm{Quord}\,(A_j, f)$.

The above assumption and the result of the previous section (Theorem 3.5) imply the validity of the next theorem.

Theorem 4.13. *If $\alpha \in \text{Quord}\,(A, f)$ and $|A/r| = 1$, then the conditions*

- *each connected component of (A, f) contains a cycle,*
- *there is $n \in \mathbb{N}$ such that each cycle of (A, f) has n elements,*
- *n is square-free,*
- *for each $a \in A$, the element $f(a)$ is cyclic*

are necessary and sufficient for the existence of a complement of α in the lattice $\text{Quord}\,(A, f)$.

5 Sufficient condition - general case

The aim of this section is to show that under the above conditions, the lattice $\text{Quord}\,(A, f)$ is complementary in general.

Assume that the above conditions are valid. Next, suppose $\alpha \in \text{Quord}\,(A, f)$ and r is as above; notice that the relation r depends on α, thus we will denote it r_α. According to the previous section the case $|J| = 1$ is solved; now let us suppose that $|J| > 1$. For $i \in J$ let c_i be a fixed cyclic element of some chosen cycle C_i in A_i. We denote by γ the following relation:

$$\gamma = \{(f^k(c_i), f^k(c_j)) : i, j \in J, k \in \mathbb{N}\}.$$

It can be easily shown that $\gamma \in \text{Quord}\,(A, f)$.

For each $i \in J$, the relation $\alpha \restriction C_i$ is a congruence of the cycle C_i, thus there is $d_i \in \mathbb{N}$ such that $\alpha \restriction C_i$ is the smallest congruence containing the pair $(c_i, f^{d_i}(c_i))$. The set of all d_i is finite, denote it by $\{d_1, d_2, \dots, d_s\}$. Without loss of generality, let $\{1, 2, \dots, s\} \subseteq J$.

Notice that, for $i \in J$, $d, l, k \in \mathbb{N}$, $(f^l(c_i), f^k(c_i)) \in \theta(c_i, f^d(c_i))$ if and only if d divides $l - k$.

In what follows, let d will be the greatest common divisor of $d_1, d_2, \dots, d_s$. This implies

Lemma 5.1. *There exist positive integers $q_1, q_2, \dots, q_s$ and q such that*

$$1 + qn = q_1\frac{d_1}{d} + q_2\frac{d_2}{d} + \cdots + q_s\frac{d_s}{d}.$$

Let $i \in J$. Put

$$\alpha_i' = \theta(c_i, f^d(c_i)) \vee \alpha_i.$$

If $\alpha' = \bigcup_{j\in J} \alpha_i'$, then $\alpha' \in \text{Quord}\,(A, f)$ and it easy to see that $r_{\alpha'} = r_\alpha$. By the results of the previous section there exists a complement β_i' of α_i' in $\text{Quord}\,(A_i, f)$. Further,from the construction of a complement on A_i we obtain

$$\beta_i' \restriction C_i = \theta(c_i, f^{\frac{n}{d}}(c_i)).$$

Lemma 5.2. *Let $i \in J$, $l, k \in \mathbb{N}$. Then $(f^l(c_i), f^k(c_i)) \in \alpha_i \vee \beta_i'$ if and only if $\frac{d_i}{d}/l - k$.*

Proof. From the notation above, $(f^l(c_i), f^k(c_i)) \in \alpha_i$ if and only if $d_i/l - k$ and $(f^l(c_i), f^k(c_i)) \in \beta_i'$ if and only if $\frac{n}{d}/l - k$. Then $(f^l(c_i), f^k(c_i)) \in \alpha_i \vee \beta_i'$ if and only if $\mathrm{g.c.d}(d_i, \frac{n}{d})/l - k$, i.e., if and only if $\frac{d_i}{d}/l - k$. □

Now we define the relation β by putting

$$\beta = \gamma \ \vee \ \bigvee_{j \in J} \beta_j'.$$

We are going to show that β is a complement of α in the lattice Quord (A, f). Since β is a join of quasiorders, it is clear that it is also a quasiorder.

