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Generalized fractional total coloring of complete graphs

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Abstract: An additive and hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphism. Let $\mathcal{P}$ and $\mathcal{Q}$ be two additive and hereditary graph properties and let r, s be integers such that $r \geq 2s$. Then $\frac{r}{s}$*-fractional* $(\mathcal{P}, \mathcal{Q})$*-total coloring* of a finite graph G is a mapping $f : V \cup E \to \binom{\{1,2,\dots,r\}}{s}$ such that for any color i all vertices of color i induce a subgraph from the property $\mathcal{P}$, all edges of color i induce a subgraph from the property $\mathcal{Q}$ and vertices and incident edges have assigned disjoint sets of colors. The minimum value of the ratio $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of G is called *fractional* $(\mathcal{P}, \mathcal{Q})$*-total chromatic number* $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \frac{r}{s}$. Let $k = \sup\{i : K_{i+1} \in \mathcal{P}\}$ and $l = \sup\{i : K_{i+1} \in \mathcal{Q}\}$. We show for a complete graph K_n that if $l \geq k + 2$ then $\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n) = \frac{n}{k+1}$ for a sufficiently large n.

1 Introduction

Total coloring of a graph is a coloring of vertices and edges such that adjacent and incident elements do not have assigned the same color. There are some papers about this theme, for example [1, 3, 6], etc.

The following conjecture is known as the Total Colouring Conjecture and was made in the 1960s by Behzad [1] and Vizing [9]. It has been verified for several special cases of graphs, including, for example, complete graphs (see [2, 6] for surveys).

Conjecture 1. *If G is a graph with maximum degree Δ, then $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$.*

The fractional version of this conjecture was proved by Kilakos and Reed [8] for any graph G, $\chi''_f(G) \leq \Delta(G) + 2$. In this paper we deal with generalized fractional total coloring of graphs. Let $\mathcal{P}$ and $\mathcal{Q}$ be sets of graphs (called graph properties). We consider a total coloring of a graph G such that adjacent elements can have assigned the same color but we require that subgraphs of G induced by the set of vertices of the same color to be from the property $\mathcal{P}$ and subgraphs of G induced by the set of edges of the same color to be from the property $\mathcal{Q}$ and incident elements can not have assigned the same color. For example if $\mathcal{P} = \mathcal{O}$ and $\mathcal{Q} = \mathcal{O}_1$ then it is ordinary total coloring of a graph. Such properties have also been studied by Borowiecki et al. in [3, 5].

A $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of a finite graph G is a mapping $f : V \cup E \to \binom{\{1,2,\ldots,r\}}{s}$ such that for any color i all vertices of color i induce a subgraph from the property $\mathcal{P}$, all edges of color i induce a subgraph from the property $\mathcal{Q}$ and vertices and incident edges have assigned disjoint sets of colors. The minimum value of the ratio of $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of G is called fractional $(\mathcal{P}, \mathcal{Q})$-total chromatic number $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \frac{r}{s}$.

We deal with $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of complete graphs by using linear programming and simplex method.

The fractional $(\mathcal{P}, \mathcal{Q})$-total chromatic number can be obtained as a solution of a problem of linear programming with $|V| + |E|$ inequalities. The main result is that this problem of linear programming for complete graphs is equivalent with another, which has only two inequalities and we can easily solve this problem by simplex method.

2 Definitions and notations

We denote the class of all finite simple graphs by $\mathcal{I}$. A *graph property* $\mathcal{P}$ is a non-empty isomorphism-closed subclass of $\mathcal{I}$. A property P is called *additive* if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$.

