P. J. ŠAFÁRIK UNIVERSITY
FACULTY OF SCIENCE
INSTITUTE OF MATHEMATICS
Jesenná 5, 040 01 Košice, Slovakia

K. Cechlárová and M. Repiský

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On the structure of the core of housing markets*

Katarína Cechlárová and Michal Repiský
[1]Institute of Mathematics, Faculty of Science, P.J. Šafárik University,
Jesenná 5, 040 01 Košice, Slovakia
email: katarina.cechlarova@upjs.sk, michal.repisky@student.upjs.sk

Abstract. The model of a housing market, useful in analysis of several real-world markets with indivisible goods, like market for researchers, study places, kidneys for transplantation etc. was introduced in 1974 by Shapley and Scarf. As in such markets monetary compensations are often impossible or illegal, a suitable solution concept is provided by the core. It is known that each housing market has a nonempty core and a core allocation can be found using the famous Top Trading Cycles algorithm. However, little is known about the structure of the core. In this paper we explore its structure in some simple markets, derived from geometric representation of the agents and show that the size of the core may grow exponentially with their number. Further, we show that it is NP-hard to decide whether the core contains an allocation in which each agent is trading, even in the case with strict preferences and everybody prefering at most two houses to his endowment.

Keywords: Housing market, Core, Algorithm, NP-completeness

AMS classification: 91A12, 91A06, 68Q25; JEL: C71

1 Introduction

The model of a housing market was introduced in the seminal paper of Shapley and Scarf [26]. In a housing market there is a finite set of agents, each one owns one unit of a unique indivisible good (traditionally called a house) and wants to exchange it for another, more preferred one. The preference relation of an agent is a linearly ordered list (possibly with ties) of a subset of houses.

Housing market is a plausible model in many real situations. This applies to various barter exchange markets in Internet described in [4], to the market with subsidized public housing in China [28], to the design of school choice mechanisms [1] or to the search of kidneys for transplantation [25], [10]. If monetary compensations are not possible or illegal, a suitable solution notion appears to be the core.

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The core of a housing market consists of allocations such that no coalition of agents may, by suitably rearranging the houses they own, make all its members strictly better off compared to the allocation in question. A core allocation always exists and can be found by an algorithm, called the Top Trading Cycles algorithm (TTC for short), attributed to Gale (see [26]).

A stronger notion is the strong core: an allocation is in the strong core if no coalition can weakly improve, i.e. its members cannot rearrange their houses in such a way that at least one member strictly improves while no member gets worse off. In the case without ties strong core always contains a unique allocation, namely the one produced by the TTC algorithm [23], but it may be empty if the agents' preferences admit ties. Quint and Wako [22] provided a polynomial algorithm for deciding the nonemptyness of the strong core in the general case. Another algorithm using a representation of a housing market by a directed graph was designed by Cechlárová and Fleiner in [11].

The structure of the core of a housing market has not been completely described yet, although some partial results have been obtained. The aim of the present paper is to make further steps in this direction. First we derive some structural properties in models where preferences of agents are derived from their geometric representation: the agents are represented by points on a line or on a circle and each agent prefers houses that are closer to him to houses that are farther. (In economic literature, such preferences are often called *single peaked*, see e.g. Brams et al. [9]). We show that in the linear case the TTC algorithm can output only allocations with trading cycle length at most 2, but still there are exponentially many such allocations. For the circular market, the TTC algorithm can produce longer cycles, but at most one for each allocation and the possible length of such a cycle is determined by the number of agents. On the other hand, the core of a linear as well as a circular market contains also exponentially many allocations that cannot be generated by the TTC algorithm.

Core allocations ensure stability, so they are a good choice in situations when it is desirable to avoid deviations of coalitions of agents. However, one can also put some other criteria on the obtained allocation. First, it is quite natural that everybody wishes to get a house that is as high in their preference list as possible. Further, longer cycles require a much more complicated logistics and can also suffer from various unexpected difficulties. For example, a very persuading and amusing artistic realisation of problems arising when people really move houses can be seen in a Czech film called Kulový Blesk (Ball Lightning, 1979), directed by Zdenek Podskalský and Ladislav Smoljak, describing a twelve-fold circular exchange of houses. In the context of kidney exchanges, the current practice is to carry out all the surgeries in an exchange cycle simultaneously, since a donor's willingness to donate a kidney might change once her intended recipient has received a successful transplant [25]. This means that in the case of a k-exchange, $2k$ simultaneous surgeries are necessary. Due to these difficulties, usually only 2-exchanges, in some cases also 3-exchanges [8] are considered. Nevertheless, the longest recorded simultaneous exchange was performed at Johns Hopkins Hospital in Maryland, U.S.A., in 2008 and involved 6 kidney donors and 6 recipients [29].

Further, real markets often involve huge numbers of participants (to name a few: 80 000 families seeking a new house in Beijing [28], over 90 000 patiens waiting for a kidney transplant [25], more than 37 000 participants in the National Residents Matching Program in the USA (see www.nrmp.org, retrieved on December 15, 2010, data concerning the year 2010, see also [24]), so it is very important to be able to find an allocation with the desired property efficiently. A suitable formalism is provided by the theory of computational complexity. Namely, an algorithm is usually considered efficient, if its number of steps can be expressed as a polynomial function of the size of the instance in question. On the other hand, a proof of NP-hardness of a problem is a very strong evidence to support the intractability of the problem.

There are several intractability results concerning the existence of some special allocation in the core of a housing market. Abraham, Blum and Sandholm [4] showed that to decide the existence of an allocation such that each agent is trading, moreover, each trading cycle is of length at most k, in the model where all the acceptable houses are tied in agents' preference lists is an NP-complete problem for each $k \geq 3$. Biró, Manlove and Rizzi [8] proved several inapproximability results for the problem of finding maximum number of trading agents on cycles of length not exceeding the given constant k. In the case with strict preferences, Irving [18] showed that it is NP-complete to decide whether the core contains a matching and Biró and Mc Dermid [7] proved that it is NP-complete to decide the existence of a core allocation where all agents trade on cycles of length exactly 3. Our intractability results extend those mentioned above. We prove that it is NP-complete to decide the existence of core allocations that have restricted trading cycle lengths or that maximize the agents' satisfaction even in the strict preferences case when each agent finds at most two houses acceptable.

