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Characterization of general classes of distributions based on independent property of transformed record values

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Characterization of general classes of distributions based on independent property of transformed record values

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Abstract

In this paper, general classes of probability distributions are characterized using the independence of suitable transformations of records in a sequence of independent, identically distributed random variables. Examples of special cases of general classes as Gumbel, Fréchet, Weibull, exponential and lognormal distributions are discussed. Further we use the theoretical results for application to simulated data.

Keywords: record values, hazard function, hazard rate, exponential distribution, Fréchet distribution, Gumbel distribution, Log-normal distribution, Weibull distribution, simulation, Hoeffding test

Mathematics Subject Classification 2000: 60E05, 62E10.

1 Introduction

The model of record values was introduced by Chandler in 1952 [8], who studied the stochastic behavior of records and defined the basic terms as record value, record time, inter-record time and frequency of records. The problem of characterization of probability distributions by some properties of record values was opened by Ahsanullah in 1982 when he characterized the exponential distribution ([3]). Ahsanullah presented in 2005 in monograph [4] the recent developments of classical records (records of independent identically distributed random variables) distributed according to exponential law, generalized extreme value distribution, generalized Pareto distribution and power law. Korean mathematicians Lee, Lim and Chang dealt with the problem of characterization of Weibull and Pareto distribution by special transformations of records ([10], [11] and [5]). Abu-Youssef [1], Malinowska and Szynal [12], Faizan and Khan [6] characterized wide classes of distributions using conditional expectation of function of record values.

In this paper, general classes of continuous probability distributions are characterized using independent property of transformed record values. Special cases of general classes as Gumbel, Fréchet, Weibull, exponential and lognormal distributions are also discussed. Some modifications of main theorems are formulated as corollaries. The theoretical results are applied for analysing simulated data. After estimating unknown parameters of chosen distributions we compare the results by Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D) goodness of fit test and Hoeffding test for transformed data.

Consider the sequence $\{X_n, n \geq 1\}$ of independent, identically distributed (iid) random variables with common absolutely continuous distribution function F and probability density function (pdf) f. Variable X_n is an upper record if $X_n > max\{X_1, X_2, \ldots, X_{n-1}\}$ and lower record if $X_n < min\{X_1, X_2, \ldots, X_{n-1}\}$. By convention X_1 is a upper and lower record value. Let $\{T_n, n \geq 1\}$ be the lower record times at which record values occur. We consider discrete time and define $T_1 = 1$ and $T_n = min\{i; i > T_{n-1}, X_i < X_{T_{n-1}}\}$. Further let $\{T^n, n \geq 1\}$ be the sequence of upper record times where again $T^1 = 1$ and $T^n = min\{i; i > T^{n-1}, X_i > X_{T^{n-1}}\}$. So $\{L_n, n \geq 1\} = \{X_{T_n}, n \geq 1\}$ is a sequence of lower record values and $\{R_n, n \geq 1\} = \{X_{T^n}, n \geq 1\}$ is a sequence of upper record values. The distribution of record values is given in terms of hazard function and hazard rate (see [4] page 2).

The function $H(x)$ defined as $H(x) = -\ln F(x)$ is called hazard function for lower records. The function $h(x) = -\frac{dH(x)}{dx} = \frac{f(x)}{F(x)}$ is called hazard rate for lower records.

The function $H(x)$ defined as $H(x) = -\ln(1 - F(x))$ is called hazard function for upper records. The function $h(x) = \frac{dH(x)}{dx} = \frac{f(x)}{1-F(x)}$ is the hazard rate for upper records.

If $F_n(x)$ is the distribution function of random variable L_n (or R_n), $n \geq 1$ and $H(x)$ is the hazard function for lower records (or for upper records) according to the distribution function $F(x)$ of $X_n, n \geq 1$, then

$$F_n(x) = \int_{-\infty}^{x} \frac{H^{n-1}(u)}{(n-1)!} dF(u) \tag{1}$$

and the corresponding density function is

$$f_n(x) = \frac{H^{n-1}(x)}{(n-1)!} f(x). \tag{2}$$

The joint probability density function of $L_1, L_2, \ldots L_n$ is given by formula

$$f_{L_1, L_2, \ldots, L_n}(x_1, x_2, \ldots, x_n) = h(x_1)h(x_2)\ldots h(x_{n-1})f(x_n), \tag{3}$$

for $x_1 > x_2 > \cdots > x_n$ where h is hazard rate for lower records. The density of $R_1, R_2, \ldots R_n$ is

$$f_{R_1, R_2, \ldots, R_n}(x_1, x_2, \ldots, x_n) = h(x_1)h(x_2)\ldots h(x_{n-1})f(x_n), \tag{4}$$

for $x_1 < x_2 < \cdots < x_n$ and h is hazard rate for upper records.

