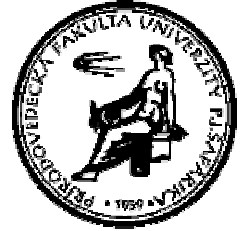




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Abstract. This paper reviews the current state of the art of the mean value theorem due to Thomas M. Flett. We present the results with detailed proofs and provide many new proofs of known results. Moreover, some new observations and yet unpublished results are included.

Key words and phrases. Flett's mean value theorem, real-valued function, differential function, Taylor's polynomial, Pawlikowska's theorem

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1 Introduction and preliminaries

Motivations and basic aim Mean value theorems of differential and integral calculus provide a relatively simple, but very powerful tool of mathematical analysis suitable for solving many diverse problems. Every student of mathematics knows the Lagrange's mean value theorem which has appeared in Lagrange's book *Theorie des fonctions analytiques* in 1797 as an extension of Rolle's result from 1691. More precisely, Lagrange's theorem says that for a continuous (real-valued) function f on a compact set $\langle a, b \rangle$ which is differentiable on (a, b) there exists a point $\eta \in (a, b)$ such that

$$f'(\eta) = \frac{f(b) - f(a)}{b - a}.$$

Geometrically Lagrange's theorem states that given a line ℓ joining two points on the graph of a differentiable function f , namely $[a, f(a)]$ and $[b, f(b)]$, then there exists a point $\eta \in (a, b)$ such that the tangent at $[\eta, f(\eta)]$ is parallel to the given line ℓ , see Fig. 1. Clearly, Lagrange's theorem reduces to Rolle's theorem

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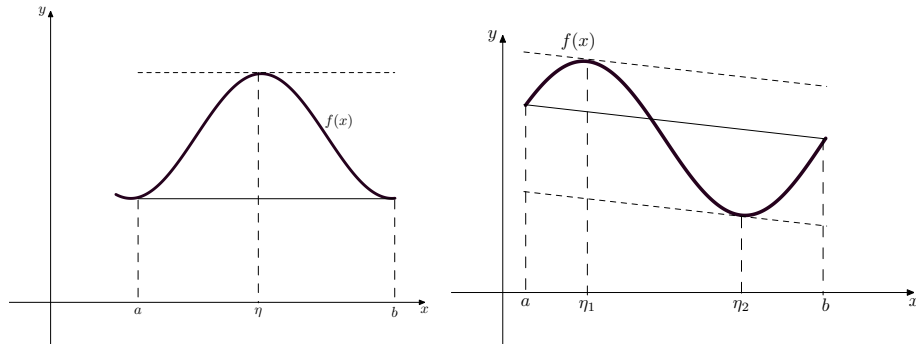


Figure 1: Geometrical interpretation of Rolle's and Lagrange's theorem

if $f(a) = f(b)$. In connection with these well-known facts the following questions may arise: *Are there changes if in Rolle's theorem the hypothesis $f(a) = f(b)$ refers to higher-order derivatives? Then, is there any analogy with the Lagrange's theorem? Which geometrical consequences do such results have?* These (and many other) questions will be investigated in this paper in which we provide a survey of known results as well as of our observations and obtained new results.

Notation Throughout this paper we will use the following unified notation: $\mathcal{C}(M)$, resp. $\mathcal{D}^n(M)$, will denote the spaces of continuous, resp. n -times differentiable real-valued functions on a set $M \subseteq \mathbb{R}$. Usually we will work with a compact set of the real line, i.e., $M = \langle a, b \rangle$ with $a < b$. Therefore, we recall that under continuity of a function on $\langle a, b \rangle$ we understand its continuity on (a, b) and one-sided continuity at the end points of the interval. Similarly we will understand the notion of differentiability on a closed interval. For functions f, g on an interval $\langle a, b \rangle$ (for which the following expression has its sense) the expressions of the form

$$\frac{f^{(n)}(b) - f^{(n)}(a)}{g^{(n)}(b) - g^{(n)}(a)}, \quad n \in \mathbb{N} \cup \{0\},$$

will be denoted by the symbol ${}^b_a\mathcal{K}(f^{(n)}, g^{(n)})$. If the denominator is equal to $b - a$, we will write only ${}^b_a\mathcal{K}(f^{(n)})$. So, Lagrange's theorem in the introduced notation has the following form: if $f \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$, then there exists $\eta \in (a, b)$ such that $f'(\eta) = {}^b_a\mathcal{K}(f)$, where we use the usual convention $f^{(0)} := f$.

Structure of this paper The organization of this paper is as follows: in Section 2 we present the original result of Flett as well as its generalization due to Riedel and Sahoo removing the boundary condition. Further sufficient conditions of Trahan and Tong for validity of assertion of Flett's theorem are described in Section 3 together with proving two new extensions and the detailed comparison of all the presented conditions. Section 4 deals with integral version

of Flett's theorem and related results. In the last Section 5 we give a new proof of higher-order generalization of Flett's mean value theorem due to Pawlikowska.