2 Related work

To stress the importance of cycle lengths used in the trading, a modification of the housing market model where these lengths are incorporated directly into agents' preferences was suggested by Cechlárová and Romero Medina in [14] and Cechlárová, Fleiner and Manlove [10] studied the application of this model in the context of kidney exchanges. Here, an agent prefers one allocation to another one if he gets a more preferred house, but also if the house he gets is equally good but his trading cycle is shorter. In this model the TTC algorithm always finds a strong core (and hence also a core) allocation, if the agents are not allowed to express ties in their preferences over houses. However, Cechlárová and Hajduková [12] proved that in the presence of ties it is NP-complete to decide whether the core as well as the strong core are nonempty. Cechlárová and Lacko [13] showed that the following problems are NP-complete in this model already in the case without ties: Does the core contain an allocation such that everybody is trading on a shorter cycle than he is trading in the TTC allocation? Does the core contain an allocation with all trading cycles having length at most 3? Does the core contain an allocation in which all agents are trading? Further, Biró and Cechlárová [6] showed that it is even hard to devise a polynomial algorithm for finding a core allocation where the number of trading agents is within the $n^{1-\varepsilon}$ bound from the maximum.

Konishi, Quint and Wako [20] studied a model with several types of goods. In this model, each agent owns and wishes to get one unit of each type (e.g. one house, one car and one boat in the case of three types). The core may be empty already in the case of just two types of goods. Cechlárová [15] showed that if the number of agents is 2, a complete description of the core can be found efficiently, irrespectively of the number of good types. However, when the number of agents is not restricted, the problem to decide the nonemptyness of the core becomes NP-hard already for two types of goods.

Finally, let us also mention another related model, used especially for studying markets for school places or working positions. In the two-sided matching model ([24], [2]) the set of agents consists of two disjoint sets (schools and students, firms and workers, men and women etc.) and each agent wants to trade with agents from the other side of the market. Agents are characterized by their preferences over agents from the other side of the market as well as by their capacity (the number of study places offered by the school, the number of working positions at the firm etc.). The number of core allocations may grow exponentially with the number of agents, but the core exhibits a lattice structure, which can be fully described in a compact form and this representation allows to find core allocations with some properties (egalitarian or maximum weight) efficiently, see the monograph by Irving and Gusfield [17]. However, Irving [19] proved that the problem to decide the existence of core allocations such that no two agents of the same side prefer to switch their assignments, is NP-complete.

The structure of the present paper is as follows. In Section 3 we formally introduce the studied model. Section 4 is devoted to the description of the core structure of housing markets where agents have *single peaked* preferences derived from their geographic location (on a line or on a circle). In this section we use the combinatorial properties of such markets and also show some interesting connections with recurrent sequences. Finally, the hardness results are contained in Section 5. Our results and the results of previous authors imply that a satisfactory succinct description of the core structure of a general housing markets is improbable.

3 Description of the model

Let A be a *set of n agents*. Each agent owns one unique indivisible good, typically called a house. Hence, by a slight abuse of notation we can identify agents and their houses. Preferences of agent a are given in the form of a linear preference list $P(a)$. The notation $i \succeq_a j$ means that agent a *prefers* house i to house j. If $i \succeq_a j$ and simultaneously $j \succeq_a i$, we say that houses i and j are *tied* in a's preference list and write $i \sim_a j$. Notation $i \succ_a j$ means that agent a prefers i to j *strictly*, i.e. $i \succeq_a j$ but $j \succeq_a i$ does not hold. We denote by $H(a)$ the set of houses $j \in A$ such that $j \succeq_a a$, $H(a)$ will be called the set of *acceptable* houses for agent a. (If the agent a is clear from the context, the subscript will be omitted.) The n-tuple of preferences $(P(a), a \in A)$ will be denoted by $\mathcal{P}$ and called the *preference profile*.

A *housing market* is the pair $\mathcal{M} = (A, \mathcal{P})$. We say that $\mathcal{M}$ is a housing market with *strict preferences* if there are no ties in $\mathcal{P}$.

$\mathcal{M}(S)$ denotes a *submarket* of $\mathcal{M}$ restricted to a subset S of agents defined in the straightforward way. The set of the most preferred houses for agent $a \in S$ according to the submarket $\mathcal{M}(S)$ is denoted by $f_S(a)$.

A bijection $x : S \to S$ for $S \subseteq A$ is an *allocation* on S. In the whole paper we consider only *individually rational* allocations, i.e. such that $x(a) \in H(a)$ for each $a \in S$. Notice that for each allocation x the set of agents can be partitioned into directed cycles (*trading cycles*) of the form $K = (a_0, a_1, \ldots, a_{\ell-1})$ in such a way that $x(a_i) = a_{i+1}$ for each $i = 0, 1, \ldots, \ell-1$ (indices for agents taken modulo ℓ). We say that agent *a is trading* in allocation x if $x(a) \neq a$, otherwise he will be said to be *single*. An agent a is said to be *happy in allocation x*, if $x(a) \in f_A(a)$. Otherwise he is *unhappy*.

We say that coalition $S \subseteq A$ *blocks* an allocation x on A, if there exists an allocation y on S such that $y(a) \succ_a x(a)$ for each $a \in S$. Coalition $S \subseteq A$ *weakly blocks* an allocation x on A, if there exists an allocation y on S such that $y(a) \succeq_a x(a)$ for each $a \in S$ and $y(a) \succ_a x(a)$ for some $a \in S$.

The *core* of a housing market $\mathcal{M}$, denoted by $Core(\mathcal{M})$, is the set of all allocations on A, which admit no blocking coalition. The *strong core*, $SCore(\mathcal{M})$, is the set of all allocations on A, which admit no weakly blocking coalition. It is known that the core of a housing market is always nonempty and contains allocations produced by the famous TTC algorithm (due to Gale [26]). For completeness, a description of the TTC algorithm in a pseudocode is given in Figure 1. Notice that the REPEAT-UNTIL cycle in Step 2 always terminates as A is finite and $a \in H(a)$ for each $a \in A$.

It is known that in the case with strict preferences the TTC algorithm outputs exactly the same trading cycles (perhaps in a different order) irrespectively of the arbitrary choices in Step 1. However, when ties are present, breaking them in different ways in Step 2 may lead to different TTC allocations. Moreover, even in the strict preferences case, $Core(\mathcal{M})$ may contain allocations that cannot be produced by the TTC algorithm – one such example was already given in [14]. As far as we know, a complete description of the core structure has not been obtained so far. In view of the results of this paper, we doubt that such a task could be satisfactorily fulfilled for general markets.