The marginal density function of L_i, L_j (R_i, R_j) is given by

$$f_{i,j}(x_i, x_j) = \frac{(H(x_i))^{i-1}}{\Gamma(i)} h(x_i) \frac{(H(x_j) - H(x_i))^{j-i-1}}{\Gamma(j-i)} f(x_j), \tag{5}$$

for $-\infty < x_j < x_i < \infty \quad (-\infty < x_i < x_j < \infty)$.

2 Main results

Theorem 1. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of iid random variables with absolutely continuous distribution function $F(x)$, $x \in (a,b)$, $(a,b) \subseteq \mathbb{R}$ with $F(a) = 0$ and $F(b) = 1$. Let function $g : (a,b) \to (0,\infty)$ with properties: g is differentiable function, $g'(x) < 0$ for all $x \in (a,b)$, $\lim_{x \to a^+} g(x) = \infty$, $\lim_{x \to b^-} g(x) = 0$. Then the distribution function of $X_1, X_2, \ldots$ is of the form $F(x) = e^{-cg(x)}$, $c > 0$, $x \in (a,b)$ if and only if random variables $g(L_n)$ and $g(L_{n+1}) - g(L_n)$, $n \geq 1$ are independent.*

Remark 1. *Before we give the proof of Theorem 1 we mention some facts. If $g'(x) < 0$ for all $x \in (a,b)$ then g is decreasing function and thus injective function and it holds $g'(g^{-1}(x))(g^{-1})'(x) = 1$. Interval (a,b) can be of the form $(-\infty, b)$, (a,∞) or $(-\infty,\infty)$ so $a = -\infty$ or $b = \infty$. From assumptions of Theorem 1 holds $\lim_{x\to\infty} g^{-1}(x) = a$ and $\lim_{x \to 0^+} g^{-1}(x) = b$.*

Proof. Let $F(x) = e^{-cg(x)}$, $x \in (a,b)$, $c > 0$. Then $f(x) = -cg'(x)e^{-cg(x)}$, $H(x) = -\ln(F(x)) = cg(x)$ and $h(x) = -cg'(x)$. If we use relation (5) then the density of L_n, L_{n+1} is of the form

$$f_{n,n+1}(x,y) = \frac{(cg(x))^{n-1}}{\Gamma(n)} cg'(x)cg'(y)e^{-cg(y)}, \quad x, y \in (a,b). \tag{6}$$

Consider the transformation

$$t : \begin{pmatrix} L_n \\ L_{n+1} \end{pmatrix} \to \begin{pmatrix} g(L_n) \\ g(L_{n+1}) - g(L_n) \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix}, \quad \tau : \begin{pmatrix} U \\ V \end{pmatrix} \to \begin{pmatrix} g^{-1}(U) \\ g^{-1}(U+V) \end{pmatrix}. \tag{7}$$

The determinant of this transformation is

$$\begin{aligned} D_\tau &= \begin{vmatrix} \frac{\partial g^{-1}(u)}{\partial u} & 0 \\ \frac{\partial g^{-1}(u+v)}{\partial u} & \frac{\partial g^{-1}(u+v)}{\partial v} \end{vmatrix} = \begin{vmatrix} (g^{-1})'(x)_{|x=u} \cdot 1 & 0 \\ (g^{-1})'(x)_{|x=u+v} \cdot 1 & (g^{-1})'(x)_{|x=u+v} \cdot 1 \end{vmatrix} \\ &= \begin{vmatrix} (g^{-1})'(u) & 0 \\ (g^{-1})'(u+v) & (g^{-1})'(u+v) \end{vmatrix} = (g^{-1})'(u)(g^{-1})'(u+v). \end{aligned} \tag{8}$$

