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Approximability of Economic Equilibrium for Housing Markets With Duplicate Houses*

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Abstract

In a modification of the classical model of housing market which includes duplicate houses, economic equilibrium might not exist. As a measure of approximation the value $\mathrm{sat}(\mathcal{M})$ was proposed: the maximum number of satisfied agents in the market $\mathcal{M}$, where an agent is said to be satisfied if, given a set of prices, he gets a most a most preferred house in his budget set. Clearly, market $\mathcal{M}$ admits an economic equilibrium if $\mathrm{sat}(\mathcal{M})$ is equal to the total number n of agents, but $\mathrm{sat}(\mathcal{M})$ is NP-hard to compute.

In this paper we give a 2-approximation algorithm for $\mathrm{sat}(\mathcal{M})$ in the case of trichotomic preferences. On the other hand, we prove that $\mathrm{sat}(\mathcal{M})$ is hard to approximate within a factor smaller than 21/19, even if each house type is used for at most two houses. If the preferences are not required to be trichotomic, the problem is hard to approximate within a factor smaller than 1.2. We also prove that, provided the Unique Games Conjecture is true, approximation is hard within factors 1.25 for trichotomic preferences, and 1.5 in the case of general preferences.

1 Introduction

A housing market consists of a finite set of agents and a finite set of houses. Each agent owns one house, considers some other houses acceptable and orders

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them according to their desirability. The aim of each agent is to get the house he finds the best possible. This model was introduced by Shapley and Scarf in [13], where the notion of the economic equilibrium in such markets was considered and its existence in all housing markets proved. A polynomial-time algorithm for finding an economic equilibrium, called the Top Trading Cycles (TTC for short) algorithm was attributed to Gale (see [13]). An asymptotically optimal implementation of the TTC algorithm was proposed in [2].

The above results rely substantially on the assumption that each agent's house is unique. A modified version of the basic model, in which some houses are equivalent (and so must have the same price), was proposed by Fekete et al. [9]. In this model it may happen that the economic equilibrium does not exist. Fekete et al. proved that it is even NP-hard to decide its existence.

Cechlárová and Fleiner [4] further narrowed the dividing line between easy and difficult cases. They proved that if agents have strict preferences over house types, a polynomial-time algorithm decides the existence of an economic equilibrium. An efficient optimal implementation of their algorithm was proposed in [5]. On the other hand, Cechlárová and Fleiner showed that the problem remains NP-complete even if each agent distinguishes only three classes of house types: desired houses, houses of the same type as his original house and unacceptable houses. They call such preferences *trichotomic*.

In general markets with indivisible goods (i. e., each agent may own several units of each good) it has been known for many years that equilibrium might not exist. A recent result of Deng, Papadimitriou and Safra states that even in the case with linear utility functions the problem to decide its existence is NP-complete [7]. These authors studied the so-called ε-approximate equilibrium, i. e., such that the market clears approximately (at most ε units of each good remain unsold) and each agent obtains a commodity bundle such that its utility is within a factor of $(1-\varepsilon)$ from the optimum in his budget set.

Cechlárová and Fleiner [4] proposed a different notion of approximate equilibrium for housing markets with duplicate houses. They studied the deficiency of housing markets, i. e., the minimum number of agents that cannot get a most preferred house in their budget set.

Cechlárová and Schlotter [6] examined deficiency from the parameterized complexity viewpoint. They proved that the deficiency problem is NP-hard even in the case when each agent prefers only one house type to his own, and the maximum number of houses of the same type is two. They further proved this problem to be W[1]-hard with the parameter α describing the desired value of deficiency, and fixed-parameter tractable when parameterized by the number of distinct house types.

In this paper, we focus on the approximability of the equilibrium in housing markets with duplicate houses when preferences contain ties, and in particular, in the case with trichotomic preferences. Technically, instead of minimizing the number of unsatisfied agents, we shall maximize $\mathrm{sat}(\mathcal{M})$, i. e., the number of satisfied agents in a housing market $\mathcal{M}$. In Section 3 we study bounds for $\mathrm{sat}(\mathcal{M})$ in trichotomic markets and present a 2-approximation algorithm. In Section 4 we prove that the problem is NP-hard to approximate within a factor smaller than 21/19, even if the maximum number of houses of the same type is two, and agents endowed with a house of the same type have the same preference lists. We further prove that if the preferences are not required to be trichotomic, then $\mathrm{sat}(\mathcal{M})$ is NP-hard to approximate within a factor smaller than 1.2.

Assuming that the Unique Games Conjecture of Khot [11] is true, we obtain stronger innaproximability results, also in the case of trichotomic preference lists whose lengths are bounded by a constant.

2 Preliminaries

Let A be the set of n *agents*, H the set of m *house types*. The *endowment function* $\omega : A \to H$ assigns to each agent the type of house he originally owns. We shall denote by $A(h)$ the set of agents endowed with a house of type $h \in H$.

Each agent has preferences over house types in H in the form of a linearly ordered list $P(a)$, possibly with ties. Notation $h \succeq_a k$ means that agent a prefers houses of type h to houses of type k. The set of house types appearing in $P(a)$ is denoted by $H(a)$, and house types in $H(a)$ are said to be *acceptable* for a. We assume that $\omega(a)$ belongs to the least preferred acceptable house types for each agent. The remaining house types are called *unacceptable*.

In the special case of *trichotomic preferences*, each agent distinguishes only three kinds of house types: house types more preferred than the type of his own house, these are called *desired*; houses of the same type as his own house and *unacceptable* house types.