2 Flett's mean value theorem

Let us begin with the following easy observation from [5]: if $g \in \mathcal{C}\langle a, b \rangle$, then from the integral mean value theorem there exists $\eta \in (a, b)$ such that

$$g(\eta) = \frac{1}{b-a} \int_a^b g(t) dt.$$

Moreover, if we consider the function $g \in \mathcal{C}\langle a, b \rangle$ with the properties

$$g(a) = 0, \quad \int_a^b g(t) dt = 0, \quad (1)$$

and define the function

$$\varphi(x) = \begin{cases} \frac{1}{x-a} \int_a^x g(t) dt, & x \in (a, b), \\ 0, & x = a, \end{cases}$$

then $\varphi \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$ and $\varphi(a) = 0 = \varphi(b)$. Thus, by Rolle's theorem there exists $\eta \in (a, b)$ such that $\varphi'(\eta) = 0$, i.e.,

$$\frac{g(\eta)}{\eta-a} - \frac{1}{(\eta-a)^2} \int_a^\eta g(t) dt = 0 \Leftrightarrow g(\eta) = \frac{1}{\eta-a} \int_a^\eta g(t) dt.$$

The latter formula resembles the one from integral mean value theorem replacing formally b by η . It is well-known that the mean value φ (known as the integral mean) of function g on the interval $\langle a, x \rangle$ is in general less irregular in its behaviour than g itself. When defining the function g we may ask whether the second condition in (1) may be replaced by a simpler condition, e.g., by the condition $g(b) = 0$. Later we will show that it is possible and the result in this more general form is a consequence of Darboux's intermediate value theorem, see the proof of Flett's theorem. If we define the function

$$f(x) = \int_a^x g(t) dt, \quad x \in \langle a, b \rangle,$$

from our considerations we get an equivalent form of result to which this paper is devoted. This result is an observation of THOMAS MUIRHEAD FLETT (1923–1976) from 1958 published in his paper [5]. Indeed, it is a variation on the theme of Rolle's theorem where the condition $f(a) = f(b)$ is replaced by $f'(a) = f'(b)$, or, we may say that it is a Lagrange's type mean value theorem with a Rolle's type condition.

Theorem 2.1 (Flett, 1958) *If $f \in \mathcal{D}\langle a, b \rangle$ and $f'(a) = f'(b)$, then there exists $\eta \in (a, b)$ such that*

$$f'(\eta) = {}^{\eta}_a \mathcal{K}(f). \quad (2)$$

For the sake of completeness (and because it is not wide known) we give the original proof of Flett's theorem adapted from [5] and rewritten in the sense of introduced notation.

Proof of Flett's theorem Without loss of generality assume that $f'(a) = f'(b) = 0$. If it is not the case we take the function $h(x) = f(x) - xf'(a)$ for $x \in \langle a, b \rangle$. Put

$$g(x) = \begin{cases} {}^x_a \mathcal{K}(f), & x \in (a, b) \\ f'(a), & x = a. \end{cases} \quad (3)$$

Obviously, $g \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$ and

$$g'(x) = -\frac{{}^x_a \mathcal{K}(f) - f'(x)}{x - a} = -{}^x_a \mathcal{K}(g) + {}^x_a \mathcal{K}(f'), \quad x \in (a, b).$$

It is enough to show that there exists $\eta \in (a, b)$ such that $g'(\eta) = 0$.

From the definition of g we have that $g(a) = 0$. If $g(b) = 0$, then Rolle's theorem guarantees the existence of a point $\eta \in (a, b)$ such that $g'(\eta) = 0$. Let $g(b) \neq 0$ and suppose that $g(b) > 0$ (similar arguments apply if $g(b) < 0$). Then

$$g'(b) = -{}^b_a \mathcal{K}(g) = -\frac{g(b)}{b - a} < 0.$$

Since $g \in \mathcal{C}\langle a, b \rangle$ and $g'(b) < 0$, i.e., g is strictly decreasing in b , then there exists $x_1 \in (a, b)$ such that $g(x_1) > g(b)$. From continuity of g on $\langle a, x_1 \rangle$ and from relations $0 = g(a) < g(b) < g(x_1)$ we deduce from Darboux's intermediate value theorem that there exists $x_2 \in (a, x_1)$ such that $g(x_2) = g(b)$. Since $g \in \mathcal{C}\langle x_2, b \rangle \cap \mathcal{D}(x_2, b)$, from Rolle's theorem we have $g'(\eta) = 0$ for some $\eta \in (x_2, b) \subset (a, b)$.
 \square

A different proof of Flett's theorem using Fermat's theorem (necessary condition for the existence of a local extremum) may be found in [18, p.225].

Geometrical meaning of Flett's theorem If a curve $y = f(x)$ has a tangent at each point of $\langle a, b \rangle$ and tangents at the end points $[a, f(a)]$ and $[b, f(b)]$ are parallel, then Flett's theorem guarantees the existence of such a point $\eta \in (a, b)$ that the tangent constructed to the graph of f at that point passes through the point $[a, f(a)]$, see Fig. 2.

Remark 2.2 Clearly, the assertion of Flett's theorem may be valid also in the cases when its assumption is not fulfilled. For instance, function $f(x) = |x|$ on

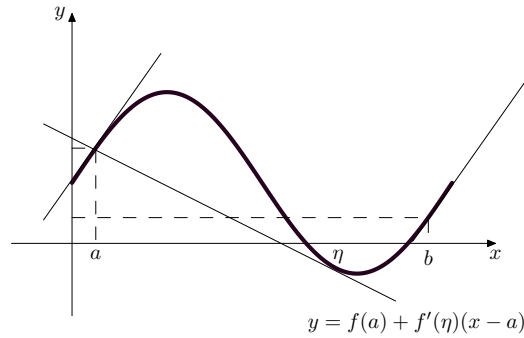


Figure 2: Geometrical interpretation of Flett's theorem

the interval $\langle a, b \rangle$, with $a < 0 < b$, is not differentiable, but there exist infinite many points $\eta \in (a, 0)$ for which the tangent constructed in the point η passes through the point $[a, -a]$ (since the tangent coincides with the graph of function $f(x)$ for $x \in (a, 0)$).