Then the density of U, V is given by formula

$$\begin{aligned} f_{U,V}(u,v) &= \frac{(cg(g^{-1}(u)))^{n-1}}{\Gamma(n)} cg'(g^{-1}(u))cg'(g^{-1}(u+v))e^{-cg(g^{-1}(u+v))}\times \\ &\times (g^{-1})'(u)(g^{-1})'(u+v) \\ &= \frac{(cu)^{n-1}}{\Gamma(n)} e^{-c(u+v)} ccg'(g^{-1}(u))(g^{-1})'(u)g'(g^{-1}(u+v))(g^{-1})'(u+v) \\ &= \frac{(cu)^{n-1}}{\Gamma(n)} e^{-c(u+v)} cc, \quad u>0, v>0, c>0. \end{aligned} \tag{9}$$

According to (2) the probability density of L_n is

$$f_n(x) = -\frac{(cg(x))^{n-1}}{\Gamma(n)} cg'(x)e^{-cg(x)}, \quad x \in (a,b), c>0 \tag{10}$$

so for density of $U = g(L_n)$ holds

$$\begin{aligned} f_U(u) &= -\frac{(cg(g^{-1}(u)))^{n-1}}{\Gamma(n)} cg'(g^{-1}(u))e^{-cg(g^{-1}(u))}|(g^{-1})'(u)| \\ &= \frac{(cu)^{n-1}}{\Gamma(n)} ce^{-cu} \quad u>0, c>0. \end{aligned} \tag{11}$$

We obtain the density of V by integration of $f_{U,V}(u,v)$ according to du. Thus

$$f_V(v) = \int_0^\infty f_{U,V}(u,v)du = ce^{-cv}, \quad v>0. \tag{12}$$

It is easy to see that $f_{U,V}(u,v) = f_U(u)f_V(v)$, so U and V are independent random variables. The necessary condition is proved.

Assume that random variables U and V are independent and consider the transformation (7) with determinant (8). Then the density of U, V can be written in general form

$$f_{U,V}(u,v) = \frac{(H(g^{-1}(u)))^{n-1}}{\Gamma(n)} h(g^{-1}(u))f(g^{-1}(u+v))|(g^{-1})'(u)(g^{-1})'(u+v)|. \tag{13}$$

We know that $U = g(L_n)$, thus for its density holds

$$f_U(u) = \frac{(H(g^{-1}(u)))^{n-1}}{\Gamma(n)} f(g^{-1}(u))|(g^{-1})'(u)|, \quad u>0. \tag{14}$$

Because U and V are independent then the density of V is

$$f_V(v) = \frac{1}{F(g^{-1}(u))} f(g^{-1}(u+v))|(g^{-1})'(u+v)| \tag{15}$$

$$= -\frac{1}{F(g^{-1}(u))} f(g^{-1}(u+v))(g^{-1})'(u+v). \tag{16}$$

By integration of equality (16) we obtain

$$\int_0^{v_1} f_V(v)dv = \int_0^{v_1} \frac{-1}{F(g^{-1}(u))} f(g^{-1}(u+v))(g^{-1})'(u+v)dv$$

$$F_V(v_1) = \frac{1}{F(g^{-1}(u))}[F(g^{-1}(u)) - F(g^{-1}(u+v_1))]$$

Consider the limit case where $u \to 0^+$, so $g^{-1}(u) \to b$ and $F(g^{-1}(u)) \to 1$. Then $F_V(v_1) = 1 - F(g^{-1}(v_1))$ and it holds

$$F(g^{-1}(v_1)F(g^{-1}(u) = F(g^{-1}(u+v_1)), \quad u > 0, v_1 > 0. \tag{17}$$

Denote $F_1(x) = F(g^{-1}(x))$, $x > 0$, then from (17) we obtain functional equation of the form

$$F_1(v_1)F_1(u) = F_1(u+v_1), \tag{18}$$

called Cauchy's functional equation (see [2]). Its nontrivial solution is $F_1(x) = e^{cx}$, where c is an arbitrary constant and $F(x) = e^{cg(x)}$. Because F is distribution function we have $F(x) = e^{-cg(x)}$, where $c > 0$. These complete the proof. □

A number of distributions can be characterized by the suitable choice of function g and interval (a, b) as we show in next examples.

Examples

1) *Gumbel distribution*

Let $g(x) = \frac{1}{c}e^{-x}$, $c > 0$, $x \in \mathbb{R} = (a, b)$. Then this distribution is characterized by independent property of variables

$$\frac{1}{c}e^{-L_n} \quad \text{and} \quad \frac{1}{c}\left(e^{-L_{n+1}} - e^{-L_n}\right) \tag{19}$$

and the distribution function is $F(x) = e^{-e^{-x}}$, $x \in \mathbb{R}$.