4 Special markets

The types of preferences used in this section model situations when agents can be associated with some geographic locations. Each agent finds all the houses in the market acceptable, prefers a closer house to a more distant one and is indifferent between houses at the same distance. (Note however that agent's own house is still his last choice.) In economic literature, such preferences are often called *single peaked* [9]. In the first model, the so-called *linear market*,

Input Housing market $\mathcal{M} = (A, \mathcal{P})$

Output Core allocation x as a collection of trading cycles

Step 0 $j := 0$

Step 1 IF $A = \emptyset$, THEN Step 4
ELSE set $i := 0$ and choose an arbitrary agent $a_i \in A$

Step 2 REPEAT
choose any $a_{i+1} \in f_A(a_i)$
$i := i + 1$
UNTIL $a_{i+1} = a_l$ for some l, $l \leq i + 1$

Step 3 $j := j + 1$
create the trading cycle $K_j = (a_l, a_{l+1}, \dots, a_i)$
reduce the market $\mathcal{M}$ by eliminating the agents from K_j
$A := A \setminus \{a_m : l \leq m \leq i\}$
Step 1

Step 4 output $K_1, K_2, \dots, K_j$

Figure 1: TTC algorithm

the agents are located on a line in equidistant points. We shall identify them with integers $1, 2, \dots, n$. In the *circular market* the agents are located in the vertices of a regular polygon. Sometimes we shall denote agent n by 0 to be able to perform computations modulo n (we shall often do it automatically without a special warning). $\mathcal{M}^L(n)$ and $\mathcal{M}^C(n)$ denote the linear and the circular market with n agents, respectively. A market which is either linear or circular will be called *geometric*.

Example 1 *Preferences of the agents in the geometric markets with $n = 4$ agents are given in Table 1 and Table 2.*

$P(1):$	$2 \succ 3 \succ 4 \succ 1$
$P(2):$	$1 \sim 3 \succ 4 \succ 2$
$P(3):$	$2 \sim 4 \succ 1 \succ 3$
$P(4):$	$3 \succ 2 \succ 1 \succ 4$

Table 1: Preferences in $\mathcal{M}^L(4)$

$P(1):$	$4 \sim 2 \succ 3 \succ 1$
$P(2):$	$1 \sim 3 \succ 4 \succ 2$
$P(3):$	$2 \sim 4 \succ 1 \succ 3$
$P(4):$	$1 \sim 3 \succ 2 \succ 4$

Table 2: Preferences in $M^C(4)$

It is easy to see that $Core(\mathcal{M}^L(4))$ consists of allocations

$$(1)\,(2,3,4)\,,(4)\,(3,2,1)\,,(1,2)\,(3,4)\,,(2,3)\,(1,4)\,,(1,2,3,4)\,,(4,3,2,1)\,,$$

while $Core(\mathcal{M}^C(4))$ contains only

$$(4,1,2,3)\,,(3,2,1,4)\,,(4,1)\,(2,3)\,,(1,2)\,(3,4)\,.$$

For brevity, we shall introduce some terminology and notation. A set of agents $S \subseteq A$ is said to be connected, if for any two distinct agents $i, j \in S$ each agent k between i and j also belongs to S; in the case of the circle, this condition is required only for one direction between the agents. For two distinct agents $i, j \in A$, $dist(i, j)$ denotes the smallest number of agents of A that are located between agents i and j (traversing the line or the circle, respectively); we set $dist(i, i) = \infty$. We keep $dist(i, j)$ the same even if the market $\mathcal{M}$ is restricted to some submarket $\mathcal{M}(Q)$. Two agents $i, j \in Q$ such that $dist(i, j)$ is the minimum of all pairs of agents in Q are called *close neighbours* in $\mathcal{M}(Q)$. A trading cycle containing exactly k agents will simply be a *k-cycle*, a 2-cycle with its members being close neighbours in a submarket $\mathcal{M}(Q)$ is a *tight cycle in $\mathcal{M}(Q)$*. If $\mathcal{K}$ is a collection of trading cycles of an allocation x, we denote the set of agents contained in these cycles by $\bigcup \mathcal{K}$. An allocation containing only 2-cycles and single agents is a *matching.*

Lemma 1 *Let $\mathcal{M} = (A, P)$ be a geometric market and $x \in Core(\mathcal{M})$. Then*

(i) x contains at most one single agent,

(ii) if i, j are close neighbours in $\mathcal{M}$ then $x(i) \in f_A(i)$ or $x(j) \in f_A(j)$.

(iii) if x is a matching, $\mathcal{K}$ a collection of some cycles of x and $|A \setminus \bigcup \mathcal{K}| \geq 2$, then x contains a tight cycle in the submarket $\mathcal{M}(A \setminus \bigcup \mathcal{K})$.

Proof. Cases (i) and (ii) follow from the definition of preferences in a geometric market. To prove (iii), let $Q = A \setminus \bigcup \mathcal{K}$ and let $i, j \in Q$ be such that $dist(i, j) = \delta$ is minimum among all pairs of agents in Q. If neither of i, j is trading with an agent at a distance equal δ, then $\{i, j\}$ blocks x. This and the assumption that x is a matching imply the existence of a tight cycle in $\mathcal{M}(Q)$. ∎

In Example 1, a little thought reveals that except for allocations $(1,2)\,(3,4)$ and $(2,3)\,(1,4)$ no other allocation in $Core(\mathcal{M}^L(4))$ can be obtained by the TTC algorithm. Similarly, the core of a circular market may contain allocations that the TTC algorithm cannot create, like $(1,2,3)(4,5)$ for $\mathcal{M}^C(5)$. Moreover, there are many allocations that cannot be obtained by the TTC algorithm. The following examples provides two constructions of such allocations.

Example 2 *Partition the set of agents A into connected sets $S_1, S_2, \ldots, S_m$, each of size at least two and at least one of size three or more. To create an allocation x, let the agents in each set $S_k = \{i, i+1, \ldots, i+l_k\}$ trade 'clockwise' on the cycle $K_k = (i, i+1, \ldots, i+l_k)$. Now suppose that $Z \subseteq A$ is a blocking set for x. Clearly, as except for the 'returning' agents $i + l_k$, all agents are happy in x, all the members of Z are returning agents. At least one of them, say $i + l_p$, must improve by trading 'anticlockwise', i.e. getting some agent $j + l_q$ who is 'before' his partition set; so in this case $dist(i, i+i_p) < dist(j+l_q, i+i_p)$ holds. Hence, no blocking set is possible.*

To obtain an exact number α_n of such allocations for the geometric market with n agents, one can use techniques described in [16], for our purposes, however, it will be sufficient to derive a simple lower bound by creating a subset

$\mathcal{A}_n$ of such allocations. First notice that $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha_4 = 1$. Now, for $n \geq 5$ take any allocation from $\mathcal{A}_{n-1}$ and modify it to obtain an allocation in $\mathcal{A}_n$ by inserting agent n into the cycle containing agent $n-1$. From each allocations in $\mathcal{A}_{n-2}$ an allocation in $\mathcal{A}_n$ will be created by adding the cycle $(n-1, n)$. Now it is clear that $|\mathcal{A}_n| = F(n-2)$, where the sequence $\{F(n)\}_{n=0}^{\infty}$ defined by

$$F(0) = 0,\ F(1) = 1,\ F(n) = F(n-1) + F(n-2) \text{ for } n \geq 3$$

is the famous Fibonacci sequence (see [16], page 73).