Another example is the function $g(x) = \operatorname{sgn} x$ and $h(x) = [x]$ (sign function and floor function) on the interval $\langle -1, 1 \rangle$ which are not differentiable on $\langle -1, 1 \rangle$. Finally, the function $k(x) = \arcsin x$ on $\langle -1, 1 \rangle$ is not differentiable at the end points, but assertion of Flett's theorem still holds (we will consider other sufficient conditions for validity of (2) in Section 3, namely k fulfills Tong's condition).

We can observe that the functions g and k have improper derivatives at the points in which are not differentiable, i.e., $g'_+(0) = g'_-(0) = k'_+(-1) = k'_-(1) = +\infty$. Therefore, we state the conjecture that Flett's theorem still holds in that case.

Conjecture 2.3 *If f has a proper or improper derivative at each point of the interval $\langle a, b \rangle$ and the tangents at the end points are parallel, then there exists $\eta \in (a, b)$ such that (2) holds.*

Remark 2.4 Assertion of Flett's theorem may be written in the following equivalent forms:

$$f'(\eta) = \frac{f(\eta) - f(a)}{\eta - a} \Leftrightarrow f(a) = T_1(f, \eta)(a) \Leftrightarrow \begin{vmatrix} f'(\eta) & 1 & 0 \\ f(a) & a & 1 \\ f(\eta) & \eta & 1 \end{vmatrix} = 0.$$

In the second expression $T_1(f, x_0)(x)$ is the first Taylor's polynomial (or, in other words a tangent) of function f at the point x_0 as a function of x . The last expression resembles an equivalent formulation of the assertion of Lagrange's theorem in the form of determinant, i.e.,

$$\begin{vmatrix} f'(\eta) & 1 & 0 \\ f(a) & a & 1 \\ f(b) & b & 1 \end{vmatrix} = 0.$$

This motivates us to state the following question:

Question 2.5 *Is it possible to find a similar proof (as a derivative of a function given in the form of determinant) of Flett's theorem?*

In connection with applicability of Flett's theorem there exists many interesting problems proposed and solved by various authors, see e.g. the problems and solutions section of journals as American Mathematical Monthly, Electronic Journal of Differential Equations, etc. A nice application of Flett's theorem for investigating some integral mean value theorems is given in [9] and similar approach is used in [6]. We give here only one representative example of this kind. The problem (2011-4 in Electronic Journal of Differential Equations) was proposed by Duong Viet Thong, Vietnam. The solution to this problem is our own.

Problem 2.6 *Let $f \in \mathcal{C}\langle 0, 1 \rangle$ and*

$$\int_0^1 f(x) dx = \int_0^1 xf(x) dx.$$

Prove that there exists $\eta \in (0, 1)$ such that

$$\eta^2 f(\eta) = \int_0^\eta xf(x) dx.$$

Solution. Consider the differentiable function

$$G(t) = \int_0^t xf(x) dx, \quad t \in \langle 0, 1 \rangle.$$

Clearly, $G'(t) = tf(t)$ for each $t \in \langle 0, 1 \rangle$. By [9, Lemma 2.8] there exists $\zeta \in (0, 1)$ such that $G(\zeta) = \int_0^\zeta xf(x) dx = 0$. Since $G(0) = 0$, then by Rolle's theorem there exists $\theta \in (0, \zeta)$ such that $G'(\theta) = 0$. From $G'(0) = 0 = G'(\theta)$ by Flett's theorem there exists $\eta \in (0, \theta)$ such that

$$G'(\eta) = {}_0^{\eta}\mathcal{K}(G) \Leftrightarrow \eta f(\eta) = \frac{G(\eta)}{\eta} \Leftrightarrow \eta^2 f(\eta) = \int_0^\eta xf(x) dx. \quad \square$$

Naturally, we may ask whether the Lagrange's idea to remove the equality $f(a) = f(b)$ from Rolle's theorem is applicable for Flett's theorem, i.e., whether the assumption $f'(a) = f'(b)$ may be removed for the purpose to obtain a more general result. First result of that kind has appeared in the book [22].

Theorem 2.7 (Riedel-Sahoo, 1998) *If $f \in \mathcal{D}\langle a, b \rangle$, then there exists $\eta \in (a, b)$ such that*

$${}_a^{\eta}\mathcal{K}(f) = f'(\eta) - {}_a^b\mathcal{K}(f') \cdot \frac{\eta - a}{2}.$$

In their original proof [22] Riedel and Sahoo consider the auxiliary function ψ given by

$$\psi(x) = f(x) - {}^b_a\mathcal{K}(f') \cdot \frac{(x-a)^2}{2}, \quad x \in \langle a, b \rangle, \quad (4)$$

and apply Flett's theorem to it. Indeed, function ψ is constructed as a difference of f and its quadratic approximation $A+B(x-a)+C(x-a)^2$ at a neighbourhood of a . From $\psi'(a) = \psi'(b)$ we get $C = \frac{1}{2} \cdot {}^b_a\mathcal{K}(f')$, and because A and B may be arbitrary, they put $A = B = 0$. Of course, the function ψ is not the only function which does this job. For instance, the function

$$\Psi(x) = f(x) - {}^b_a\mathcal{K}(f') \cdot \left(\frac{x^2}{2} - ax \right), \quad x \in \langle a, b \rangle,$$

does the same, because $\Psi'(x) = \psi'(x)$ for each $x \in \langle a, b \rangle$. In what follows we provide a different proof of Riedel-Sahoo's theorem with an auxiliary function of different form.