We use the following standard notations for specific hereditary properties:

$$\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\},$$

$$\mathcal{O}^k = \{G \in \mathcal{I} : \chi(G) \leq k\},$$

$$\mathcal{D}_k = \{G \in \mathcal{I} : \text{each subgraph of } G \text{ contains a vertex of degree at most } k\},$$

$$\mathcal{T} = \{G \in \mathcal{I} : G \text{ is a planar graph}\},$$

$$\mathcal{I}_k = \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\},$$

$$\mathcal{J}_k = \{G \in \mathcal{I} : \chi'(G) \leq k\},$$

$$\mathcal{O}_k = \{G \in \mathcal{I} : \text{ each component of } G \text{ has at most } k+1 \text{ vertices}\},$$

$$\mathcal{S}_k = \{G \in \mathcal{I} : \Delta(G) \leq k\},$$

where $\chi(G)$ is the *chromatic number*, $\chi'(G)$ the *chromatic index* and $\Delta(G)$ the *maximum degree* of the graph $G = (V, E)$.

Borowiecki and Mihók ([5]) dealt with graph properties and showed that the set of all additive and hereditary properties is a complete distributive lattice $(\mathbb{L}^a, \subseteq)$, where $\mathcal{O}$ is the smallest element of it and $\mathcal{I}$ is the gareatest one. The set of properties $\mathcal{P} \in \mathbb{L}^a$ with $c(\mathcal{P}) = k, k \in \mathbb{N}$ is also a complete distributive lattice $(\mathbb{L}^a_k, \subseteq)$, with the smallest element $\mathcal{O}_k$ and the greatest $\mathcal{I}_k$.

A *total coloring* of a graph G is a coloring of the vertices and edges (together the *elements* of G) such that all adjacent and incident elements obtain distinct colors. The minimum number of colors of a total coloring of G is called total chromatic number $\chi''(G)$ of G.

Definition 1. *Let $G = (V, E)$ be a graph. Let $r, s \in \mathbb{N}$ and $2s \le r$. A $\frac{r}{s}$-fractional ($\mathcal{P}$,$\mathcal{Q}$)-total coloring of G is a mapping $f : V \cup E \to \binom{\{1,2,\ldots,r\}}{s}$ such that for each color i all vertices of color i induce a subgraph from the property $\mathcal{P}$, all edges of color i induce a subgraph from the property $\mathcal{Q}$, moreover the vertices and incident edges have assigned disjoint sets of colors. The fractional ($\mathcal{P}$,$\mathcal{Q}$)-total chromatic number is $\chi''_{1,f,\mathcal{P},\mathcal{Q}}(G) = \inf\{\frac{r}{s}$ such that there exists $\frac{r}{s}$ -fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of $G\}$.*

Definition 2. *($\mathcal{P}$,$\mathcal{Q}$)-independent set is a subset of $V \cup E$ such that the vertices in this set induce a graph from the property $\mathcal{P}$, edges induce a graph from the property $\mathcal{Q}$ and moreover vertices and edges are not incident.*

Definition 3. *Let $I_1, I_2, \ldots, I_t,\ \ t \in \mathbb{N}$ be all (maximal) ($\mathcal{P}$,$\mathcal{Q}$)-independent sets in G. A fractional ($\mathcal{P}$,$\mathcal{Q}$)-total coloring of G is a mapping g, which assigns to each set $I_j,\ \ j = 1, \ldots, t$ a non-negative weight $g(I_j)$ such that $\sum_{u \in I_j} g(I_j) \ge 1$ for each element $u \in V \cup E$. The fractional ($\mathcal{P}$,$\mathcal{Q}$)-total chromatic number $\chi''_{2,f,\mathcal{P},\mathcal{Q}}(G)$ of G is the least total weight of the fractional ($\mathcal{P}$,$\mathcal{Q}$)-total coloring of G.*

Note that definitions 1 and 3 of the fractional total chromatic number $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ are equivalent. Proof is shown in [4].