2) *Fréchet distribution*

If we take $g(x) = \frac{1}{c}\left(\frac{\beta}{x}\right)^{\alpha}$, $\alpha > 0$, $\beta > 0$, $c > 0$ and $(a, b) = (0, \infty)$, then the independence of

$$\frac{\beta^{\alpha}}{c} L_n^{-\alpha} \quad \text{and} \quad \frac{\beta^{\alpha}}{c}\left(L_{n+1}^{-\alpha} - L_n^{-\alpha}\right) \tag{20}$$

characterizes Fréchet distribution with $F(x) = e^{-\left(\frac{\beta}{x}\right)^{\alpha}}$, where $x \in (0, \infty)$, $\alpha > 0$, $\beta > 0$, $c > 0$.

3) *Weibull distribution for extreme values*

Proper function and interval are $g(x) = \frac{1}{c}(-x)^{-\alpha}$, $\alpha < 0$, $c > 0$ and $(a, b) = (-\infty, 0)$. Then this distribution is characterized by independence of

$$\frac{1}{c}(-L_n)^{-\alpha} \quad \text{and} \quad \frac{1}{c}\left[(-L_{n+1})^{-\alpha} - (-L_n)^{-\alpha}\right], \tag{21}$$

and the distribution function is $F(x) = e^{-(-x)^{-\alpha}}$, $x \in (-\infty, 0)$, $\alpha < 0$.

4) *Exponential distribution*

If we consider $g(x) = -\frac{1}{c}\ln(1 - e^{-\lambda x})$, $c > 0$, $\lambda > 0$ and $(a, b) = (0, \infty)$, then independence of

$$-\frac{1}{c}\ln(1 - e^{-\lambda L_n}) \quad \text{and} \quad -\frac{1}{c}\ln\left(\frac{1 - e^{-\lambda L_n}}{1 - e^{-\lambda L_{n+1}}}\right) \tag{22}$$

characterizes exponential distribution $F(x) = 1 - e^{-\lambda x}$, $x > 0$, $\lambda > 0$.

5) *Weibull distribution*

Let $g(x) = -\frac{1}{c}\ln\left(1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)$, $x \in (0, \infty) = (a, b)$, $\alpha > 0$, $\beta > 0$, $c > 0$. Independence of variable

$$-\frac{1}{c}\ln\left(1 - e^{-\left(\frac{L_n}{\beta}\right)^{\alpha}}\right) \quad \text{and} \quad \frac{1}{c}\ln\left(\frac{1 - e^{-\left(\frac{L_n}{\beta}\right)^{\alpha}}}{1 - e^{-\left(\frac{L_{n+1}}{\beta}\right)^{\alpha}}}\right) \tag{23}$$

characterizes this distribution with $F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$, $x \in (0, \infty)$, $\alpha > 0$, $\beta > 0$.

6) *Lognormal distribution*

If $g(x) = -\frac{1}{c}\ln\Phi\left(\frac{\ln x - \mu}{\sigma}\right)$, $x \in (0, \infty) = (a, b)$, $\mu \in \mathbb{R}$, $\sigma > 0$, then lognormal distribution with $F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$, where $x \in (0, \infty)$, $\mu \in \mathbb{R}$, $\sigma > 0$ and Φ is standard normal distribution function, is characterized by independence of

$$-\frac{1}{c}\ln\Phi\left(\frac{\ln L_n - \mu}{\sigma}\right) \text{ and } \frac{1}{c}\ln\left(\frac{\Phi\left(\frac{\ln L_n - \mu}{\sigma}\right)}{\Phi\left(\frac{\ln L_{n+1} - \mu}{\sigma}\right)}\right). \tag{24}$$

Remark 2. *Theorem 1 can be modified for $m \geq 2$ independent random variables as we can see in following corollary.*

Corollary 1. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of iid random variables with absolutely continuous distribution function $F(x)$, $x \in (a,b)$, $(a,b) \subseteq \mathbb{R}$ with $F(a) = 0$ and $F(b) = 1$. Let function $g : (a,b) \to (0,\infty)$ with properties: g is differentiable function, $g'(x) < 0$ for all $x \in (a,b)$, $\lim_{x \to a^+} g(x) = \infty$, $\lim_{x \to b^-} g(x) = 0$. Then the distribution function of $X_1, X_2, \ldots$ is of the form $F(x) = e^{-cg(x)}$, $c > 0$, $x \in (a,b)$ if and only if random variables $g(L_1)$, $g(L_2) - g(L_1), \ldots,$ $g(L_m) - g(L_{m-1})$, are independent for $m \geq 2$.*