New proof of Riedel-Sahoo's theorem. Let us consider the function F defined by

$$F(x) = \begin{vmatrix} f(x) & x^2 & x & 1 \\ f(a) & a^2 & a & 1 \\ f'(a) & 2a & 1 & 0 \\ f'(b) & 2b & 1 & 0 \end{vmatrix}, \quad x \in \langle a, b \rangle.$$

Clearly, $F \in \mathcal{D}\langle a, b \rangle$ and

$$F'(x) = \begin{vmatrix} f'(x) & 2x & 1 & 0 \\ f(a) & a^2 & a & 1 \\ f'(a) & 2a & 1 & 0 \\ f'(b) & 2b & 1 & 0 \end{vmatrix}, \quad x \in \langle a, b \rangle.$$

Thus, $F'(a) = F'(b) = 0$, and by Flett's theorem there exists $\eta \in (a, b)$ such that $F'(\eta) = {}^a_\eta\mathcal{K}(F)$, which is equivalent to the assertion of Riedel-Sahoo's theorem. \square

Remark 2.8 As in the case of Flett's theorem it is easy to observe that the assertion of Riedel-Sahoo's theorem may be equivalently written as follows

$$\begin{vmatrix} f'(\eta) & 1 & 0 \\ f(a) & a & 1 \\ f(\eta) & \eta & 1 \end{vmatrix} = {}^b_a\mathcal{K}(f') \cdot \frac{(\eta-a)}{2}.$$

The geometrical fact behind Flett's theorem is a source of interesting study in [4] we would like to mention here in connection with Riedel-Sahoo's theorem.

Following [4] we will say that *the graph of $f \in \mathcal{C}\langle a, b \rangle$ intersects its chord in the extended sense* if either there is a number $\eta \in (a, b)$ such that

$${}^{\eta}_a\mathcal{K}(f) = {}^b_a\mathcal{K}(f), \quad \text{or} \quad \lim_{x \rightarrow a^+} {}^x_a\mathcal{K}(f) = {}^b_a\mathcal{K}(f).$$

Now, for $f \in \mathcal{C}\langle a, b \rangle$ denote by M the set of all points $x \in \langle a, b \rangle$ in which f is non-differentiable and put $\mathfrak{m} = |M|$. Define the function

$$\mathcal{F}(x) := \frac{1}{x-a} (f'(x) - {}^x_a\mathcal{K}(f)), \quad x \in (a, b) \setminus M.$$

Then the assertion of Flett's theorem is equivalent to $\mathcal{F}(\eta) = 0$. Clearly, if $\mathfrak{m} = 0$, then by Riedel-Sahoo's theorem there exists $\eta \in (a, b)$ such that

$$\mathcal{F}(\eta) = \frac{1}{2} \cdot {}^b_a\mathcal{K}(f').$$

So, what if $\mathfrak{m} > 0$?

Theorem 2.9 (Powers-Riedel-Sahoo, 2001) *Let $f \in \mathcal{C}\langle a, b \rangle$.*

(i) *If $\mathfrak{m} \leq n$ for some non-negative integer and $a \notin M$, then there exist $n + 1$ points $\eta_1, \dots, \eta_{n+1} \in (a, b)$ and $n + 1$ positive numbers $\alpha_1, \dots, \alpha_{n+1}$ with $\sum_{i=1}^{n+1} \alpha_i = 1$ such that*

$$\sum_{i=1}^{n+1} \alpha_i \mathcal{F}(\eta_i) = \frac{1}{b-a} ({}^b_a\mathcal{K}(f) - f'(a)).$$

(ii) *If \mathfrak{m} is infinite and the graph of f intersects its chord in the extended sense, then there exist $\eta \in (a, b)$ and two positive numbers δ_1, δ_2 such that*

$$\text{either } \mathcal{F}_1(\eta, h) \leq 0 \leq \mathcal{F}_2(\eta, k), \quad \text{or} \quad \mathcal{F}_2(\eta, k) \leq 0 \leq \mathcal{F}_1(\eta, h),$$

holds for $h \in (0, \delta_1)$ and $k \in (0, \delta_2)$, where

$$\mathcal{F}_1(\eta, h) := (\eta - a) ({}^{\eta-h}_\eta\mathcal{K}(f) - {}^{\eta}_a\mathcal{K}(f)),$$

$$\mathcal{F}_2(\eta, k) := (\eta - a) ({}^{\eta+k}_\eta\mathcal{K}(f) - {}^{\eta}_a\mathcal{K}(f)).$$

In item (i) we note that if $f'(a) = {}^b_a\mathcal{K}(f)$, i.e., the second condition for the graph of f intersecting its chord in the extended sense holds, then the convex combination of values of \mathcal{F} at points η_i , $i = 1, \dots, n + 1$, is simply zero. If, in item (ii), f is differentiable at η , then

$$\lim_{h \rightarrow 0^+} \frac{\mathcal{F}_1(\eta, h)}{(\eta - a)^2} = \lim_{k \rightarrow 0^+} \frac{\mathcal{F}_2(\eta, k)}{(\eta - a)^2} = \mathcal{F}(\eta).$$

The proof of item (i) can be found in [17] and the proof of (ii) is given in [4]. Note that in the paper [17] authors extended the results of Theorem 2.9 in the context of topological vector spaces X, Y for a class of Gateaux differentiable functions $f : X \rightarrow Y$.