The n-tuple $\mathcal{P}$ of all preference lists is called the *preference profile*. The quadruple $\mathcal{M} = (A, H, \omega, \mathcal{P})$ is called a *housing market*. A market $\mathcal{M}' = (A', H', \omega', \mathcal{P}')$ is a *submarket* of a market $\mathcal{M} = (A, H, \omega, \mathcal{P})$ if $A' \subseteq A$, $\omega(A') \subseteq H' \subseteq H$, $H'(a) = H(a) \cap H'$ and the endowment function ω' and preference profile $\mathcal{P}'$ are restrictions of ω and $\mathcal{P}$ to A'.

In a housing market $\mathcal{M}$, we want to assign prices to house types and design trading consistent with prices so that each agent ends up with exactly one acceptable house. More formally, we say that a triple $\mathcal{T} = (x, \pi, p)$ is a *solution for* $\mathcal{M}$ if

(*i*) $x : A \to H$ is a function and $\pi : A \to A$ is a bijection such that $x(a) = \omega(\pi(a))$ and $x(a) \in H(a)$ for each $a \in A$,

(*ii*) $p : H \to \mathbb{R}$ is a *price function* such that $p(x(a)) \leq p(\omega(a))$ for each $a \in A$.

Condition (*ii*) ensures that each agent can afford a house of type $x(a)$. Function x and bijection π define who gets whose house. Moreover, they partition A

into cycles of the form $(a_0, \dots, a_{l-1})$, where $x(a_i) = \omega(a_{i+1})$ for all $i = 0, \dots, l-1$ (modulo l), called *trading cycles*. We say that an agent a is *trading* in solution $\mathcal{T}$ if $a \neq \pi(a)$.

Given a price function $p : H \to \mathbb{R}^+$, the *budget set* of agent a with respect to p is the set of house types that a can afford, i. e., $\{h \in H : p(h) \leq p(\omega(a))\}$. An agent a is *satisfied* in a solution $\mathcal{T}$ of $\mathcal{M}$ if $x(a)$ is among the most preferred house types in the budget set of a. (Notice that in the case of trichotomic preferences, an agent a is satisfied if and only if a is either trading or $p(h) > p(\omega(a))$ for each $h \in H(a)$ such that $h \neq \omega(a)$; in other words, of all acceptable houses a can only afford houses of the same type as his endowment). Otherwise, we say that a is *dissatisfied*. By $\mathrm{Sat}(\mathcal{T})$ and $\mathrm{Dissat}(\mathcal{T})$ we denote the sets of satisfied and dissatisfied agents in $\mathcal{T}$, respectively, and by $\mathrm{sat}(\mathcal{T})$ and $\mathrm{dissat}(\mathcal{T})$ the cardinalities of these sets. The minimum of values $\mathrm{dissat}(\mathcal{T})$ over all solutions for a market $\mathcal{M}$ is called the *deficiency* of $\mathcal{M}$ [4]. As a dual notion, the maximum of $\mathrm{sat}(\mathcal{T})$ over all solutions $\mathcal{T}$ is denoted by $\mathrm{sat}(\mathcal{M})$; a solution achieving this value is called *optimal*.

A solution $\mathcal{T}$ is called an *economic equilibrium for* $\mathcal{M}$ if all agents are satisfied in $\mathcal{T}$.

A simple observation follows directly from the definitions (see also [4, 9]).

Lemma 1. If $\mathcal{T} = (x, \pi, p)$ is a solution for a market $\mathcal{M}$ then $p(x(a)) = p(\omega(a))$ for each agent $a \in A$.

We study the following problems.

> MAX-SHDTIES (Maximum Satisfied Housing with Duplicate houses and Ties): Given a market $\mathcal{M}$, find a solution $\mathcal{T}$ that maximizes $\mathrm{sat}(\mathcal{T})$.
>
> MAX-SHDTRI (Maximum Satisfied Housing with Duplicate houses and Trichotomic preferences): Given a market $\mathcal{M}$ with trichotomic preferences, find a solution $\mathcal{T}$ that maximizes $\mathrm{sat}(\mathcal{T})$.

For our purpose it is very convenient to view a housing market $\mathcal{M}$ as a digraph $G_{\mathcal{M}} = (A, E)$. Vertices of $G_{\mathcal{M}}$ correspond to agents and each vertex is colored according to the type of house this agent is endowed with—hence the color classes correspond to the sets $A(h)$. An arc $ab \in E$ means that $\omega(b) \succ_a \omega(a)$. In case of a general housing market $\mathcal{M}$, the arcs may be labelled with numbers that express the preference order.

When there is no danger of confusion, we identify the digraph $G_{\mathcal{M}}$ with the market $\mathcal{M}$, and the vertices of $G_{\mathcal{M}}$ with agents of $\mathcal{M}$. We then speak of *in-neighbors* and *out-neighbors* of agents, of *directed cycles in* $\mathcal{M}$, and we say that $\mathcal{M}$ is *acyclic* if $G_{\mathcal{M}}$ does not contain any directed cycle (note that in $G_{\mathcal{M}}$ there are no loops and arcs between agents of the same type). Finally, as it will be clear from the context, we usually say simply *cycle* also for a directed cycle.