When we want to find $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ according to Definition 3, we have to solve the following problem of the linear programming:

$$\sum_{j=1}^{t} f(I_j) \to \min$$

$$\sum_{u \in I_j} f(I_j) \ge 1, \quad \forall u \in V \cup E \tag{1}$$

$$f(I_j) \ge 0, \quad \forall j = 1, \ldots, t$$

3 Main results

Let $\mathcal{P} \supseteq \mathcal{O}$ and $\mathcal{Q} \supseteq \mathcal{O}_1$ be two additive and hereditary graph properties and $G = (V, E)$ be a graph with n vertices. We denote a completness $k = c(\mathcal{P}) = \sup\{i : K_{i+1} \in \mathcal{P}\}$ correspond to the property $\mathcal{P}$ for the vertices of the graph G and analogously $l = c(\mathcal{Q}) = \sup\{i : K_{i+1} \in \mathcal{Q}\}$ correspond to the property $\mathcal{Q}$ for the edges. Suppose $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \frac{r}{s}$, where r is the number of used colors, from which we choose s-element subset for every element of graph G. We denote the number of colors used for i vertices by x_i, $\;i \in \{0, \ldots, k+1\}$ and a maximum number of edges, which can be colored by the same color as is used on i vertices as a_i. We can consider a_i as a function of $n - i$ and the property $\mathcal{Q}$. Denote it as $a(n - i, Q) = a_i$. It is easy to see that sequence $\{a_i\}_{i=0}^{k+1}$ is decreasing.

In the following we will often use only notation a_i. Obviously $a_i \geq \lfloor \frac{n-i}{l+1} \rfloor \binom{l+1}{2} + \binom{n-i-\lfloor \frac{n-i}{l+1} \rfloor (l+1)}{2}$, because when we use color c for i vertices, we get $n - i$ vertices without color c. Separate these $n - i$ vertices according to the property $\mathcal{Q}$ into sets with $l + 1$ vertices. These arising subgraphs have at most $\binom{l+1}{2}$ edges. So all subgraphs consist of at most $\lfloor \frac{n-i}{l+1} \rfloor \binom{l+1}{2}$ edges,which induce subgraph from the property $\mathcal{Q}$. The number of the residual edges is $\binom{n-i-\lfloor \frac{n-i}{l+1} \rfloor (l+1)}{2}$. The equality holds for $\mathcal{Q} = \mathcal{O}_l$ and $\mathcal{O}_l$ is the smallest graph property for edges. It means that for other properties strict inequality holds.

The cardinality of multiset of colors used for fractional total coloring of a graph G with n vertices is at least $(n + \binom{n}{2})s$, otherwise we do not have sufficient number of multicolors (colors with their multiplicities) for a correct coloring. So we need at least ns multicolors for the vertices and $\binom{n}{2}s$ for the edges. We get the following two inequalities adequate for finding out of value of $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$:

$$h : \sum_{i=0}^{k+1} x_i \rightarrow \min$$

$$\sum_{i=0}^{k+1} i x_i \geq ns \tag{2}$$

$$\sum_{i=0}^{k+1} a_i x_i \geq \frac{n(n-1)}{2} s$$

$$x_i \geq 0, \quad \forall i = 0, \ldots, k+1$$

Let $x'_i := \frac{x_i}{s}$. Then problem (2) and the following problem are equivalent.

$$h' : \sum_{i=0}^{k+1} x'_i \rightarrow \min$$

$$\sum_{i=0}^{k+1} i x'_i \geq n \tag{3}$$

$$\sum_{i=0}^{k+1} a_i x'_i \geq \frac{n(n-1)}{2}$$

$$x'_i \geq 0, \quad \forall i = 0, \ldots, k+1$$

Now we show that problem (1) and problem (3) are equivalent for complete graphs. It follows that it is sufficient to solve problem (3) only with two inequalities.

Theorem 1. *There exists feasible solution* $\boldsymbol{x'} = (x'_0, x'_1, \ldots, x'_{k+1})$ *of the problem (3) for each optimal solution of the problem (1)* $\boldsymbol{f(I)} = (f(I_1), f(I_2), \ldots, f(I_t))$, $t \in \mathbb{N}$, *moreover,* $h'(\boldsymbol{x'}) \leq h(\boldsymbol{f(I)})$. *There exists feasible solution of the problem (1)* $\boldsymbol{f(I)} = (f(I_1), f(I_2), \ldots, f(I_t))$, $t \in \mathbb{N}$ *for each optimal solution of the problem (3)* $\boldsymbol{x'} = (x'_0, x'_1, \ldots, x'_{k+1})$, *moreover,* $h'(\boldsymbol{x'}) \geq h(\boldsymbol{f(I)})$.