Example 3 *Let $n \geq 5$ and $m = \lfloor \frac{n}{5} \rfloor$. Let us partition the set of agents A into successive alternating triples and pairs, i.e. $A = \{1, 2, 3\} \cup \{4, 5\} \cup \{6, 7, 8\} \cup \{9, 10\} \cup \dots$, ending with a set of size 3 followed by one or several pairs, depending on n modulo 5. Let the agents in pairs trade with each other, thus creating tight cycles. The agents in triples have two possibilities to trade: either clockwise along the cycle $(i, i+1, i+2)$ or anticlockwise creating the cycle $(i+2, i+1, i)$. To see that such allocations are in the core, let us realize that each 3-cycle contains exactly one unhappy agent, who got his second choice agent. However, his both neighbours got their most preferred agent, so no blocking coalition can arise. The number of such allocations is 2^m.*

In what follows, we shall turn back to the study of the TTC allocations.

Lemma 2 *Let $\mathcal{M}$ be a geometric market and $x \in Core(\mathcal{M})$ a matching. Then x can be obtained by a suitable realisation of the TTC algorithm.*

Proof. Lemma 1 with $\mathcal{K} = \emptyset$ implies that there is a tight cycle $K_1 = (i_1, j_1)$ in $\mathcal{M}$, that means $i_1 \in f_A(j_1)$ and $j_1 \in f_A(i_1)$. Let i_1 be chosen in Step 1 and let i_1 and j_1 choose each other in Step 2 of the TTC algorithm to create K_1. Suppose that we have already created cycles $K_1 = (i_1, j_1), \dots, K_p = (i_p, j_p)$ of x. Let $Q = A \setminus \bigcup\{K_1, \dots, K_p\}$. Again, using Lemma 1$(iii)$, x contains a cycle (i_{p+1}, j_{p+1}), which is tight in $\mathcal{M}(Q)$, hence $i_{p+1} \in f_Q(j_{p+1})$ and $j_{p+1} \in f_Q(i_{p+1})$. This cycle can be created as the $(p+1)$-st cycle in the TTC algorithm by choosing agent i_p in Step 1 and by letting these agents point to each other in Step 2. The assertion follows by induction. ■

To get all the TTC allocations in a geometric market one can break the ties arbitrarily and run the algorithm, which now outputs a unique allocation. However, this would be a very inefficient approach as the number of markets with strict preferences obtained in this way is too high. Consider linear markets at first. Agents 1 and n have no ties in their preference lists, but agents 2 and $n-1$ have one tie, hence 2 possibilities to break it. Agents 3 and $n-2$ have 2 ties, so 2^2 possibilities and so on, up to agent $\frac{n+1}{2}$, who has $\frac{n-1}{2}$ ties if n is odd and to agents $\frac{n}{2}$ and $\frac{n}{2}+1$ with $\frac{n-2}{2}$ ties for n even. Hence the total number of different strict preferences markets derived from $\mathcal{M}^L(n)$ is equal to

$$\left(2 \times 2^2 \times 2^3 \times \dots \times 2^{(n-3)/2}\right)^2 \times 2^{(n-1)/2} = 2^{(n-1)(n-1)/4} \text{ if } n \text{ is odd and}$$

$$\left(2 \times 2^2 \times 2^3 \times \dots \times 2^{(n-2)/2}\right)^2 = 2^{(n-2)n/4} \text{ if } n \text{ is even.}$$

The number of all possible strict circular markets generated by breaking all ties in $\mathcal{M}^C(n)$ can be derived in a similar way. For n odd there are $2^{n(n-1)/2}$ different strict circular markets, for n even their number is $2^{n(n-2)/2}$. However, we will see later that the number of the TTC allocations is much smaller.

The following assertions already show some differences between the linear and the circular market.

Theorem 1 *Any application of the TTC algorithm to a linear market outputs a matching.*

Proof. Let x be a TTC allocation and let a trading cycle $K = (i, j, \dots, k) \in x$ be such that i, j, k are mutually different. Without loss of generality we can suppose that i is minimum of all agents in K. In particular, $i < k$. The definition of the TTC algorithm implies that $i \in f_K(k)$, so $j > k$. This means that agent i prefers agent k to agent j, hence cycle K cannot be produced by the TTC algorithm. ■

To obtain the number $T(n)$ of TTC allocations for $\mathcal{M}^L(n)$, let us first consider agents 1 and 2. Since they are close neighbours in $\mathcal{M}^L(n)$, Lemma 1 (*ii*) implies that there are just two possibilities for a matching x in the core:

(*i*) x contains cycle $(1, 2)$. We can suppose that this cycle was obtained as the first cycle during the application of the TTC algorithm to $\mathcal{M}^L(n)$. The remaining course of the TTC is the same as its application to $\mathcal{M}^L(n-2)$, so the number of TTC allocations in the market $\mathcal{M}^L(n)$ containing cycle $(1, 2)$ is equal to $T(n-2)$.

(*ii*) If x does not contain cycle $(1, 2)$ then it necessarily contains cycle $(2, 3)$. We show that if y is any TTC allocation of the linear market $\mathcal{M}^L(n-3)$ with agents denoted by $4, \dots, n$, then y corresponds to a unique TTC allocation on $\mathcal{M}^L(n)$ containing cycle $(2, 3)$. Let $x = (1)(2, 3)y$. If $x \in Core(\mathcal{M}^L(n))$ then we are done. Otherwise there is an agent j such that y contains either a cycle (j) or a cycle (j, l) such that $j < l$ and $dist(j, 1) < dist(j, l)$. Let us take the minimum j with this property and modify x by adding cycle $(1, j)$ and either creating cycle (l) or cycle (l, s) if there was a single agent $s > l$ in y. It is easy to see that the modified x can be obtained by the TTC algorithm applied to $\mathcal{M}^L(n)$ (just create cycles $(1, j)$ and (l) or (l, s) as the last two cycles). Hence the number of TTC allocations containing $(2, 3)$ is at least $T(n-3)$.

The above arguments lead to the following recursive inequality

$$T(n) \geq T(n-2) + T(n-3) \text{ for } \ n \geq 4, \tag{1}$$

while we know that $T(1) = 1$, $T(2) = 1$ and $T(3) = 2$.

Replacing the inequality in (1) by equality, the well-known Padovan sequence (see [27]) will be obtained. Hence we have

Lemma 3 *The number $T(n)$ of TTC allocations of $\mathcal{M}^L(n)$ increases as least as fast as the Padovan sequence.*

It is known that Padovan sequence increases exponentially, however, we shall show that this growth is of a much smaller order than the growth of the number of all possible linear markets with strict preferences generated by $\mathcal{M}^L(n)$. To derive the upper bound for $T(n)$, let us realize that each TTC allocation for $n = 2m$ agents can be represented as a collection of m arcs connecting m pairs of points on a line, drawn in such a way that the arcs do not intersect. The number of different ways to draw such arcs for n points is equal to the Catalan number C_m (see [21], page 211). Figure 2 illustrates the case $m = 3$, i.e. 6 agents. However, not all such arrangements correspond to core allocations; this is clearly the case in the first arrangement, where the set $\{1, 2\}$ will block the allocation.