Recall that a set S of vertices of an undirected graph is called a *vertex cover* (VC for short) if each edge has at least one endpoint in S. A set of vertices in a directed graph is a *feedback vertex set* (FVS for short) if its removal leaves an acyclic graph. Both problems MIN-VC and MIN-FVS, asking for finding the size of a minimum vertex cover VC(G) and the size of a minimum feedback vertex set FVS(G), respectively, in a given (di)graph G are well-known NP-complete problems.

3 Bounds for $\mathrm{sat}(\mathcal{M})$

In this section we deal with markets that exhibit special structure. For example, we show that each acyclic market admits an equilibrium and that $\mathrm{sat}(\mathcal{M})$ can be computed easily in markets with only two house types . Then we prove that in each trichotomic market at least half of the agents can always be satisfied. The proof of this theorem also provides a simple 2-approximation algorithm for $\mathrm{sat}(\mathcal{M})$. Finally, we show that the approximation guarantee 2 of this algorithm is tight.

Lemma 2. Any acyclic market $\mathcal{M}$ admits an economic equilibrium, i. e., $\mathrm{sat}(\mathcal{M}) = n$. Moreover, an optimal solution can be found in time $O(L)$, where $L = \sum_{a\in A} |H(a)|$.

Proof. Let us denote by $G^*_{\mathcal{M}}$ the digraph whose vertices correspond to house types and a pair hk is an arc if and only if there exists an agent a with $\omega(a) = h$ such that $k \succ_a \omega(a)$.

If $G_{\mathcal{M}}$ is acyclic then $G^*_{\mathcal{M}}$ is acyclic too, and the argument is as follows: Suppose that $(h_0, h_1, \ldots, h_{k-1})$ is a cycle in $G^*_{\mathcal{M}}$. This means that there are agents $a_0, a_1, \ldots, a_{k-1}$ such that $\omega(a_i) = h_i$ and a_i desires h_{i+1} (modulo k). It is easy to see that $(a_0, a_1, \ldots, a_{k-1})$ is a cycle in $G_{\mathcal{M}}$. Now assign prices to house types according to any topological ordering of vertices in $G^*_{\mathcal{M}}$ so that $hk \in E(G^*_{\mathcal{M}})$ implies $p(h) < p(k)$. Such prices enable no trading, but all agents are satisfied.

Finally, the digraph $G^*_{\mathcal{M}}$ can easily be constructed from preference lists of agents in time $O(L)$, the number of its arcs is $O(L)$ too, so the topological ordering of its vertices can also be performed in time $O(L)$. □

The main significance of the above lemma is in the possibility to extend a solution obtained for an acyclic submarket to a solution of the whole market while preserving the number of satisfied agents.

Lemma 3. Let $\mathcal{M}'$ be a submarket of $\mathcal{M}$. Every solution $\mathcal{T}'$ of $\mathcal{M}'$ can be extended to a solution $\mathcal{T}$ of $\mathcal{M}$ such that $\mathrm{sat}(\mathcal{T}) \geq \mathrm{sat}(\mathcal{T}')$. Hence, $\mathrm{sat}(\mathcal{M}) \geq \mathrm{sat}(\mathcal{M}')$.

Proof. Let the trading relations in $\mathcal{T}$ be the same as in $\mathcal{T}'$; agents of $\mathcal{M} \setminus \mathcal{M}'$ are not trading. All house types present in $\mathcal{M}'$ are given the same price as in $\mathcal{T}'$. House types that are only in $\mathcal{M} \setminus \mathcal{M}'$ are all given equal price, higher than that of any house type in $\mathcal{M}'$.

It is straightforward to check that any agent satisfied in $\mathcal{T}'$ is satisfied in $\mathcal{T}$ as well, because trading is the same and he can afford exactly the same house types as before. Thus, $\text{sat}(\mathcal{T}) \geq \text{sat}(\mathcal{T}')$. □

The following lemma is an immediate corollary of Lemma 2 and Lemma 3.

Lemma 4. Let $\mathcal{M}$ be a market. If a set $F \subseteq A$ is a FVS in $G_{\mathcal{M}}$, then there exists a solution $\mathcal{T}$ for $\mathcal{M}$ such that $A \setminus F \subseteq \text{Sat}(\mathcal{T})$. Hence, $\text{sat}(\mathcal{M}) \geq |A \setminus F|$.

The following theorem provides a lower bound for the number of satisfied agents in each trichotomic market.

Theorem 1. In each trichotomic market $\mathcal{M}$ with n agents at least $n/2$ agents can be satisfied.

Proof. Let $\mathcal{C}$ be any maximum cycle packing of $G_{\mathcal{M}}$, i. e., a set of vertex-disjoint directed cycles that contains the maximum possible number of vertices. Let us denote by $A_{\mathcal{C}}$ the set of agents contained in $\mathcal{C}$.

If $|A_{\mathcal{C}}| \geq n/2$, a possible solution is as follows: all house types in $\mathcal{M}$ receive the same price, and the cycles of $\mathcal{C}$ are trading cycles. Then all trading agents are satisfied, hence $\text{sat}(\mathcal{T}) \geq n/2$.

Now consider the case $|A_{\mathcal{C}}| < n/2$. As $\mathcal{C}$ is maximal, $A_{\mathcal{C}}$ is a FVS of $\mathcal{M}$ and so the submarket generated by $A \setminus A_{\mathcal{C}}$ acyclic. Using Lemma 4 we obtain a solution $\mathcal{T}$ satisfying all the agents in $A \setminus A_{\mathcal{C}}$, i. e., $\text{sat}(\mathcal{T}) \geq n/2$. □

Notice that there is a folklore polynomial algorithm for finding a maximum cycle packing in a digraph, see e. g. [1]. So the proof of Theorem 1 immediately provides the following corollary.