Proof. Let $I_1, \ldots, I_t$, $t \in \mathbb{N}$ be all $(\mathcal{P}, \mathcal{Q})$-independent subsets of $V \cup E$. Suppose that there exists an optimal solution of problem (1) $\mathbf{f(I)} = (f(I_1), f(I_2), \ldots, f(I_t))$, $t \in \mathbb{N}$. Let $x'_i = \sum_{|I_j \cap V| = i} f(I_j)$. According to the assumptions

$$\forall u \in V \cup E : \quad \sum_{u \in I_j} f(I_j) \geq 1 \qquad \text{and} \qquad f(I_j) \geq 0$$

we obtain the following inequality

$$\sum_{\forall v \in V} \sum_{I_j \ni v} f(I_j) \geq n.$$

Then the first inequality that we need holds:

$$\sum_{i=0}^{k+1} i x'_i = \sum_{i=0}^{k+1} i \sum_{|I_j \cap V| = i} f(I_j) = \sum_{v \in V} \sum_{I_j \ni v} f(I_j) \geq n.$$

The last equality holds because on both sides is the sum of all weights over all vertices.
Analogously we show inequality for the edges where we have the following constraints from the previous:

$$\sum_{\forall e \in E} \sum_{I_j \ni e} f(I_j) \geq \binom{n}{2};$$

$$\sum_{i=0}^{k+1} a_i x'_i = \sum_{i=0}^{k+1} a_i \sum_{|I_j \cap V| = i} f(I_j) \geq \sum_{e \in E} \sum_{I_j \ni e} f(I_j) \geq \binom{n}{2}.$$

Now suppose that we have an optimal solution of problem (3) $\mathbf{x}' = (x'_0, x'_1, \ldots, x'_{k+1})$ and according to the previous we know that

$$\sum_{i=0}^{k+1} i x'_i \geq n$$

$$\sum_{i=0}^{k+1} x'_i a_i \geq \binom{n}{2}.$$

Denote $I^i := \{I_j : |I_j \cap V| = i \wedge |I_j \cap E| = a_i\}$, $m_i := |I^i|$ and let

$$f(I_j) = \begin{cases} \frac{x'_i}{m_i}, & \text{if } I_j \in I^i, \\ 0, & \text{otherwise.} \end{cases}$$

Then the following statement holds:

$$\forall v \in V: \quad \sum_{I_j \ni v} f(I_j) = \sum_{i=0}^{k+1} \sum_{|I_j \cap E| = a_i} \frac{x'_i}{m_i} = \sum_{i=0}^{k+1} \frac{x'_i}{m_i} \cdot \frac{i m_i}{n} \geq 1$$

The last equality holds, because when we count all vertices over all I_j from I^i we get a number im_i and independent sets in I^i are symmetric. Therefore we know that each vertex $v \in V$ belongs to $\frac{im_i}{n}$ subsets $I_j \in I^i$ for each $i \in \{0, 1, \ldots, k+1\}$. Analogously we count all edges over all sets $I_j \in I^i$ with non-zero weights and again we use the fact that these sets are symmetric. Consequently each edge belongs to $\frac{a_i m_i}{\binom{n}{2}}$ subsets $I_j \in I^i$ for each $i \in \{0, 1, \ldots, k+1\}$ and so the following statement is also satisfied.

$$\forall e \in E: \quad \sum_{I_j \ni e} f(I_j) = \sum_{i=0}^{k+1} \sum_{|I_j \cap E| = a_i} \frac{x'_i}{m_i} = \sum_{i=0}^{k+1} \frac{x'_i}{m_i} \cdot \frac{a_i m_i}{\binom{n}{2}} \geq 1$$

Values of the objective functions are the same:

$$h'(\mathbf{x}') = \sum_{i=0}^{k+1} x'_i = \sum_{i=0}^{k+1} f(I^i) = \sum_{i=0}^{k+1} \sum_{|I_j \cap V| = i} f(I_j) = \sum_{j=1}^{t} f(I_j) = h(\mathbf{f}(\mathbf{I})).$$

□

We should consider all maximal independent sets I_j in the one way of the last proof, but we can see that it is sufficient to consider only all maximum from maximal sets.