Proof. The idea of proof is similar as in Theorem 1 so we give only the most important steps. If we assume that $F(x) = e^{-cg(x)}$, $c > 0$, $x \in (a,b)$ then the joint pdf of $L_1, \ldots, L_m$ is

$$f_{L_1,\ldots,L_m}(x_1, \ldots, x_n) = (-c)^m g'(x_1) \ldots g'(x_m) e^{-cg(x_m)}. \tag{25}$$

Now the transformation is of the form

$$\begin{gathered} t : \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{pmatrix} \to \begin{pmatrix} g(L_1) \\ g(L_2) - g(L_1) \\ \vdots \\ g(L_m) - g(L_{m-1}) \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{pmatrix}, \\ \tau : \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{pmatrix} \to \begin{pmatrix} g^{-1}(U_1) \\ g^{-1}(U_1 + U_2) \\ \vdots \\ g^{-1}(U_1 + \cdots + U_m) \end{pmatrix}, \end{gathered} \tag{26}$$

and its determinant $D_\tau = (g^{-1})'(u_1)(g^{-1})'(u_1 + u_2) \ldots (g^{-1})'(u_1 + \cdots + u_m)$. So pdf of $U_1, U_2, \ldots, U_m$ is

$$f_{U_1,U_2,\ldots,U_m}(u_1, u_2, \ldots, u_m) = c^m e^{-c(u_1+u_2+\cdots+u_m)}, \tag{27}$$

for $u_1 > 0, u_2 > 0, \ldots, u_m > 0$, and for marginal pdf holds $f_{U_1}(u_1) = ce^{-cu_1}$, $f_{U_2}(u_2) = ce^{-cu_2}, \ldots, f_{U_m}(u_m) = ce^{-cu_m}$, thus $U_1, U_2, \ldots, U_m$ are independent.

Let $U_1, U_2, \ldots, U_m$ are independent and consider the transformation (26). Then we get

$$f_{U_m}(u_m) = -\frac{f(g^{-1}(u_1 + u_2 + \cdots + u_m))(g^{-1})'(u_1 + u_2 + \cdots + u_m)}{F(g^{-1}(u_1 + u_2 + \cdots + u_{m-1}))} \tag{28}$$

By integration of (28) we obtain that

$$F_{U_m}(u_{m_1}) = \frac{F(g^{-1}(u_1+\cdots+u_{m-1})) - F(g^{-1}(u_1+\cdots+u_{m-1}+u_{m_1}))}{F(g^{-1}(u_1+\cdots+u_{m-1}))}. \quad (29)$$

Limit cases $u_1 \to 0^+, u_2 \to 0^+, \ldots, u_{m-1} \to 0^+$ lead to the same functional equation as in proof of Theorem 1, so $F(x) = e^{-cg(x)}$, where $c > 0$. □

Theorem 2. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of iid random variables with absolutely continuous distribution function $F(x)$, $x \in (a,b)$, $(a,b) \subseteq \mathbb{R}$ with $F(a) = 0$ and $F(b) = 1$. Let function $g : (a,b) \to (0,\infty)$ with properties: g is differentiable function, $g'(x) > 0$ for all $x \in (a,b)$, $\lim_{x\to a^+} g(x) = 0$, $\lim_{x\to b^-} g(x) = \infty$. Then the distribution function of $X_1, X_2, \ldots$ is of the form $F(x) = 1 - e^{-cg(x)}$, $c > 0$, $x \in (a,b)$ if and only if random variables $g(R_n)$ and $g(R_{n+1}) - g(R_n)$, $n \geq 1$ are independent.*

Proof. The proof is similar to the proof of Theorem 1, we only use the hazard function and hazard rate for upper records. □

We also describe some special cases of Theorem 2 by proper choice of function g and interval (a,b).

Examples

1) *Gumbel distribution*

   Let $g(x) = -\frac{1}{c}\ln\left(1 - e^{-e^{-x}}\right)$, $c > 0$, $x \in \mathbb{R} = (a,b)$. Then the independent property of variables

$$-\frac{1}{c}\ln\left(1 - e^{-e^{-R_n}}\right) \quad \text{and} \quad \frac{1}{c}\ln\left(\frac{1 - e^{-e^{-R_n}}}{1 - e^{-e^{-R_{n+1}}}}\right) \quad (30)$$

   characterizes Gumbel distribution $F(x) = e^{-e^{-x}}$, $x \in \mathbb{R}$.