Corollary 1. There is a polynomial 2-approximation algorithm for Max-SHDTri.

Example 1. Consider the following market $\mathcal{M}$.

$$\mathcal{M}: \quad \begin{aligned} & A = \{a_1, a_2, \ldots, a_q, b_1, b_2, \ldots, b_q, c\} \\ & \omega(a_i) = \omega(b_i) = h_i;\ \omega(c) = h_{q+1} \\ & P(a_i) = P(b_i) = h_{i+1} \succ h_i;\ P(c) = h_1 \succ h_{q+1} \end{aligned}$$

Here, $|A| = 2q + 1$ and any maximum cycle packing contains a single cycle of length $q+1$: $(c, a_1 \text{ or } b_1, a_2 \text{ or } b_2, \ldots, a_q \text{ or } b_q)$. Thus, the algorithm satisfies $q+1$ agents only, and since the house types of the rest of agents are already present in the trading cycle, q agents remain unsatisfied. On the other hand, if agent c is removed, an acyclic market with $2q$ satisfied agents is obtained.

If we let q grow indefinitely, this example shows that the bound of the above approximation algorithm cannot be tightened to $2-\varepsilon$ for any $\varepsilon > 0$.

Example 2. Suppose that housing market $\mathcal{M}$ fulfills $|H| = 2$ and $H(a) \setminus \{\omega(a)\} \neq \emptyset$ for each agent. We show that $\mathrm{sat}(\mathcal{M}) = \max\{2\min\{n_1, n_2\}, n_1, n_2\}$, where $|A(h_1)| = n_1$, $|A(h_2)| = n_2$.

In this case, $G_{\mathcal{M}}$ is a complete bipartite digraph. For any solution $\mathcal{T}$, there are three possibilities. If $p_1 = p_2$ then each trading cycle is even and contains alternately agents from $A(h_1)$ and $A(h_2)$, hence the same number of agents of the two types. So $\mathrm{sat}(\mathcal{T}) = 2\min\{n_1, n_2\}$. If $p_1 < p_2$ then there is no trading, but all the agents from $A(h_1)$ are satisfied, as they cannot afford a house of type h_2. So $\mathrm{sat}(\mathcal{T}) = n_1$. Finally, if $p_2 < p_1$ then $\mathrm{sat}(\mathcal{T}) = n_2$ by a similar argument.

This example gives an infinite number of housing markets with $\mathrm{sat}(\mathcal{M}) = \frac{2}{3}n$ if we set $n_2 = 2n_1$. We remark that it remains an open problem whether a housing market exists where the number of dissatisfied agents is more than one third of all agents.

4 Inapproximability

We derive our results from the hardness results for MIN-VC. In Subsection 4.1 we describe a transformation from MIN-VC to housing markets and in Subsection 4.2 we then derive several consequences for the hardness of approximation of $\mathrm{sat}(\mathcal{M})$.

4.1 The Transformation

Proposition 1. For every integer $k \geq 1$, there is a polynomial-time transformation $\tilde{T}_k$ from MIN-VC to MAX-SHDTIES such that each graph G with $|V(G)|$ vertices is transformed into a housing market $\tilde{\mathcal{M}}_k = \tilde{T}_k(G)$ with $n = (2k+1)|V(G)|$ agents, such that $\mathrm{sat}(\tilde{\mathcal{M}}_k) = (2k+1)|V(G)| - \mathrm{VC}(G)$.

Proof. Consider an instance G of MIN-VC. We construct a market $\tilde{\mathcal{M}}_k$.

For each vertex $v \in G$, there is a set of vertices A_v consisting of $k+1$ *incoming* agents $I_v = \{i_{v,1}, i_{v,2}, \ldots, i_{v,k+1}\}$, and k *outgoing* agents $O_v = \{o_{v,1}, o_{v,2}, \ldots, o_{v,k}\}$. Their endowments and preferences are as follows.

$$\omega(i_{v,1}) = \omega(i_{v,2}) = \cdots = \omega(i_{v,k+1}) = h_v$$
$$\omega(o_{v,1}) = h^*_{v,1};\ \omega(o_{v,2}) = h^*_{v,2};\ \ldots\ ;\ \omega(o_{v,k}) = h^*_{v,k}$$

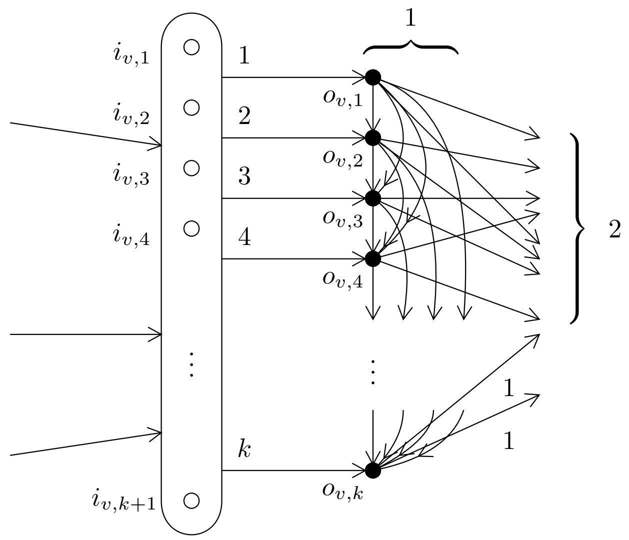

Figure 1: Agents in $\tilde{\mathcal{M}}_k$ corresponding to a vertex v of G'.