In the following two theorems we show that for arbitrary additive and hereditary $\mathcal{P}$ is $\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n) = \frac{n}{c(\mathcal{P})+1}$ if $\mathcal{Q} = \mathcal{I}_l$ or if $\mathcal{Q}$ is also arbitrary and hereditary property with $c(\mathcal{Q}) \geq c(\mathcal{P}) + 2$.

Theorem 2. *Let $\mathcal{P}, \mathcal{Q}$ be two additive and hereditary properties and $c(\mathcal{Q}) \geq c(\mathcal{P})+2$. Then there exists $T(\mathcal{P},\mathcal{Q})$ such that $\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n) = \frac{n}{c(\mathcal{P})+1}$ holds for each $n \geq T(\mathcal{P},\mathcal{Q})$.*

Proof. Denote $k = c(\mathcal{P})$ and $l = c(\mathcal{Q})$. According to the previous theorem it is sufficient to solve problem (3) to prove this theorem. We rewrite this problem of linear programming to the standard form. We need to use two slack variables $p_1, p_2 \geq 0$:

$$\begin{aligned} h' : \sum_{i=0}^{k+1} x'_i \rightarrow \min \\ -\sum_{i=0}^{k+1} i x'_i + p_1 = -n \\ -\sum_{i=0}^{k+1} a_i x'_i + p_2 = -\frac{n(n-1)}{2} \\ p_1, p_2, x'_i \geq 0, \quad \forall i = 0, \ldots, k+1 \end{aligned} \tag{4}$$

We can see that the variables p_1 and p_2 in the basis and we are able to solve the dual problem:

	x'_0	x'_1	x'_2	$\ldots$	x'_{k-1}	x'_k	x'_{k+1}	p_1	p_2	
	1	1	1	$\ldots$	1	1	1	0	0	0
p_1	0	-1	-2	$\ldots$	$-(k-1)$	$-k$	$-(k+1)$	1	0	$-n$
p_2	$-a_0$	$-a_1$	$-a_2$	$\ldots$	$-a_{k-1}$	$-a_k$	$-a_{k+1}$	0	1	$-\frac{n(n-1)}{2}$

A pivot in the first row of this table is $-(k+1)$. We need to get a number 1 instead of the pivot and zeros instead of other numbers in the column with pivot.

	x'_0	x'_1	$\ldots$	x'_k	x'_{k+1}	p_1	p_2	
	1	$\frac{k}{k+1}$	$\ldots$	$\frac{1}{k+1}$	0	$\frac{1}{k+1}$	0	$-\frac{n}{k+1}$
x'_{k+1}	0	$\frac{1}{k+1}$	$\ldots$	$\frac{k}{k+1}$	1	$-\frac{1}{k+1}$	0	$\frac{n}{k+1} \geq 0$
p_2	$-a_0$	$\frac{a_{k+1}}{k+1} - a_1$	$\ldots$	$\frac{ka_{k+1}}{k+1} - a_k$	0	$-\frac{a_{k+1}}{k+1}$	1	$\frac{na_{k+1}}{k+1} - \frac{n(n-1)}{2}$

If $\frac{na_{k+1}}{k+1} - \frac{n(n-1)}{2} \geq 0$ then the optimal value is $h'(x'^*_0, \ldots, x'^*_{k+1}) = h'(0, \ldots, 0, \frac{n}{k+1}) = \frac{n}{k+1}$. Now we want to find a relationship between $c(\mathcal{P}) = k$ and $c(\mathcal{Q}) = l$ such that $a_{k+1} \geq \frac{(k+1)(n-1)}{2}$. We know:

$$\binom{n-(k+1)-\lfloor\frac{n-(k+1)}{l+1}\rfloor(l+1)}{2} \geq 0$$

$$a_{k+1} = \lfloor\frac{n-(k+1)}{l+1}\rfloor\binom{l+1}{2} + \binom{n-(k+1)-\lfloor\frac{n-(k+1)}{l+1}\rfloor(l+1)}{2} \geq$$