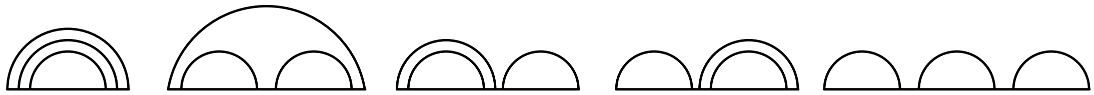


Figure 2: Representation of C_3

The following inequalities are straightforward to see, taking into account that for the odd number of agents exactly one (but arbitrary) agent remains single.

Lemma 4 *Let $m \geq 1$ be arbitrary. Then*

$$\begin{aligned} T(2m) &\leq C_m \\ T(2m-1) &\leq (2m-1)C_{m-1}. \end{aligned}$$

As the formula for Catalan numbers is $C_m = \frac{1}{m+1}\binom{2m}{m}$ (see [21], page 11), it is easy to show that $C_m \leq 4^m$ for each m. So we have

Lemma 5 *The number of TTC allocations for the market $\mathcal{M}^L(n)$ is bounded from above by 2^n if n is even and by $n2^n$ if n is odd.*

Does $Core\mathcal{M}^L(n)$ remain exponentially large in strict linear markets (notice that in this case there is only one TTC allocation). As a representative case let us take the linear market where all ties are broken in the favour of the agent with the bigger index. In this *directed linear market*, denoted by $\mathcal{M}^{DL}(n)$, agent i prefers agent j to agent k if $dist(i, j) < dist(i, k)$ or $dist(i, j) = dist(i, k)$ and $j > k$.

Theorem 2 *For any $n \geq 1$, the following holds:*

(i) $|Core(\mathcal{M}^{DL}(n))| \geq F(n)$.

(ii) $|SCore(\mathcal{M}^{DL}(n))| = 1$.

Proof. We show (i) by constructing a subset $\mathcal{F}(n) \subseteq Core(\mathcal{M}^{DL}(n))$ such that $|\mathcal{F}(n)| \geq F(n)$ for each n by induction.

First, it is easy to see that $|Core(\mathcal{M}^{DL}(n))| = 1$ for $n = 1, 2$ and $|Core(\mathcal{M}^{DL}(3))| = 2$. Let us set $\mathcal{F}(n) = Core(\mathcal{M}^{DL}(n))$ for $n = 1, 2, 3$. Now suppose that $n \geq 4$ and that we have already constructed $\mathcal{F}(k)$ for all $k < n$. Let us construct $\mathcal{F}(n)$ in the following way:

- Take the set $\mathcal{F}(n-2)$ and append the trading cycle $(n-1, n)$ to each allocation in $\mathcal{F}(n-2)$. The obtained allocations will belong to $Core(\mathcal{M}^{DL}(n))$ because $n-1$ and n are happy.

- For each allocation $x \in \mathcal{F}(n-1)$ get an allocation x' for $x\mathcal{F}(n)$in this way:

  $$x'(n-1) = n, \; x'(n) = x(n-1) \text{ and } \; x'(i) = x(i) \text{ for each } \; i \neq n, n-1,$$

  i.e. agent n was inserted into the trading cycle of x containing agent $n-1$. Should allocation x' admit a blocking set Z, then necessarily $n \in Z$ and $n-1 \notin Z$ as $n-1$ is happy in x', so $Z' = Z \setminus \{n\} \cup \{n-1\}$ would be a blocking set for allocation x. Hence allocation $x' \in Core(\mathcal{M}^{DL}(n))$.

To show (ii), let us realise that in each market with strict preferences the strong core consists of a unique TTC allocation [23]. It is easy to see that TTC outputs the allocation $(1,2)(3,4)\dots(n-1,n)$ for n even, and allocation $(1)(2,3)(4,5)\dots(n-1,n)$ for n odd. ■

Unlike linear markets, some circular may admit TTC allocations with longer cycles. However, their form is precisely determined by the number of agents. We say that a trading cycle $(i_0, i_1, \dots, i_{k-1})$ is *regular* in $\mathcal{M}^C(n)$, if $dist(i_j, i_{j+1}) = \frac{n}{k} - 1$ for each $j = 0, 1, \dots, k-1$. (Notice that n must be divisible by k to allow a regular cycle of length k.)

Lemma 6 *Let x be a TTC allocation in $\mathcal{M}^C(n)$. Then the following holds.*

(i) If x contains a k-cycle K for $k \geq 3$, then K is regular.

(ii) x contains at most one regular cycle.

(iii) If x contains a regular k-cycle K for $2 < k < n$ then $n = (2\ell + 1)k$; moreover, the 2ℓ agents between two consecutive agents on K trade on ℓ 2-cycles.

Proof. (i) Suppose that x contains a k-cycle $K = (i_0, i_1, \dots, i_{k-1})$ for some $k \geq 3$, which is not regular. Then there exists $j \in \{0, 1, \dots, k-1\}$ such that $dist(i_j, i_{j+1}) < dist(i_{j+1}, i_{j+2})$ (indices are taken modulo k, if necessary). Then agent i_{j+1} prefers agent i_j to i_{j+2}, so such a cycle could not have been obtained by the TTC algorithm.
(ii) Suppose that x contains two different regular cycles, say $K_1 = (i^1_0, i^1_1, \dots, i^1_{k-1})$ and $K_2 = (i^2_0, i^2_1, \dots, i^2_{l-1})$. Suppose further that K_1 was created before K_2 in the course of the TTC algorithm. Then there must exist j, s such that

$dist(i_j^1, i_{j+1}^1) > dist(i_j^1, i_s^2)$, so agent i_j^1 could not choose agent i_{j+1}^1 when agent i_s^2 was still available, hence the TTC algorithm could not produce K_1, a contradiction.
(iii) Suppose that x contains a regular k-cycle $K = (i_0, i_1, \ldots, i_{k-1})$. Since i_s chose i_{s+1} during the TTC algorithm, all the agents between i_s and i_{s+1} must have been unavailable at that moment. As K is the only cycle with length ≥ 3, it is clear that all these agents must be trading on 2-cycles, so their number must be even, say equal to 2ℓ. Now a simple counting argument and (ii) imply the assertion. ■

To get a better picture, Table 3 and Table 4 present all the possible types of TTC allocations in circular markets for some small values of n. Notice that a cycle with length $k, 2 < k < n$ appears for the first time for $n = 9$.