$$P(i_{v,1}) = \cdots = P(i_{v,k+1}) = h^*_{v,1} \succ h^*_{v,2} \succ \cdots \succ h^*_{v,k} \succ h_v$$
$$P(o_{v,1}) = (h^*_{v,2}, \ldots, h^*_{v,k}) \succ (\text{all } h_w \text{ such that } \{vw\} \in E(G)) \succ h^*_{v,1}$$
$$P(o_{v,2}) = (h^*_{v,3}, \ldots, h^*_{v,k}) \succ (\text{all } h_w \text{ such that } \{vw\} \in E(G)) \succ h^*_{v,2}$$
$$\vdots$$
$$P(o_{v,k}) = (\text{all } h_w \text{ such that } \{vw\} \in E(G)) \succ h^*_{v,k}.$$

For illustration, see Figure 1. Incoming (white) agents have the same in-arcs and out-arcs, which is for simplicity illustrated by just one copy of the arcs coming into and going out of the oval shape. Numbers accompanying arcs express the preference ordering.

Clearly, this construction can be performed in polynomial time.

For a vertex subset B of G, we define the corresponding set O_B of outgoing agents: $O_B = \{o_{v,j} : v \in B, j = 1, \ldots, k\}$. We prove several properties of $\tilde{\mathcal{M}}_k$.

Lemma 5. *F is a vertex cover in G if and only if O_F is a FVS in $\tilde{\mathcal{M}}_k$.*

Proof. First notice that each cycle C in $\tilde{\mathcal{M}}_k$ has the following form: it goes through one incoming agent corresponding to v_1, then through one or more outgoing agents corresponding to v_1, through one incoming agent corresponding to v_2, one or more outgoing agents corresponding to v_2, etc., moreover, there is always an edge $\{uv\}$ in G if there is an arc from some outgoing agent in O_u to some incoming agent I_v.

Now suppose that F is a vertex cover in G and C any cycle in $\tilde{\mathcal{M}}_k$. As said above, C corresponds to a cycle $(v_1, v_2, \ldots, v_\ell)$ in G. As F is a vertex cover in G

and $\{v_1, v_2\} \in E$, we have that either $v_1 \in F$ or $v_2 \in F$. Then it is easy to see that leaving out agents in O_F will destroy C, so O_F is a FVS in $\tilde{\mathcal{M}}_k$.

For the converse implication realize that for each edge $\{u, v\}$ in G, the market $\tilde{\mathcal{M}}_k$ contains a cycle crossing I_u, then O_u, then I_v, O_v and coming back to I_u. If O_F is a FVS in $\tilde{\mathcal{M}}_k$ then the set F must contain either v or u, which means that F is a vertex cover in G. □

Lemma 6. There exists an optimal solution for $\tilde{\mathcal{M}}_k$ with no trading.

Proof. Assume that every optimal solution has at least one trading cycle, and let $\mathcal{T}$ be an optimal solution with the minimum number of trading cycles. Let C be any trading cycle of $\mathcal{T}$. By Lemma 1, houses of all agents on C have the same price; we denote this price by p_C.

Now consider the set A_v for any v such that at least one agent $a \in I_v$ belongs to C. Then the price of the house of each agent from O_v who is trading must also be equal to p_C, as some of the agents preceding him on his trading cycle must belong to I_v. Thus, each agent of A_v is either trading with price p_C or not trading.

Let A_C^+ denote the union of all sets A_v such that at least one agent in A_v is trading with price p_C.

We construct another solution $\mathcal{T}'$ from $\mathcal{T}$ by modifying prices of houses according to their owner and their price in $\mathcal{T}$ as follows.

1. Owner in $A \setminus A_C^+$, price $\leq p_C$. These houses receive the same price in $\mathcal{T}'$ as in $\mathcal{T}$.
2. Owner in $A \setminus A_C^+$, price $> p_C$. These houses receive their price in $\mathcal{T}$ increased by $k+1$.
3. Owner in A_C^+. For each v such that $A_v \subseteq A_C^+$, we set $p(h_v) = p_C$, $p(h^*_{v,1}) = p_C + 1$, $p(h^*_{v,2}) = p_C + 2$, $\ldots$, $p(h^*_{v,k}) = p_C + k$.

It is immediate that prices of the houses are defined consistently. Trading cycles from $\mathcal{T}$ with price different from p_C are in $\mathcal{T}'$ as well, while agents of A_C^+ are not trading in $\mathcal{T}'$. This concludes the construction of $\mathcal{T}'$.

Clearly, the number of trading cycles in $\mathcal{T}'$ is strictly smaller than the number of trading cycles in $\mathcal{T}$. Now we show that $\text{sat}(\mathcal{T}') \geq \text{sat}(\mathcal{T})$.

Let us fist consider an agent outside A_C^+. His budget set in $\mathcal{T}'$ has not increased, and his trading/nontrading status has not changed compared to $\mathcal{T}$, so if this agent was satisfied in $\mathcal{T}$, he remains satisfied in $\mathcal{T}'$ as well.