$$\geq \lfloor\frac{n-(k+1)}{l+1}\rfloor\binom{l+1}{2} > (\frac{n-(k+1)}{l+1}-1)\binom{l+1}{2}$$

$$a_{k+1} > \frac{(n-(k+1)-(l+1))l}{2}$$

So we want to show that $\frac{(n-(k+1)-(l+1))l}{2} \geq \frac{(k+1)(n-1)}{2}$ if $l \geq k+2$:

$$\frac{(n-(k+1)-(l+1))l}{2} \geq \frac{(k+1)(n-1)}{2} \quad \Leftrightarrow \quad n(l-k-1)-l(k+2)-l^2+k+1 \geq 0$$

If $l-k-1>0$ we get $n \geq \frac{l^2+l(k+2)-k-1}{l-k-1}$. We can take this lower bound as $T(\mathcal{P},\mathcal{Q})$ but the exact value of $T(\mathcal{P},\mathcal{Q})$ can be lower. If $l-k-1<0$ we get $n < \frac{l^2+l(k+2)-k-1}{l-k-1} < 0$ and so this case cannot occur. If $l-k-1=0$ then the last inequality is equivalent to the expression $-2l^2 \geq 0$ for each n therefore this case cannot occur, too.

□

We get $(k+2)$-tuple $h'(x_0'^*,\dots,x_{k+1}'^*) = h'(0,\dots,0,\frac{n}{k+1})$ as a optimal solution of problem of linear programming in the previous proof. It means that we use each color on exactly $k+1$ vertices and a_{k+1} edges. When we do not get any optimal solution after the first step, we get two nonzero variables $x_{k+1}'^* = \frac{n}{k+1}$ and some $x_i'^* > 0$. In this case we use only colors, which are used exactly on $k+1$ vertices and colors that are used exactly on i vertices. When we consider $\frac{r}{s}$- coloring, the number of used colors on exactly i vertices is $x_i'^*.s$.

Corollary 1. *Let $\mathcal{P},\mathcal{Q}$ be two additive and hereditary properties and $c(\mathcal{Q}) \geq c(\mathcal{P})+2$. Then $\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n) = \frac{n}{c(\mathcal{P})+1}$ if and only if there exists a graph from the property $\mathcal{Q}$ on $n-(c(\mathcal{P})+1)$ vertices with at least $\frac{(n-1)(c(\mathcal{P})+1)}{2}$ edges.*

The result in the following theorem was proved by A.Kemnitz et al. in [7] by constructing of coloring. We prove this result by using results from this paper.

Theorem 3. *Let $\mathcal{P}$ be additive and hereditary property and $\mathcal{Q} = \mathcal{I}_l$. There exists $T(\mathcal{P},\mathcal{I}_l)$ such that for each $n \geq T(\mathcal{P},\mathcal{I}_l)$ it holds that $\chi''_{f,\mathcal{P},\mathcal{I}_l}(K_n) = \frac{n}{c(\mathcal{P})+1}$.*

Proof. Denote $k = c(\mathcal{P})$. This proof is the same as proof of the theorem 2, but we know precise value of a_i for $0 \leq i \leq k+1$ according to well known Turan's theorem. Whereas each $(l+1)-$partite graph does not contain any complete graph on $l+2$ vertices as a subgraph, we can divide vertices into $l+1$ equable sets. We have $n-i-(l+1)\left\lfloor\frac{n-i}{l+1}\right\rfloor < l+1$ sets with $\left\lceil\frac{n-i}{l+1}\right\rceil$ vertices and $l+1-\left(n-i-(l+1)\left\lfloor\frac{n-i}{l+1}\right\rfloor\right)$ sets with $\left\lfloor\frac{n-i}{l+1}\right\rfloor$ vertices. Therefore a_i is number of edges such complete $(l+1)$- partite graph:

$$a_i = \left\lceil\frac{n-i}{l+1}\right\rceil^{n-i-(l+1)\left\lfloor\frac{n-i}{l+1}\right\rfloor} \cdot \left\lfloor\frac{n-i}{l+1}\right\rfloor^{l+1-(n-i-(l+1)\left\lfloor\frac{n-i}{l+1}\right\rfloor)}$$