Flett's and Riedel-Sahoo's theorem give an opportunity to study the behaviour of intermediate points from different points of view. Recall that points η (depending on the interval $\langle a, b \rangle$) from Flett's, or Riedel-Sahoo's theorem are called *the Flett's*, or *the Riedel-Sahoo's points of function f on the interval $\langle a, b \rangle$* , respectively.

The questions of stability of Flett's points was firstly investigated in [3], but the main result therein was shown to be incorrect. In paper [8] the correction was made and the following results on Hyers-Ulam's stability of Riedel-Sahoo's and Flett's points were proved.

Theorem 2.10 (Lee-Xu-Ye, 2009) *Let $f \in \mathcal{D}\langle a, b \rangle$ and η be a Riedel-Sahoo's point of f on $\langle a, b \rangle$. If f is twice differentiable at η and*

$$f''(\eta)(\eta - a) - 2f'(\eta) + 2_a^n \mathcal{K}(f) \neq 0,$$

then to any $\varepsilon > 0$ and any neighborhood $N \subset (a, b)$ of η , there exists a $\delta > 0$ such that for every $g \in \mathcal{D}\langle a, b \rangle$ satisfying $|g(x) - g(a) - (f(x) - f(a))| < \delta$ for $x \in N$ and $g'(b) - g'(a) = f'(b) - f'(a)$, there exists a point $\xi \in N$ such that ξ is a Riedel-Sahoo's point of g and $|\xi - \eta| < \varepsilon$.

As a corollary we get the Hyers-Ulam's stability of Flett's points.

Theorem 2.11 (Lee-Xu-Ye, 2009) *Let $f \in \mathcal{D}\langle a, b \rangle$ with $f'(a) = f'(b)$ and η be a Flett's point of f on $\langle a, b \rangle$. If f is twice differentiable at η and*

$$f''(\eta)(\eta - a) - 2f'(\eta) + 2_a^n \mathcal{K}(f) \neq 0,$$

then to any $\varepsilon > 0$ and any neighborhood $N \subset (a, b)$ of η , there exists a $\delta > 0$ such that for every $g \in \mathcal{D}\langle a, b \rangle$ satisfying $g(a) = f(a)$ and $|g(x) - f(x)| < \delta$ for $x \in N$, there exists a point $\xi \in N$ such that ξ is a Flett's point of g and $|\xi - \eta| < \varepsilon$.

Another interesting question is the limit behaviour of Riedel-Sahoo's points (Flett's points are not interesting because of the condition $f'(a) = f'(b)$). We demonstrate the main idea on the following easy example: let $f(t) = t^3$ for $t \in \langle 0, x \rangle$ with $x > 0$. By Riedel-Sahoo's theorem for each $x > 0$ there exists a point $\eta_x \in (0, x)$ such that

$$3\eta_x^2 = \frac{\eta_x^3}{\eta_x} + \frac{3x^2}{x} \cdot \frac{\eta_x}{2} \Leftrightarrow 4\eta_x^2 = 3x\eta_x \Leftrightarrow \eta_x = \frac{3}{4}x.$$

Thus, we have obtained a dependence of Riedel-Sahoo's points on x . If we shorten the considered interval, we get

$$\lim_{x \rightarrow 0^+} \frac{\eta_x - 0}{x - 0} = \lim_{x \rightarrow 0^+} \frac{\frac{3}{4}x}{x} = \frac{3}{4}.$$

So, how do Flett's points behave for the widest class of function? In paper [16] authors proved the following result.

Theorem 2.12 (Powers-Riedel-Sahoo, 1998) *Let $f \in \mathcal{D}\langle a, a+x \rangle$ be such that*

$$f(t) = p(t) + (t-a)^\alpha g(t), \quad \alpha \in (1, 2) \cup (2, +\infty),$$

where p is a polynomial at most of second order, g' is bounded on the interval $(a, a+x)$ and $g(a) = \lim_{x \rightarrow 0^+} g(a+x) \neq 0$. Then

$$\lim_{x \rightarrow 0^+} \frac{\eta_x - a}{x} = \left(\frac{\alpha}{2(\alpha-1)} \right)^{\frac{1}{\alpha-2}},$$

where η_x are the corresponding Riedel-Sahoo's points of f on $\langle a, a+x \rangle$.

Problem 2.13 *Enlarge the Power-Riedel-Sahoo's family of functions for which it is possible to state the exact formula for limit properties of corresponding intermediate points.*

3 Further sufficient conditions for validity of (2)

3.1 Trahan's inequalities

Probably the first study about Flett's result and its generalization is dated to the year 1966 by DONALD H. TRAHAN [24]. He provides a different condition for the assertion of Flett's theorem under some inequality using a comparison of slopes of secant line passing through the end points and tangents at the end points.