Lemma 5.3. *If $(x, y) \in \alpha \wedge \beta$, then $x = y$*

Proof. Let $(x, y) \in \alpha \wedge \beta$, $x \neq y$. The relation $(x, y) \in \alpha$ implies that there is $i \in J$ such that $x, y \in A_i$, $(x, y) \in \alpha_i$. Then $(x, y) \in \alpha_i'$. We have $\alpha_i \cap \beta_i' = \alpha_i' \cap \beta_i'$, which, since β_i' is a complement to α_i', is the smallest quasiorder of (A_i, f). The assumption $x \neq y$ yields that $(x, y) \notin \beta_i'$. There is the shortest chain of elements $x = u_0, u_1, \ldots, u_m = y$ with $m > 1$ such that either $(u_k, u_{k+1}) \in \gamma$ or $(u_k, u_{k+1}) \in \bigvee_{j \in J} \beta_j'$, for any k. Obviously, the elements $u_0, u_1, \ldots, u_m$ are distinct and if $(u_k, u_{k+1}) \in \gamma$, then $(u_{k+1}, u_{k+2}) \in \bigvee_{j \in J} \beta_j'$, and similarly for the second possibility. For each k there is $i_k \in J$ with $u_k \in A_{i_k}$. From the definition of β we get

$$(u_k, u_{k+1}) \in \gamma \implies u_k = f^{t_k}(c_{i_k}), u_{k+1} = f^{t_{k+1}}(c_{i_{k+1}}), i_k \neq i_{k+1}, t_k = t_{k+1}, \quad (1)$$

$$(u_k, u_{k+1}) \in \beta_j' \implies i_k = i_{k+1}, \quad (2)$$

$$u_k = f^{t_k}(c_{i_k}), u_{k+1} = f^{t_{k+1}}(c_{i_{k+1}}), (u_k, u_{k+1}) \in \beta_j' \implies i_k = j, \frac{n}{d}/t_k - t_{k+1}. \quad (3)$$

We have either

$$x = u_0 \ \gamma \ u_1 \ \beta_j' \ u_2 \ \gamma \ u_3 \ \beta_j' \ u_4 \ \ldots, \quad (4)$$

or

$$x = u_0 \ \beta_j' \ u_1 \ \gamma \ u_2 \ \beta_j' \ u_3 \ \gamma \ u_4 \ \ldots. \quad (5)$$

We have $m > 1$, thus between the elements of the chain, the quasiorder γ is used at least twice.

Assume that (5) holds. Also, assume that $u_{m-1} \in A_i$. (The remaining cases are similar, but more simple.) Then m is odd. By the definition of γ, for each $0 < k \leqq m$ there exists a positive integer t_k such that $u_k = f^{t_k}(c_{i_k})$. In view of (1)-(3), $t_1 = t_2$, $\frac{n}{d}/t_2 - t_3$, $t_3 = t_4$, $\frac{n}{d}/t_4 - t_5$, $\ldots$, $t_{m-2} = t_{m-1}$. Then

$$\frac{n}{d}/(t_1 - t_2) + (t_2 - t_3) + (t_4 - t_5) + \ldots + (t_{m-3} - t_{m-2}) + (t_{m-2} - t_{m-1}) = t_1 - t_{m-1},$$

hence $(u_1, u_{m-1}) \in \beta'_{i_0}$. This, together with the relations

$$(u_0, u_1) \in \beta'_{i_0},\ (u_{m-1}, u_m) \in \beta'_{i_0} \text{ implies } (x, y) = (u_0, u_m) \in \beta'_{i_0},$$

which is a contradiction. □

Lemma 5.4. $\alpha \vee \beta = A \times A$.

Proof. To prove that $\alpha \vee \beta$ is the greatest quasiorder of Quord (A, f) we must show that $(x, y) \in \alpha \vee \beta$ for every $x, y \in A$. We will prove that there are $m \in \mathbb{N} \cup \{0\}$ and a chain of elements $x = u_0,\ u_1,\ u_2,\ \ldots,\ u_m = y$ of the set A such that

$$\text{either } (u_k, u_{k+1}) \in \gamma \text{ or } (u_k, u_{k+1}) \in \alpha_j \vee \beta'_j \text{ for some } j \in J \tag{6}$$

is valid for each $0 \leq k < m$.

Assume that $x \neq y$. We will investigate the following four cases and we will use the previous cases for the proof of a new one (we omit the case symmetric to the third one, because these cases are similar):

1. $x \in C_1$, $y = f(x)$,
2. $i \in J$, $x, y \in C_i$,
3. $i \in J$, $x \in A_i$, $y \in C_i$,
4. $i, j \in J$, $x \in A_i$, $y \in A_j$.