2) *Fréchet distribution*

   If we take $g(x) = -\frac{1}{c}\ln\left(1 - e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right)$, $\alpha > 0$, $\beta > 0$,$c > 0$ and $(a,b) = (0,\infty)$, then independence of

$$-\frac{1}{c}\ln\left(1 - e^{-\left(\frac{\beta}{R_n}\right)^{\alpha}}\right) \quad \text{and} \quad \frac{1}{c}\ln\left(\frac{1 - e^{-\left(\frac{\beta}{R_n}\right)^{\alpha}}}{1 - e^{-\left(\frac{\beta}{R_{n+1}}\right)^{\alpha}}}\right) \quad (31)$$

   characterizes Fréchet distribution with $F(x) = e^{-\left(\frac{\beta}{x}\right)^{\alpha}}$, where $x \in (0,\infty)$, $\alpha > 0$, $\beta > 0$, $c > 0$.

3) *Weibull distribution for extreme values*

Proper function and interval are $g(x) = -\frac{1}{c}\ln\left(1 - e^{-(-x)^{\alpha}}\right)$, $\alpha < 0$, $c > 0$ and $(a,b) = (-\infty, 0)$. Then the independence of

$$-\frac{1}{c}\ln\left(1 - e^{-(-R_n)^{\alpha}}\right) \quad \text{and} \quad \frac{1}{c}\ln\left(\frac{1 - e^{-(-R_n)^{\alpha}}}{1 - e^{-(-R_{n+1})^{\alpha}}}\right) \tag{32}$$

characterizes Weibull distribution for extremes with $F(x) = e^{-(-x)^{-\alpha}}$, $x \in (-\infty, 0)$, $\alpha < 0$.

4) *Exponential distribution*

If we consider $g(x) = \frac{1}{c}\lambda x$, $c > 0$, $\lambda > 0$ and $(a,b) = (0,\infty)$, then the independence of

$$\frac{1}{c}\lambda R_n \quad \text{and} \quad \frac{1}{c}\lambda(R_{n+1} - R_n) \tag{33}$$

characterizes exponential distribution $F(x) = 1 - e^{-\lambda x}$, $x > 0$, $\lambda > 0$.

5) *Weibull distribution*

Let $g(x) = \frac{1}{c}\left(\frac{x}{\beta}\right)^{\alpha}$, $x \in (0,\infty) = (a,b)$, $\alpha > 0$, $\beta > 0$, $c > 0$. Independence of variables

$$\frac{1}{c}\left(\frac{R_n}{\beta}\right)^{\alpha} \quad \text{and} \quad \frac{1}{c\beta^{\alpha}}(R_{n+1}^{\alpha} - R_n^{\alpha}) \tag{34}$$

characterizes this distribution with $F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$, $x \in (0,\infty)$, $\alpha > 0$, $\beta > 0$.

6) *Lognormal distribution*

If $g(x) = -\frac{1}{c}\ln\left(1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)\right)$, $x \in (0,\infty) = (a,b)$, $\mu \in \mathbb{R}$, $\sigma > 0$, then lognormal distribution $F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$, where $x \in (0,\infty)$, $\mu \in \mathbb{R}$, $\sigma > 0$ and Φ is standard normal distribution function, is characterized by independence of

$$-\frac{1}{c}\ln\left[1 - \Phi\left(\frac{\ln R_n - \mu}{\sigma}\right)\right] \text{ and } \frac{1}{c}\ln\left[\frac{1 - \Phi\left(\frac{\ln R_n - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{\ln R_{n+1} - \mu}{\sigma}\right)}\right]. \tag{35}$$

Remark 3. *The following Corollary is modified version of Theorem 2.*

Corollary 2. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of iid random variables with absolutely continuous distribution function $F(x)$, $x \in (a,b)$, $(a,b) \subseteq \mathbb{R}$ with $F(a) = 0$ and $F(b) = 1$. Let function $g : (a,b) \to (0,\infty)$ with properties: g is differentiable* *function,* $g'(x) > 0$ *for all* $x \in (a,b)$, $\lim_{x\to a^+} g(x) = 0$, $\lim_{x\to b^-} g(x) = \infty$. *Then the distribution function of* $X_1, X_2, \ldots$ *is of the form* $F(x) = 1 - e^{-cg(x)}$, $c > 0$, $x \in (a,b)$ *if and only if random variables* $g(R_1)$, $g(R_2)-g(R_1), \ldots,$ $g(R_m)-g(R_{m-1})$, *are independent for* $m \geq 2$.