Now take a set $A_v \subseteq A_C^+$. Notice that all agents from I_v have the same strict preferences and in $\mathcal{T}$ they have the same nonempty budget set. Hence, since at most one of them can get their common unique most preferred affordable house from O_v, at most one agent from I_v is satisfied in $\mathcal{T}$. This implies that $|\operatorname{Sat}(\mathcal{T}) \cap A_v| \leq k+1$. In $\mathcal{T}'$ no agent from A_v is trading, but since agents from I_v cannot afford any house from O_v, they are all satisfied. This implies $|\operatorname{Sat}(\mathcal{T}') \cap A_v| \geq k+1$.

Summarizing, $\operatorname{sat}(\mathcal{T}') \geq \operatorname{sat}(\mathcal{T})$. Hence, $\mathcal{T}'$ is another optimal solution having less trading cycles than $\mathcal{T}$, which is a contradiction with the choice of $\mathcal{T}$. □

Lemma 7. Let $\mathcal{T}$ be any optimal solution of $\tilde{\mathcal{M}}_k$ with no trading and let $v \in G$ be arbitrary. Then $I_v \subseteq \operatorname{Sat}(\mathcal{T})$ and either $O_v \subseteq \operatorname{Sat}(\mathcal{T})$ or $O_v \subseteq \operatorname{Dissat}(\mathcal{T})$.

Proof. Let $\mathcal{T}$ be an optimal solution with no trading and v any vertex from $V(G)$. First we claim that

$$p(h_v) < p(h^*_{v,j}) \text{ for each } j \in \{1, \dots, k\}. \tag{1}$$

Otherwise, if $p(h_v) \geq p(h^*_{v,j})$ for some j, then all the agents in I_v can afford the acceptable house $h^*_{v,j}$ and are dissatisfied. By increasing $p(h^*_{v,j})$ to $p(h_v)+1$ for every such j, all $k+1$ agents in I_v become satisfied, while at most k agents from O_v may become dissatisfied. A contradiction with optimality of $\mathcal{T}$. Hence, (1) holds, and this implies that $I_v \subseteq \operatorname{Sat}(\mathcal{T})$.

Now distinguish two cases.

Case 1. $p(h_v) < p(h_u)$ for each $u \in V$ such that $\{v,u\} \in E$. Let us denote $p^* = \min\{p(h_u); \{v,u\} \in E\}$ and define new prices p' fulfilling the following inequality:

$$p(h_v) < p'(h^*_{v,1}) < p'(h^*_{v,2}) < \cdots < p'(h^*_{v,k}) < p^*.$$

Thanks to (1), the satisfaction of any agent outside A_v w. r. t. new prices does not differ from his satisfaction according to old prices. Moreover, all agents in O_v are satisfied. So they all were satisfied in $\mathcal{T}$ too, or $\mathcal{T}$ was not optimal.

Case 2. There exists $u \in V$ such that $\{v,u\} \in E$ and $p(h_v) \geq p(h_u)$. Due to (1), all agents from O_v can afford the house h_u and so they are all dissatisfied, as there is no trading in $\mathcal{T}$. □

To prove the Proposition, we show that $\operatorname{VC}(G) = l$ if and only if $\operatorname{sat}(\tilde{\mathcal{M}}_k) = n - kl$. Assume that F is a vertex cover of size l in G. Then by Lemma 5 the set O_F is a feedback vertex set in $\tilde{\mathcal{M}}_k$ and by Lemma 4, $\operatorname{sat}(\tilde{\mathcal{M}}_k) \geq |A \setminus O_F| = n - kl$.

Now let $\mathcal{T}$ be an optimal solution in $\tilde{\mathcal{M}}_k$ with no trading (which exists by Lemma 6). Lemma 7 implies that $\operatorname{Dissat}(\mathcal{T}) = \bigcup_{v \in B} O_v$ for some $B \subseteq V$, and hence $\operatorname{dissat}(\mathcal{T}) = kl$ for some l. For any directed cycle C in $\tilde{\mathcal{M}}_k$, at least one agent will have maximum price among the agents of C, and because there is no trading, this agent is dissatisfied. Moreover, (1) implies that the dissatisfied agent is outgoing. Hence, the set O_B contains at least one agent out of every directed cycle in $\tilde{\mathcal{M}}_k$, and thus O_B is a FVS of $\tilde{\mathcal{M}}_k$. Lemma 5 directly implies that B is a vertex cover in G and hence $\operatorname{VC}(G) \leq l$.

Therefore $\operatorname{sat}(\tilde{\mathcal{M}}_k) = n - kl$ if and only if $\operatorname{VC}(G) = l$. □

In case that $k = 1$, the constructed market $\tilde{\mathcal{M}}_k$ is trichotomic. Hence, we get the following proposition as a corollary.

Proposition 2. There is a polynomial-time transformation T from Min-VC to Max-SHDTri such that each graph G with $|V(G)|$ vertices is transformed into a housing market $\mathcal{M} = T(G)$ with $n = 3|V(G)|$ agents, such that $\text{sat}(\mathcal{M}) = 3|V(G)| - \text{VC}(G)$.

4.2 Inapproximability for Max-SHDTri

Halldórsson et al. [10], when studying the approximability of the problem to find a stable matching of maximum size in the stable marriage problem with incomplete lists and ties (SMTI for short), presented a construction that assigns to each graph $G = (V, E)$ an SMTI instance I such that the number of men as well as the number of women in I is equal to $3|V(G)|$, and $\text{opt}(I) = 3|V(G)| - \text{VC}(G)$. Since the quantitative relations in our Proposition 2 and in the proof of their Theorem 3.2. are the same, we get by exactly the same argument the following analogies of Theorem 3.2., its Corollary 3.4. and Remark 3.6. of [10].