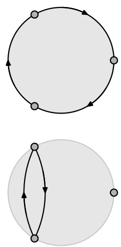
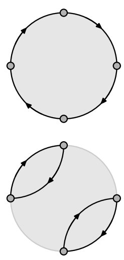
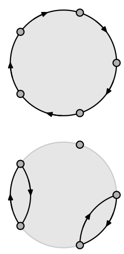
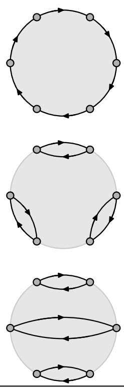
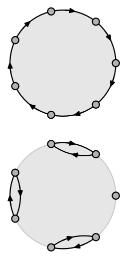
$\|A\| = n$	3	4	5	6	7
types of TTC allocations					

Table 3: Types of TTC allocations of $\mathcal{M}^C(n)$

Finally, unlike the core, the strict core of the geometric markets is very simple.

Theorem 3 *For $n \geq 2$, the strong core of a geometric market is nonempty if and only if n is even. In the case of the linear market the strong core contains the unique allocation $x = (1,2)\,(3,4)\ldots(n-1,n)$; the strong core of the circular market contains x and also allocation $x' = (2,3)\,(4,5)\ldots(n,1)$.*

Proof. It is easy to see that if i is unhappy then $\{i, i+1\}$ or $\{i, i-1\}$ is a weakly blocking set. Hence each agent must be happy in $x \in SCore\mathcal{M}^L(n)$. The only allocation making everybody happy is the one consisting solely of tight cycles. ■

5 Hardness results

In this section we return to the study of general housing markets. Extending the results of [4], [7], [8], [13] and [18], we explore the computational complexity of

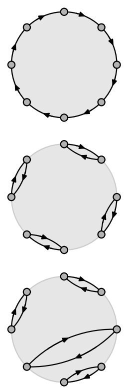
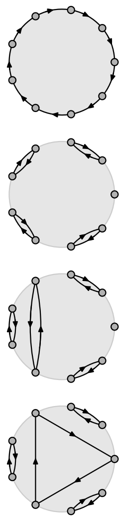
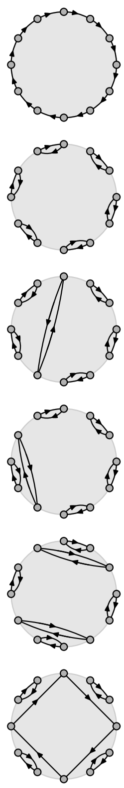
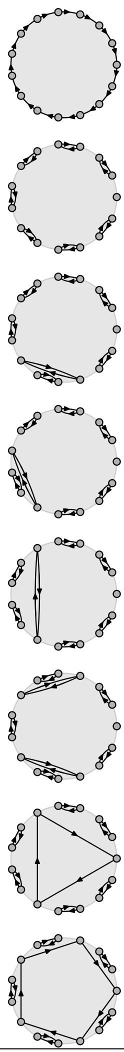
$\lvert A \rvert = n$	8	9	12	15
types of TTC allocations				

Table 4: Types of TTC allocations of $\mathcal{M}^C(n)$

some decision problems concerning the structure of the core. In each problem, the instance is a housing market $\mathcal{M}$.

Problem everybody trading
Does $Core(\mathcal{M})$ contain an allocation in which all agents are trading?

Problem k-choice core
Does $Core(\mathcal{M})$ contain an allocation x such that every agent of $\mathcal{M}$ has at worst his k-th preferred house?

Problem k-cycle core
Does $Core(\mathcal{M})$ contain an allocation x such that every trading cycle of x is of length at most k?

Given an allocation x in a market $\mathcal{M}$, it can be decided in polynomial time whether x belongs to the core: simply construct a directed graph $G = (V, E)$

(sometimes called the envy-graph, see [3]) whose vertices are agents and the pair (i, j) belongs to E if and only if agent i prefers agent j to $x(i)$. It is easy to see that $x \in Core(\mathcal{M})$ if and only if G is acyclic. This means that all the above problems belong to NP.

We prove that all the three problems are NP-complete. Moreover, our results are in a sense strongest possible, as they hold in a very restricted case: when each agent accepts at most two houses (except for his own) and is not indifferent between them. Namely, if each agents accepts only one house, these problems can be solved in polynomial time by constructing the directed graph of the first choices. This graph has a very special structure: each vertex has outdegree 1, so it is simply a collection of disjoint cycles with some paths leading to them.

Theorem 4 Everybody Trading *is NP-complete even for markets with strict preferences and with each agent having at most two preferred houses.*

Proof. We provide a polynomial reduction from a special case of 3-SAT called (3,B2)-SAT. NP-completeness of (3,B2)-SAT was proved by Berman, Karpinski and Scott in [5] and we give here the precise formulation of this problem.

Problem (3,B2)-SAT
INSTANCE: Boolean function B in CNF, in which each clause has exactly three literals and each variable occurs exactly twice nonnegated and twice negated.
QUESTION: Is B satisfiable?

For each instance B of (3,B2)-SAT we construct a housing market $\mathcal{M}(B) = (A, \mathcal{P})$ with strict preferences such that B satisfiable if and only if $Core(\mathcal{M}(B))$ contains such an allocation x, in which every agent is trading.

For each clause of B we will have a *clause cell* containing 6 agents and a *variable cell* containing one agent for each occurrence of a literal. The agents of $\mathcal{M}(B)$ and their preferences are given in Table 5 and Table 6 (the last choices of agents, their own houses, are omitted in preference lists). Agents ℓ_{i_1}, ℓ_{i_2}, ℓ_{i_3} correspond to the first, second and third position in clause C_i, respectively. Agent $v(\ell_{i_j})$ denotes the agent of the variable cell corresponding to the literal ℓ_{i_j}. Symbols $\ell(v^j)$ and $\ell(\bar{v}^j)$ correspond to the agents v^j, $\bar{v}^j$ of a clause cell.

This completes the construction of housing market $\mathcal{M}(B)$. In Figure 3 a clause cell and a variable cell are illustrated. First choices of agents are represented by thick lines, second choices by thin lines. Note that agents $\ell(v^j)$ and $\ell(\bar{v}^j)$ are some ℓ-agents corresponding to positions containing the respective literals.