Proof. The procedure is similar as in Corollary 1, we only use the hazard rate for upper records. □

3 Application to simulated data

In this section we apply theoretical results from Theorem 1 to simulated data. The algorithm is following: Consider independent identically distributed random variables $X_1, X_2, \ldots, X_{500}$. Each random variable X_i, get values $X_{i,1}, X_{i,2}, \ldots, X_{i,1000}$ for $i = 1, \ldots, 500$, which are generated from Fréchet distribution with parameters $\alpha = 3$, $\beta = 2$. We estimate unknown parameters of Weibull, exponential, lognormal and also Fréchet distribution based on observations $X_{1,1}, X_{1,2}, \ldots, X_{1,1000}$. In data $X_{1,j}, \ldots, X_{500,j}$, $j = 1, \ldots, 1000$ we find the realization of recods L_1, L_2. We use Hoeffding's test (see [14]), Theorem 1 and some transformations from first example to test equality of sample and theoretical distributions (to test independence of considered variables). We compare results with K-S and A-D goodness of fit tests.

Remark 4. *The unknow parameters were estimated by maximum-likelihood method and the results are in Table 1. The density plots with estimated parameters we can see on Figure 1. The generated data was transformed by Theorem 1 with proper choice of function* g *as in first example. The test results are given bellow.*

Distribution	$\boldsymbol{\lambda}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\mu}$	$\boldsymbol{\sigma}$
Exponential	0.37	–	–	–	–
Fréchet	–	2.96	2.01	–	–
Weibull	–	2.67	3.04	–	–
Lognormal	–	–	–	0.89	0.43

Table 1: Estimated unknow parameters

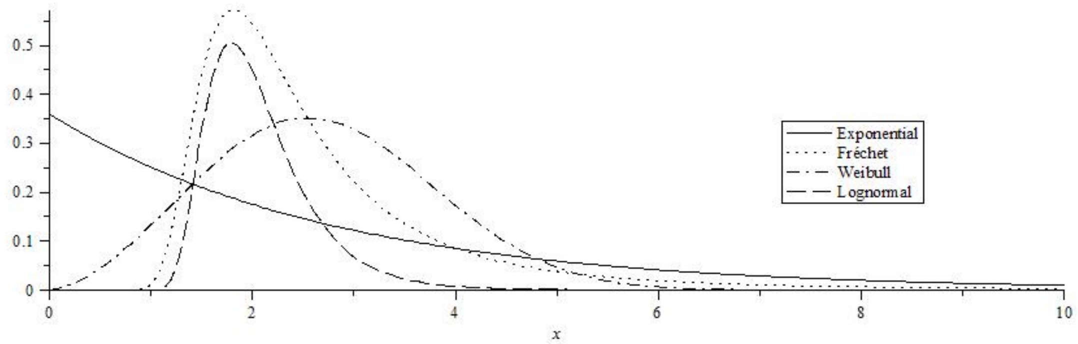

Figure 1: Probability density plots

Test results

Exp. distribution	K-S test	A-D test	Hoeffding test
Statistics	0.3549	162.9127	4.4136
p-Value	0	$6e^{-7}$	0
Fréchet distribution	K-S test	A-D test	Hoeffding test
Statistics	0.0197	0.3094	0.0127
p-Value	0.8342	0.9309	0.9353
Weibull distribution	K-S test	A-D test	Hoeffding test
Statistics	0.1415	52.412	3.1249
p-Value	0	$6e^{-7}$	0
Lognorm. distribution	K-S test	A-D test	Hoeffding test
Statistics	0.064	10.1832	0.9848
p-Value	0.00055	$5.37e^{-6}$	0

Decisions wheater we reject hypothesis H_0 or not are based on p-Values. We establish that only Fréchet distribution with estimated parameters is suitable model for generated data. According to test results we can deduce that Hoeffding test together with Theorem 1 gives identical results as another two goodness of fit tests. The same simulation can be realised also using Theorem 2 and choice proper function g from second example. Also data can be generated from other distributions as we chose.

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