According to the previous proof we need to find out if $\frac{na_{k+1}}{k+1} - \frac{n(n-1)}{2} \geq 0$. We want to know if for each l and k there exists $T(\mathcal{P},\mathcal{I}_l)$ such that for each $n \geq T(\mathcal{P},\mathcal{I}_l)$ the following inequality is satisfied:

$$\left\lceil\frac{n-(k+1)}{l+1}\right\rceil^{n-(k+1)-(l+1)\left\lfloor\frac{n-(k+1)}{l+1}\right\rfloor} \cdot \left\lfloor\frac{n-(k+1)}{l+1}\right\rfloor^{l+1-(n-(k+1)-(l+1)\left\lfloor\frac{n-(k+1)}{l+1}\right\rfloor)}$$
$$\geq \frac{(n-1)(k+1)}{2}$$

We know that the following inequalities hold:

$$\left\lceil\frac{n-(k+1)}{l+1}\right\rceil^{n-(k+1)-(l+1)\left\lfloor\frac{n-(k+1)}{l+1}\right\rfloor} \cdot \left\lfloor\frac{n-(k+1)}{l+1}\right\rfloor^{l+1-(n-(k+1)-(l+1)\left\lfloor\frac{n-(k+1)}{l+1}\right\rfloor)}$$
$$\geq \left\lfloor\frac{n-(k+1)}{l+1}\right\rfloor^{l+1} \geq \left(\frac{n-(k+1)}{l+1}-1\right)^{l+1} = \left(\frac{n-(k+1)-(l+1)}{l+1}\right)^{l+1}$$
$$\geq \frac{(n-1)(k+1)}{2}$$

The last inequality is obvious, because there exists $T(\mathcal{P},\mathcal{I}_l)$ such that for each $n \geq T(\mathcal{P},\mathcal{I}_l)$ this polynomial of unknown n is non-negative. □

Theorem 4. $\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_n) = \frac{n(n+1)}{2(n-1)}$ *for each integer* $n \geq 3$.

Proof. We have here $a_i = (n-i)-1$ for $i = 0,1,2$, because graph property $\mathcal{D}_1$ is a class of forests. Every a_i is non-negative. According to the previous we solve the following problem of linear programming:

$$x'_0 + x'_1 + x'_2 \to \min$$
$$x'_1 + 2x'_2 \geq n$$
$$(n-1)x'_0 + (n-2)x'_1 + (n-3)x'_2 \geq \frac{n(n-1)}{2}$$
$$x'_0, x'_1, x'_2 \geq 0$$

	x'_0	x'_1	x'_2	p_1	p_2	
	1	1	1	0	0	0
p_1	0	-1	**-2**	1	0	-n
p_2	$1-n$	$2-n$	$3-n$	0	1	$-\frac{n(n-1)}{2}$

After pivoting according to -2 in the p_1-row:

	x'_0	x'_1	x'_2	p_1	p_2	
	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	-$\frac{n}{2}$
x'_2	0	$\frac{1}{2}$	1	-$\frac{1}{2}$	0	$\frac{n}{2}$
p_2	$1-n$	$\frac{1-n}{2}$	0	$\frac{3-n}{2}$	1	-n

After second pivoting according to $1-n$ in p_2-row:

	x'_0	x'_1	x'_2	p_1	p_2	
	0	0	0	$\frac{1}{n-1}$	$\frac{1}{n-1}$	$-\frac{n(n+1)}{2(n-1)}$
x'_2	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{n}{2}$
x'_0	1	$\frac{1}{2}$	0	$\frac{3-n}{2(1-n)}$	$\frac{1}{1-n}$	$\frac{n}{n-1}$

For $n \geq 3$ there are satisfied all conditions that are required for table in optimum. □

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