Theorem 3.1 (Trahan, 1966) *Let $f \in \mathcal{D}\langle a, b \rangle$ and*

$$(f'(b) - {}^b_a\mathcal{K}(f)) \cdot (f'(a) - {}^b_a\mathcal{K}(f)) \geq 0. \quad (5)$$

Then there exists $\eta \in (a, b)$ such that (2) holds.

Donald Trahan in his proof again considers the function g given by (3). Then $g \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$ and

$$g'(x) = \frac{1}{x-a} (f'(x) - {}^x_a\mathcal{K}(f)), \quad x \in (a, b).$$

Since

$$[g(b) - g(a)]g'(b) = \frac{-1}{b-a} (f'(b) - {}^b_a\mathcal{K}(f)) \cdot (f'(a) - {}^b_a\mathcal{K}(f)),$$

then by (5) we get $[g(b) - g(a)]g'(b) \leq 0$. Now Trahan concludes that $g'(\eta) = 0$ for some $\eta \in (a, b)$, which is equivalent to (2).

The only step here is to prove **Trahan's lemma**, i.e., the assertion of Rolle's theorem under the conditions $g \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$ and $[g(b) - g(a)]g'(b) \leq 0$. Easily, if $g(a) = g(b)$, then Rolle's theorem gives the desired result. If $g'(b) = 0$, putting $\eta = b$ we have $g'(\eta) = 0$. So, let us assume that $[g(b) - g(a)]g'(b) < 0$. This means that either $g'(b) < 0$ and $g(b) > g(a)$, or $g'(b) > 0$ and $g(b) < g(a)$. In the first case, since $g \in \mathcal{C}\langle a, b \rangle$, $g(b) > g(a)$ and g is strictly decreasing in b , then g has its maximum at $\eta \in (a, b)$ and by Fermat's theorem we get $g'(\eta) = 0$. Similarly, in the second case g has minimum at the same point $\eta \in (a, b)$, thus $g'(\eta) = 0$.

Remark 3.2 Obviously, the class of Trahan's functions, i.e., differentiable functions on $\langle a, b \rangle$ satisfying Trahan's condition (5), is wider than the class of Flett's functions $f \in \mathcal{D}\langle a, b \rangle$ satisfying Flett's condition $f'(a) = f'(b)$. Indeed, for $f'(a) = f'(b)$ Trahan's condition (5) is trivially fulfilled. On the other hand the function $y = x^3$ for $x \in \langle -\frac{1}{2}, 1 \rangle$ does not satisfy Flett's condition, and it is easy to verify that it satisfies Trahan's one.

Geometrical meaning of Trahan's condition Clearly, Trahan's inequality (5) holds if and only if

$$[f'(b) \geq {}^b_a\mathcal{K}(f) \wedge f'(a) \geq {}^b_a\mathcal{K}(f)] \vee [f'(b) \leq {}^b_a\mathcal{K}(f) \wedge f'(a) \leq {}^b_a\mathcal{K}(f)].$$

Since ${}^b_a\mathcal{K}(f)$ gives the slope of the secant line between $[a, f(a)]$ and $[b, f(b)]$, Trahan's condition requires either both slopes of tangents at the end points are greater or equal, or both are smaller or equal to ${}^b_a\mathcal{K}(f)$. We consider two cases:

- (i) if $f'(b) = {}^b_a\mathcal{K}(f)$, then the tangent at b is parallel to the secant, and the tangent at a may be arbitrary (parallel to the secant, lying above or under the graph of secant on (a, b)), analogously for $f'(a) = {}^b_a\mathcal{K}(f)$;
- (ii) if $f'(b) \neq {}^b_a\mathcal{K}(f)$ and $f'(a) \neq {}^b_a\mathcal{K}(f)$, then one of the tangents at the end points has to lie above and the second one under the graph of secant line on (a, b) , or vice versa, see Fig. 3. More precisely, let tangent at a intersect the line $x = b$ at the point $Q = [b, y_Q]$ and tangent at b intersect the line $x = a$ at the point $P = [a, y_P]$. Then either $y_Q > f(b)$ and $y_P < f(a)$, or $y_Q < f(b)$ and $y_P > f(a)$. For parallel tangents at the end points, i.e., for $f'(a) = f'(b)$, this geometrical interpretation provides a new insight which leads to the already mentioned paper [4].

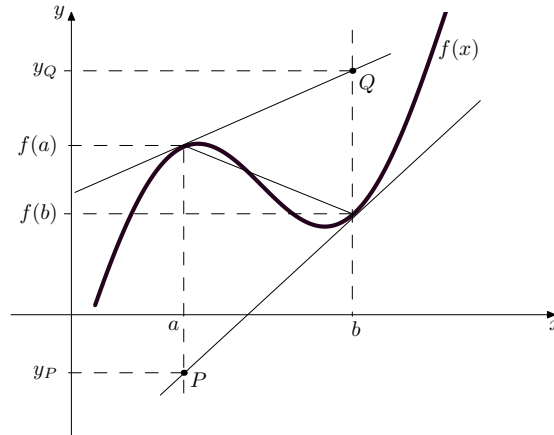


Figure 3: Geometrical interpretation of Trahan's condition

Moreover, Trahan in his paper [24] provides other generalization of Flett's theorem. Namely, he proves certain „Cauchy form“ of his result for two functions which will be a source of our results later in Section 3.3.