Lemma 7 *If B is satisfiable, then there exists an allocation $x \in Core(\mathcal{M}(B))$, where every agent of $\mathcal{M}(B)$ is trading.*

Proof. Since B is satisfiable every clause has at least one true literal. Consider the clause cell corresponding to C_i. Let l_{i_j} correspond to a true literal in C_i. We let l_{i_j} trade on cycle $\big(l_{i_j}, v(l_{i_j}), f_A(v(l_{i_j}))\big)$. Such a cycle contains one agent

agent	preference list
c_{i_1} :	$c_{i_2} \succ \ell_{i_2}$
c_{i_2} :	$c_{i_3} \succ \ell_{i_3}$
c_{i_3} :	$c_{i_1} \succ \ell_{i_1}$
ℓ_{i_1} :	$v(\ell_{i_1}) \succ c_{i_1}$
ℓ_{i_2} :	$v(\ell_{i_2}) \succ c_{i_2}$
ℓ_{i_3} :	$v(\ell_{i_3}) \succ c_{i_3}$

Table 5: Preferences in the cell of C_i

agent	preference list
v^1 :	$\bar{v}^2 \succ \ell(\bar{v}^1)$
v^2 :	$\bar{v}^1 \succ \ell(\bar{v}^2)$
$\bar{v}^1$:	$v^1 \succ \ell(v^2)$
$\bar{v}^2$:	$v^2 \succ \ell(v^1)$

Table 6: Preferences in the cell of v

of a clause cell and two agents of a variable cell – we will call this type of trading cycle a triangle. Agents corresponding to false literals will form a trading cycle with agents c_{i_1}, c_{i_2} and c_{i_3} according to Figure 4. Here, the solid lines represent trading, dotted lines lead to other acceptable choices of agents. Figure 4a) shows trading of agents when l_{i_1} is true, in 4b) the true literals are l_{i_1} and l_{i_2} and 4c) corresponds to the case when all 3 literals of the clause are true. As this construction is symmetric, all the possible cases are thus covered.

For every variable v either v^1 and v^2 are true or $\bar{v}^1$ and $\bar{v}^2$ are true. This means that every agent of a variable cell is trading in x on a triangle. Figure 4 d) shows trading of variable cell agents in the case that v^1 and v^2 are true. It remains to show, that $x \in Core(\mathcal{M}(B))$.

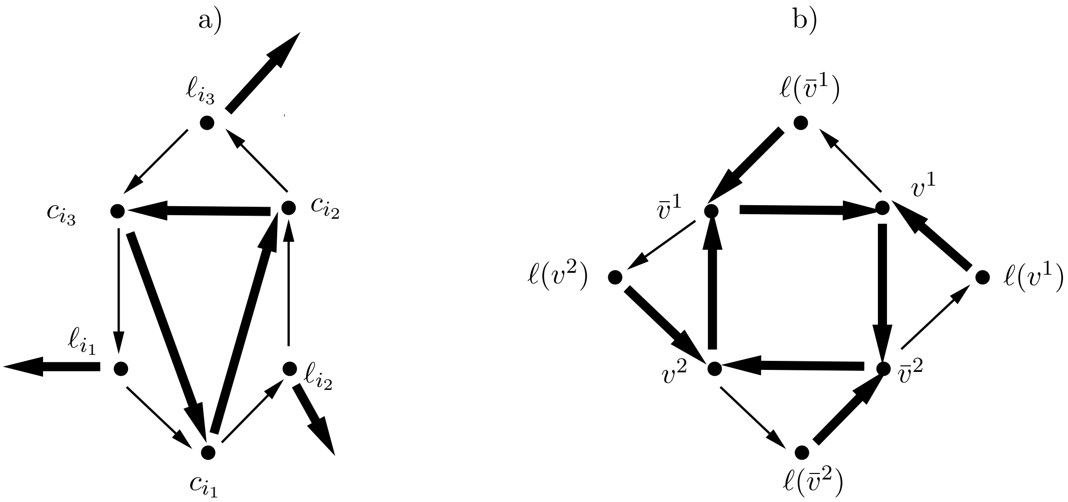


Figure 3: Clause cell and variable cell

Consider the variable cell corresponding to a variable v. W.l.o.g. assume that v^1, v^2 are true. Agents v^1, v^2 have their first choices in x, so they will not belong to any blocking coalition. Agents $\bar{v}^1$ and $\bar{v}^2$ could only improve by getting v^1, v^2, respectively, but v^1, v^2 cannot strictly improve. This means no variable agent will be a member of a blocking coalition of x.