Proposition 3. For any $\varepsilon > 0$ and $p < \frac{3-\sqrt{5}}{2}$, given an instance $\mathcal{M}$ of Max-SHDTri with n agents, it is NP-hard to distinguish between the following two cases:

1. $\text{sat}(\mathcal{M}) \geq \frac{2+p-\varepsilon}{3}n$, and
2. $\text{sat}(\mathcal{M}) < \frac{2+\max\{p^2,4p^3-3p^4\}+\varepsilon}{3}n$.

Theorem 2. It is NP-hard to approximate Max-SHDTri within a factor smaller than 21/19.

Theorem 3. If Min-VC is NP-hard to approximate within a factor of $2-\varepsilon$ then Max-SHDTri is NP-hard to approximate within a factor smaller than 1.25.

The *Unique Games Conjecture* (UGC) was introduced by Khot [11]. For the statement of the Conjecture and all necessary definitions, see [11] and [12]. Khot and Regev [12] proved that if UGC is true, then Min-VC is NP-hard to approximate within a ratio smaller than 2. Thus, the assumption of Theorem 3 would be fulfilled. We get the following corollary.

Corollary 2. If UGC is true, then Max-SHDTri is NP-hard to approximate within a factor smaller than 1.25.

The results of Austrin et al. [3, page 3 and Theorem 4.1] may be stated in the following way.

Proposition 4. If UGC is true, then for every sufficiently large integer d it is NP-hard to distinguish between the following two cases:

1. $\text{VC}(G) \leq (1/2 + \Theta(\frac{\log\log d}{\log d}))|V(G)|$, and

2. $\mathrm{VC}(G) > (1 - \frac{1}{\log d})|V(G)|$,

even on graphs of maximum degree d.

It is easy to see that in the transformation of Proposition 2, the resulting preference list lengths are bounded in terms of degrees of G. We therefore define a restricted variant of the Max-SHDTri problem with preference lists of length at most d; we call it Max-SHDTri$_d$. From Proposition 4 we derive the following result.

Theorem 4. If UGC is true, then for every sufficiently large integer d it is NP-hard to approximate Max-SHDTri$_d$ within a ratio $1.25 - \Theta(\frac{\log\log d}{\log d})$.

Proof. To a Min-VC instance with maximum degree d, we apply the transformation T of Proposition 2 and obtain a market $\mathcal{M}$ with preference lists of length at most d (not counting the agents' own house types). By the properties of T, it holds that $\mathrm{sat}(\mathcal{M}) = 3|V(G)| - \mathrm{VC}(G)$, and the two cases of Proposition 4 become the following.

1. $\mathrm{sat}(\mathcal{M}) \geq \frac{5}{6}n - \frac{1}{3}\Theta(\frac{\log\log d}{\log d})n$, and

2. $\mathrm{sat}(\mathcal{M}) < \frac{2}{3}n + \frac{1}{3\log d}n$.

Assume that UGC is true. Then, by Proposition 4, it is also NP-hard to distinguish the two cases above, which could be done by an approximation algorithm with ratio

$$\frac{\frac{5}{6}n - \frac{1}{3}\Theta\left(\frac{\log\log d}{\log d}\right)n}{\frac{2}{3}n + \frac{1}{3\log d}n} =$$

$$= \frac{\frac{5}{4} + \frac{5}{4}\cdot\frac{1}{2\log d}}{1 + \frac{1}{2\log d}} - \frac{\frac{5}{4}\cdot\frac{1}{2\log d}}{1 + \frac{1}{2\log d}} - \frac{\frac{1}{2}\Theta\left(\frac{\log\log d}{\log d}\right)}{1 + \frac{1}{2\log d}} =$$

$$= \frac{5}{4} - \Theta\left(\frac{1}{\log d}\right) - \Theta\left(\frac{\log\log d}{\log d}\right) = 1.25 - \Theta\left(\frac{\log\log d}{\log d}\right).$$

□

4.3 Inapproximability for Max-SHDTies

As the problem Max-SHDTri is a restricted version of Max-SHDTies, all inapproximability results for Max-SHDTri apply to Max-SHDTies as well. However, due to the general Proposition 1, we obtain even stronger inapproximability results here.

As in the reasoning that leads to Proposition 3, we use the following assertion for Min-VC due to Dinur and Safra [8].

Proposition 5. For any $\varepsilon > 0$ and $p < \frac{3-\sqrt{5}}{2}$, given a graph G, it is NP-hard to distinguish between the following two cases:

1. $\mathrm{VC}(G) \leq (1 - p + \varepsilon)|V(G)|$, and
2. $\mathrm{VC}(G) > (1 - \max\{p^2, 4p^3 - 3p^4\} - \varepsilon)|V(G)|$.

Theorem 5. The problem Max-SHDTies is

1. NP-hard to approximate within a factor smaller than 1.2, and
2. NP-hard to approximate within a factor smaller than 1.5, if UGC is true.

Proof. We apply the transformation $\tilde{T}_k$ of Proposition 1 to a Min-VC instance G, and obtain a market $\tilde{\mathcal{M}}_k$ with $n = (2k+1)|V(G)|$ agents. By Proposition 5 with $p = 1/3$, for any $\varepsilon > 0$ it is NP-hard to distinguish between the following cases:

1. $\mathrm{sat}(\tilde{\mathcal{M}}_k) \geq \frac{\frac{4}{3}k+1-\varepsilon k}{2k+1} n$, and
2. $\mathrm{sat}(\tilde{\mathcal{M}}_k) < \frac{\frac{10}{9}k+1+\varepsilon k}{2k+1} n$.