Theorem 3.3 (Trahan, 1966) *Let $f, g \in \mathcal{D}\langle a, b \rangle$, $g'(x) \neq 0$ for each $x \in \langle a, b \rangle$ and*

$$\left(\frac{f'(a)}{g'(a)} - {}^b_a\mathcal{K}(f, g) \right) \left({}^b_a\mathcal{K}(g) f'(b) - {}^b_a\mathcal{K}(f) g'(b) \right) \geq 0.$$

Then there exists $\eta \in (a, b)$ such that $\frac{f'(\eta)}{g'(\eta)} = {}^{\eta}_a\mathcal{K}(f, g)$.

Its proof is based on application of Trahan's lemma [24, Lemma 1] for function

$$h(x) = \begin{cases} {}^x_a\mathcal{K}(f, g), & x \in (a, b) \\ \frac{f'(a)}{g'(a)}, & x = a. \end{cases}$$

3.2 Tong's discrete and integral means

Another sufficient condition for validity of (2) was provided by JINGCHEONG TONG in the beginning of 21st century in his paper [23]. Tong does not require differentiability of function f at the end points of the interval $\langle a, b \rangle$, but he uses certain means of that function. Indeed, for a function $f : M \rightarrow \mathbb{R}$ and two distinct points $a, b \in M$ denote by

$$A_f(a, b) = \frac{f(a) + f(b)}{2} \quad \text{and} \quad I_f(a, b) = \frac{1}{b-a} \int_a^b f(t) dt$$

the *arithmetic* (discrete) and *integral* (continuous) mean of f on the interval $\langle a, b \rangle$, respectively.

Theorem 3.4 (Tong, 2004) *Let $f \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$. If $A_f(a, b) = I_f(a, b)$, then there exists $\eta \in (a, b)$ such that (2) holds.*

In his proof Tong defines the auxiliary function

$$h(x) = \begin{cases} (x - a)[A_f(a, x) - I_f(a, x)], & x \in (a, b) \\ 0, & x = a. \end{cases}$$

Easily, $h \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$ and $h(a) = 0 = h(b)$. Then Rolle's theorem for h on $\langle a, b \rangle$ finishes the proof.

Geometrical meaning of Tong's condition The condition $A_f(a, b) = I_f(a, b)$ is not so evident geometrically in comparison with the Flett's condition $f'(a) = f'(b)$. In some sense we can demonstrate it as "the area under the graph of f on $\langle a, b \rangle$ is exactly the volume of a rectangle with sides $b - a$ and $\frac{f(a)+f(b)}{2}$ ".

Let us analyze Tong's condition $A_f(a, b) = I_f(a, b)$ for $f \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$ in detail. It is important to note that this equality does not hold in general for each $f \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$. Indeed, for $f(x) = x^2$ on $\langle 0, 1 \rangle$ we have

$$A_f(0, 1) = \frac{0^2 + 1^2}{2} = \frac{1}{2}, \quad I_f(0, 1) = \frac{1}{1-0} \int_0^1 x^2 dx = \frac{1}{3}.$$

A natural question is *how large is the class of such functions?* For $f \in \mathcal{C}(M) \cap \mathcal{D}(M)$ denote by F a primitive function to f on an interval M and let a, b be interior points of M . Then the condition $A_f(x, b) = I_f(x, b)$, $x \in M$, is equivalent to the condition

$$\frac{f(x) + f(b)}{2} = {}^x_a\mathcal{K}(F), \quad x \neq a. \quad (6)$$

Since $f \in \mathcal{D}(M)$, then $F \in \mathcal{D}^2(M)$ and $f'(t) = F''(t)$ for each $t \in M$. Differentiating the equality (6) with respect to x we get

$$\frac{f'(x)}{2} = \frac{F'(x) - {}^x_a\mathcal{K}(F)}{x - a},$$

which is equivalent to the equation

$$F''(x)(x - a)^2 = 2(F'(x)(x - a) + F(a) - F(x)).$$

Solving this differential equation on intervals $(-\infty, b) \cap M$ and $(b, +\infty) \cap M$, and using the second differentiability of F at b we have

$$F(x) = \frac{\alpha}{2}x^2 + \beta x + \gamma, \quad x \in M, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad \alpha \neq 0,$$

and therefore

$$f(x) = \alpha x + \beta, \quad x \in M.$$

So, the class of functions fulfilling Tong’s condition $A_f(a, b) = I_f(a, b)$ for each interval $\langle a, b \rangle$ is quite small (affine functions, in fact). Of course, if we do not require the condition ”on each interval $\langle a, b \rangle$ ”, then we may use, e.g., the function $y = \arcsin x$ on the interval $\langle -1, 1 \rangle$ which does not satisfy neither Flett’s nor Trahan’s condition (because it is not differentiable at the end points).