Now we consider a clause cell C_i. First assume that exactly one literal of

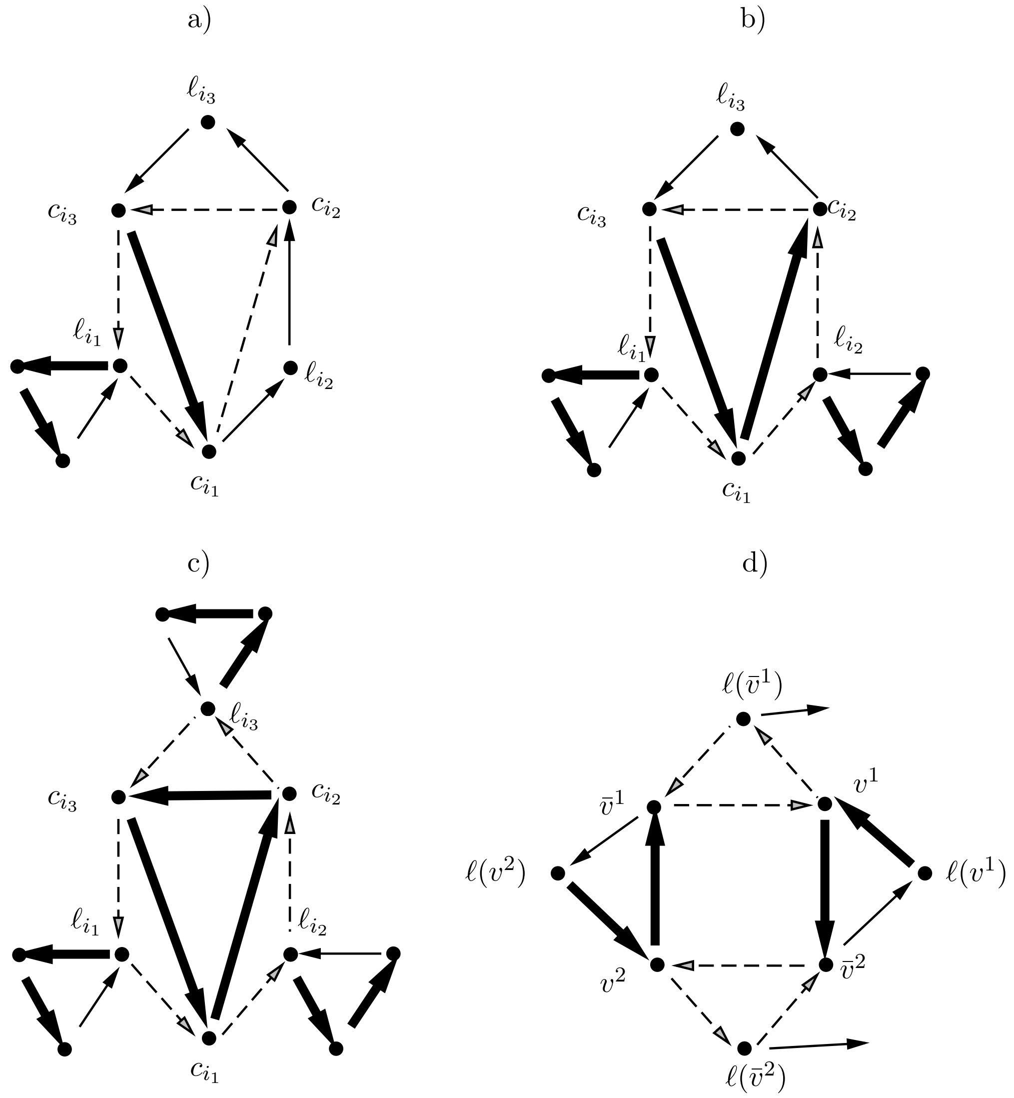


Figure 4: Possible trading of clause and variable cell agents

C_i is true, w.l.o.g. let it be ℓ_{i_1}. Agent ℓ_{i_1} has his first choice. Agents ℓ_{i_2}, ℓ_{i_3} could improve by getting a variable cell agent, but we showed before that no variable cell agent is a member of a blocking coalition. Agent c_{i_3} has his first choice. Agents c_{i_1}, c_{i_2} could improve only on the trading cycle $(c_{i_1}, c_{i_2}, c_{i_3})$, but c_{i_3} would not improve. So in this case no clause cell agent is blocking. The cases of 2 and 3 true clause literals can be analyzed similarly.

Hence no blocking coalition exists for allocation x, which concludes the proof. ■

Lemma 8 *If there exists an allocation $x \in Core(\mathcal{M}(B))$, where every agent of $\mathcal{M}(B)$ is trading, then B is satisfiable.*

Proof. Let x be a core allocation of $\mathcal{M}(B)$, where every agent is trading. First we show that no trading cycle can contain agents from 2 different clause cells. Consider clause cell C_i. Agents c_{i_1}, c_{i_2}, c_{i_3} can trade only with agents of C_i, so a cycle containing agents from C_i and some $C_j \neq C_i$ (we will call such a cycle an *intercell cycle*) has to use literal agents to enter and exit the clause cell.

An intercell cycle Γ must contain exactly one literal agent of each clause cell. To see this, suppose that Γ uses at least 2 literals of C_i. Suppose that w.l.o.g. (clause cells are symmetric) the intercell cycle Γ enters C_i via agent ℓ_{i_1}. Thens Γ goes to c_{i_1} to reach ℓ_{i_2} or ℓ_{i_3} and exit C_i. This means that in x both acceptable agents for c_{i3}, namely ℓ_{i_1} and c_{i_1}, are already trading on Γ with other agents, so c_{i_3} cannot trade.

Hence the only possible intercell cycle is of the form

$$\Gamma = \left(\ell(v^1), v^1, \ell(\bar{v}^1), \bar{v}^1, \ell(v^2), v^2, \ell(\bar{v}^2), \bar{v}^2\right).$$

However, allocation containing Γ would be blocked by $(v^1, \bar{v}^2, v^2, \bar{v}^1)$. (See Figure 3 b).

Now we show that at least one literal agent from every clause cell is trading with an agent from a variable cell. Suppose that every agent of a clause cell C_i is trading only with an agent of C_i. Then the only possible way would be trading cycle $\Delta = (\ell_{i_1}, c_{i_1}, \ell_{i_2}, c_{i_2}, \ell_{i_3}, c_{i_3})$. However, allocation containing Δ would be blocked by $(c_{i_1}, c_{i_2}, c_{i_3})$. Since only literal agents can trade with variable cell agents, we get that at least one of literal agent is trading with a variable cell agent.

Assume w.l.o.g. that literal $\ell(v^1)$ from some C_i is trading with variable cell agents. Since no intercell cycle exists, agent $\ell(v^1)$ must trade on the triangle $(l(v^1), v^1, \bar{v}^2)$. The remaining 2 variable cell agents then must trade on triangle $(l(v^2), v^2, \bar{v}^1)$ (see Figure 4 d). So we have: Literal $\ell(v^1)$ is trading on triangle $(l(v^1), v^1, \bar{v}^2)$ if and only if literal $\ell(v^2)$ is trading on triangle $(l(v^2), v^2, \bar{v}^1)$, literal $\ell(\bar{v}^1)$ is trading on triangle $(l(\bar{v}^1), \bar{v}^1, v^2)$ if and only if literal $\ell(\bar{v}^2)$ is trading on triangle $(l(\bar{v}^2), \bar{v}^2, v^1)$. If no agent of the variable cell is trading with clause agents, then the agents are trading on the cycle $(v^1, \bar{v}^2, v^2, \bar{v}^1)$.

We now construct a truth assignment for B. If in allocation x there are triangles $(l(v^1), v^1, \bar{v}^2)$ and $(l(v^2), v^2, \bar{v}^1)$ we set variable v to be TRUE. If there are triangles $(l(\bar{v}^1), \bar{v}^1, v^2)$ and $(l(\bar{v}^2), \bar{v}^2, v^1)$ we set variable v to be FALSE. If x contains cycle $(v^1, \bar{v}^2, v^2, \bar{v}^1)$ we can choose randomly for v to be TRUE or FALSE. This is a correct truth assignment since both occurrences of v or $\bar{v}$ are either true or false. Now we claim that this truth assignment satisfies B. We have shown before that every clause cell has at least one literal agent trading with a variable cell. According to our truth assignment this literal is set to be TRUE. Hence every clause contains a TRUE literal and B is satisfied. ■

Lemma 7 and lemma 8 complete the proof of Theorem 4. ■

Since in the allocation x constructed in proof of Theorem 4 every agent receives at worst his second choice and every trading cycle has length at most 5, we get the following interesting corollaries.

Corollary 1 2-Choice Core *is NP-complete even for markets with strict preferences and with each agent having at most two preferred choices.*

Corollary 2 5-Cycle Core *is NP-complete even for markets with strict preferences and with each agent having at most two preferred choices.*

6 Conclusion

The housing market model is a basis for the study of the exchange of indivisible goods. For its extensive applications in real markets, which often do not allow monetary transfers, the concept of the core plays a crucial role. However, in spite of the classical positive results, ensuring the nonemptyness of the core, its complete structure has so far not been described. In this paper we succeeded to obtain quite a detailed picture for the special cases of markets with single peaked preferences, corresponding to agents located in equidistant points on a line or on a circuit. On the other hand, we proved some intractability results showing that this task is computationally really difficult.

Possible extensions of this research can be seen in the applications of the parameterized complexity approaches. Further, given an allocation, the agents of a blocking set must find each other and coordinate their action to be able to really improve, which might be difficult if their number is big. So when seeking allocations of a special form, one might weaken the stability requirement and look for allocations which admit a very small number of agents who could improve or conversely, for allocations that require a very large number of agents to ensure an improvement for each of them.

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