Consider an approximation algorithm for Max-SHDTies with approximation ratio $1.2 - \delta$, which is equal to $\frac{6}{5} - \delta$. In the first case the algorithm gives a solution with at least $\frac{1}{(6/5)-\delta} \frac{\frac{4}{3}k+1-\varepsilon k}{2k+1} n$ satisfied agents. If we choose k large enough, there exists a positive ε such that $\varepsilon < \frac{50\delta k + 45\delta - 9}{99k - 45\delta k}$. Then

$$\frac{6}{5} - \delta < \frac{\frac{4}{3}k + 1 - \varepsilon k}{\frac{10}{9}k + 1 + \varepsilon k},$$

and the number of satisfied agents is larger than $\frac{\frac{10}{9}k+1+\varepsilon k}{2k+1} n$. In the second case, however, the approximation algorithm finds a solution with less than $\frac{\frac{10}{9}k+1+\varepsilon k}{2k+1} n$ satisfied agents, and hence, it can distinguish between the first and the second case.

To prove the second part of the theorem, we use the following special case of the results of Khot and Regev [12, Section 4].

Proposition 6. If UGC is true, then for any $\varepsilon > 0$, given a graph G, it is NP-hard to distinguish between the following two cases:

1. $\mathrm{VC}(G) \leq (\frac{1}{2} + \varepsilon)|V(G)|$, and
2. $\mathrm{VC}(G) > (1 - \varepsilon)|V(G)|$.

By using the cases of Proposition 6 for a market $\tilde{\mathcal{M}}_k$, and assuming that UGC is true, we get that it is NP-hard to distinguish

1. $\text{sat}(\tilde{\mathcal{M}}_k) \geq \frac{\frac{3}{2}k+1+\varepsilon k}{2k+1}n$, and
2. $\text{sat}(\tilde{\mathcal{M}}_k) < \frac{k+1-\varepsilon k}{2k+1}n$.

Analogously to the reasoning above, we get the desired result for a large enough k and for ε small enough. □

5 Conclusion and Open Problems

In this paper, we have presented a simple 2-approximation algorithm for Max-SHDTri. We have also shown that the number of agents satisfiable in any instance of Max-SHDTri is at least $n/2$.

Based on a reduction from Min-VC, we have shown several inapproximability results for Max-SHDTri, Max-SHDTri$_d$ (where preference lists have length at most d), and Max-SHDTies (where preference lists are not required to be trichotomic).

All the markets constructed in this paper have a special property: all agents that own the same house type have the same preference lists. Markets with this property may be called *coherent*.

One could expect that for coherent markets, stronger results could be obtained. Further, no approximation algorithm is known for Max-SHDTies, nor for markets where preference lists are strictly ordered.

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References

[1] D. Abraham, A. Blum, T. Sandholm, Clearing algorithms for barter exchange markets: Enabling nationwide kidney exchanges, EC'07, San Diego, California (2007).

[2] D. Abraham, K. Cechlárová, D. Manlove, and K. Mehlhorn. Pareto optimality in house allocation problems. In *Algorithms and Computation, ISAAC 2004, Hong Kong, December 2004, Eds. R. Fleischer, G. Trippen*, volume 3341 of *Lecture Notes in Computer Science*, pages 3–15, 2004.

[3] P. Austrin, S. Khot, and M. Safra. Inapproximability of vertex cover and independent set in bounded degree graphs. In *CCC '09: Proceedings of the 2009 24th Annual IEEE Conference on Computational Complexity*, pages 74–80, Washington, DC, USA, 2009. IEEE Computer Society.

[4] K. Cechlárová and T. Fleiner. Housing markets through graphs. *Algorithmica*, 58/1, pages 19–33, 2010.

[5] K. Cechlárová and E. Jelínková. An efficient implementation of the equilibrium algorithm for housing markets with duplicate houses. *IM Preprint A2-2010*, Institute of Mathematics, P. J. Šafárik University, Košice, Slovakia, 2010.

[6] K. Cechlárová and I. Schlotter. Computing the deficiency of housing markets with duplicate houses. *IPEC 2010, LNCS 6478, 72–83*, 2010.

[7] X. Deng, Ch. Papadimitriou, S. Safra. On the complexity of price equilibria. *J. Computer and System Sciences*, 67:311–324, 2003.

[8] I. Dinur and S. Safra. On the hardness of approximating minimum vertex cover. *Annals of Mathematics*, 162:439–485, 2005.

[9] S. Fekete, M. Skutella, and G. Woeginger. The complexity of economic equilibria for house allocation markets. *Information Processing Letters*, 88(5):219–223, 2003.

[10] M. M. Halldórsson, K. Iwama, S. Miyazaki, and H. Yanagisawa. Improved approximation results for the stable marriage problem. *ACM Trans. Algorithms*, 3(3):30, 2007.

[11] S. Khot. On the power of unique 2-prover 1-round games. In *Proc. 34th ACM Symp. on Theory of Computing, STOC 2002*, pages 767 – 775, 2002.

[12] S. Khot and O. Regev. Vertex cover might be hard to approximate to within $2 - \varepsilon$. *Journal of Computer and System Sciences*, 74(3):335 – 349, 2008. Computational Complexity 2003.

[13] L. Shapley and H. Scarf. On cores and indivisibility. *J. Math. Econ.*, 1:23–37, 1974.

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