Let the case 1 be valid. There is $k \in \mathbb{N}$ with $x = f^k(c_1)$. In view of Lemmas 5.2 and 5.1 we obtain

$$\begin{aligned} x = f^k(c_1)\ (\alpha_1 \vee \beta'_1)\ f^{k+q_1\frac{d_1}{d}}(c_1)\ \gamma\ f^{k+q_1\frac{d_1}{d}}(c_2)\ (\alpha_2 \vee \beta'_2) \\ f^{k+q_1\frac{d_1}{d}+q_2\frac{d_2}{d}}(c_2)\ \ldots\ (\alpha_s \vee \beta'_s)\ f^{k+q_1\frac{d_1}{d}+q_2\frac{d_2}{d}+\cdots+q_s\frac{d_s}{d}}(c_s) \\ = f^{k+1+qn}(c_s) = f^{k+1}(c_s)\ \gamma\ f^{k+1}(c_1) = f(x) = y. \end{aligned}$$

Hence $x\ (\alpha \vee \beta)\ y$.

Assume that the case 2 occurs. Then $x = f^k(c_i)$, $y = f^l(c_i)$. By Lemma 5.2 and by the case 1,

$$\begin{aligned} x = f^k(c_i)\ \gamma\ f^k(c_1)\ (\alpha \vee \beta)\ f(f^k(c_1))\ (\alpha \vee \beta)\ f(f^{k+1}(c_1))\ \ldots \\ (\alpha \vee \beta) f^l(c_1)\ \gamma\ f^l(c_i) = y. \end{aligned}$$

Now let the case 3 be valid. Since β'_i is a complement to α'_i, it yields that $(x, y) \in \alpha'_i \vee \beta'_i$ and there exist $m \in \mathbb{N}$ and a chain $x = v_0, v_1, \ldots, v_m = y$ such that for each $0 \leq k < m$ either $(v_k, v_{k+1}) \in \alpha'_i$ or $(v_k, v_{k+1}) \in \beta'_i$ holds. If k is such that $(v_k, v_{k+1}) \in \alpha'_i$ and $(v_k, v_{k+1}) \notin \alpha_i$, then $v_{k+1} \in C_i$ and there is $v'_{k+1} \in C_i$

such that $(v_k, v'_{k+1}) \in \alpha_i$. By the case 2, $(v'_{k+1}, v_{k+1}) \in \alpha \vee \beta$. This implies that $x\ (\alpha \vee \beta)\ y$.

Finally, suppose that the case 4 holds. Using the case 3 (and the dual to it) we obtain

$$x\ (\alpha \vee \beta)\ c_i\ \gamma\ c_j\ (\alpha \vee \beta)\ y,$$

therefore $x\ (\alpha \vee \beta)\ y$.

□

Now, according to Theorem 4.13 and Lemmas 5.3, 5.4 the main result is obtained:

Theorem 5.5. *Let (A, f) be a monounary algebra. Then the conditions*

- *each connected component of (A, f) contains a cycle,*
- *there is $n \in \mathbb{N}$ such that each cycle of (A, f) has n elements,*
- *n is square-free,*
- *for each $a \in A$, the element $f(a)$ is cyclic*

are necessary and sufficient for the lattice Quord (A, f) *to be complementary.*

In [7] it was shown that the next theorem is valid.

Theorem 5.6. *Let (A, f) be a monounary algebra. The following conditions are equivalent:*

(i) *The lattice* Quord(A, f) *is modular.*

(ii) *The lattice* Quord(A, f) *is distributive.*

(iii) *Either $|A| \leq 2$ or (A, f) is connected and there exists a cycle C of (A, f) such that $|A| \leq |C| + 1$.*

According to the above results, we summarize:

Theorem 5.7. *Let (A, f) be a monounary algebra. The lattice* Quord (A, f) *is Boolean if and only if either $|A| \leq 2$ or (A, f) is connected with a cycle C of (A, f) such that $|A| \leq |C| + 1$ and $|C|$ is square-free.*

Example 5.8. *Let an algebra (A, f) be such that $A = \{0, a, b\}$, $a \neq b \neq 0$ and $f(a) = f(b) = f(0) = 0$. The lattice* Quord (A, f) *consists of all quasiorders on the set A. This lattice is complementary, but complements need not be unique. E.g., the both quasiorders $\beta_1 = I_A \cup \{(0, a), (0, b), (b, 0), (b, a)\}$ and $\beta_2 = I_A \cup \{(0, a), (0, b), (b, a)\}$ are complements to the quasiorder $\alpha = I_A \cup \{(a, 0)\}$ in the lattice* Quord(A, f). *This implies that the lattice* Quord (A, f) *is not Boolean.*

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