Remark 3.5 The relations among the classes of Flett’s, Trahan’s and Tong’s functions are visualized in Fig. 4, where each class is displayed as a rectangle with the corresponding name below the left corner. Moreover, $\Delta_i, i = 1, \dots, 6$, are the classes of functions of possible relationships of Flett’s, Trahan’s and Tong’s classes of functions. For instance, Δ_6 denotes a class of (not necessarily differentiable or continuous) functions on $\langle a, b \rangle$ for which none of the three conditions is fulfilled, but the assertion of Flett’s theorem still holds. Immediately, Δ_1 is non-empty, because it contains all affine functions on $\langle a, b \rangle$. Thus,

- (i) Flett’s and Trahan’s conditions were compared in Remark 3.2 yielding that Trahan’s class of functions is wider than Flett’s one, i.e., $\Delta_1 \cup \Delta_2 \subset \Delta_3 \cup \Delta_4$;
- (ii) Tong’s condition and Flett’s condition are independent each other, because for

$$f(x) = \sin x, \quad x \in \left\langle \frac{\pi}{2}, \frac{5}{2}\pi \right\rangle,$$

we have $f'(\frac{\pi}{2}) = f'(\frac{5}{2}\pi) = 0$, but $1 = A_f(\frac{\pi}{2}, \frac{5}{2}\pi) \neq I_f(\frac{\pi}{2}, \frac{5}{2}\pi) = 0$; on the other hand $f(x) = \arcsin x$ for $x \in \langle -1, 1 \rangle$ fulfills Tong’s condition, but does not satisfy Trahan’s one; also for $f(x) = x^3, x \in \langle -1, 1 \rangle$, we have $A_f(-1, 1) = I_f(-1, 1)$ and $f'(-1) = f'(1)$ which yields that the classes of functions Δ_1, Δ_2 and Δ_3 are non-empty;

- (iii) similarly, Trahan’s condition and Tong’s condition are independent, e.g., the function $f(x) = x^3$ on the interval $\langle -\frac{1}{2}, 1 \rangle$ satisfies Trahan’s condition, but $A_f(-\frac{1}{2}, 1) \neq I_f(-\frac{1}{2}, 1)$, so Δ_4 is non-empty, and for $f(x) = \arcsin x, x \in \langle -1, 1 \rangle$ we have $f \in \Delta_5$;
- (iv) for the function sgn on $\langle -2, 1 \rangle$ none of the three conditions is fulfilled, but the assertion (2) still holds, i.e., Δ_6 is non-empty.

Question 3.6 *Is each class $\Delta_i, i \in \{3, 5, 6\}$, non-empty when considering the stronger condition $f \in \mathcal{D}\langle a, b \rangle$ in Tong’s assumption?*

Removing the condition $A_f(a, b) = I_f(a, b)$ Tong obtained the following result which no more corresponds to the result of Riedel-Sahoo’s theorem.

Theorem 3.7 (Tong, 2004) *Let $f \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$. Then there exists $\eta \in (a, b)$ such that*

$$f'(\eta) = \eta_a \mathcal{K}(f) + \frac{6[A_f(a, b) - I_f(a, b)]}{(b - a)^2}(\eta - a).$$

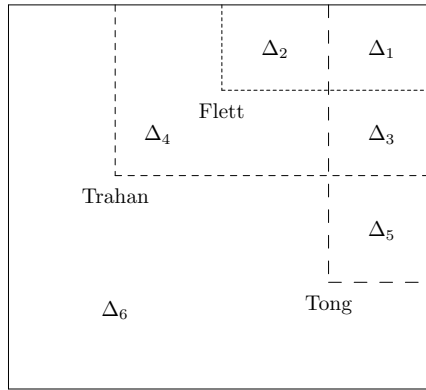


Figure 4: The relations among Flett's, Trahan's and Tong's families of functions

Tong's proof uses the auxiliary function

$$H(x) = f(x) - \frac{6[A_f(a, b) - I_f(a, b)]}{(b - a)^2}(x - a)(x - b), \quad x \in \langle a, b \rangle.$$

It is easy to verify that $H \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$, $H(a) = f(a)$ and $H(b) = f(b)$. Thus, $A_H(a, b) = A_f(a, b)$ and $I_H(a, b) = A_H(a, b)$. Then by Theorem 3.4 there exists $\eta \in (a, b)$ such that $H'(\eta) = {}^{\eta}_a\mathcal{K}(H)$ which is equivalent to the assertion of theorem.

Question 3.8 Analogously to Riedel-Sahoo's Theorem 2.7 we may ask the following: What is the limit behaviour of Tong's points η of a function f on the interval $\langle a, b \rangle$?

3.3 Two new extensions: inequalities in action

In this section we present other sufficient conditions for validity of (2) and its extension. As far as we know they are not included in any literature we were able to find. The basic idea is a mixture of Trahan's results with (although not explicitly stated) Diaz-Výborný's concept of intersecting the graphs of two functions [4]. We will also present nice geometrical interpretations of these results. A particular case of our second result is discussed in the end of this section.

Lemma 3.9 If $f, g \in \mathcal{D}\langle a, b \rangle$, $g(b) \neq g(a)$ and

$$[f'(a) - {}^b_a\mathcal{K}(f, g)g'(a)] \cdot [f'(b) - {}^b_a\mathcal{K}(f, g)g'(b)] \geq 0, \tag{7}$$

then there exists $\xi \in (a, b)$ such that

$$f(\xi) - f(a) = {}^b_a\mathcal{K}(f, g)(g(\xi) - g(a)). \tag{8}$$

Proof. Let us consider the function

$$\varphi(x) = f(x) - f(a) - {}^b_a\mathcal{K}(f, g) \cdot (g(x) - g(a)), \quad x \in \langle a, b \rangle.$$

Then

$$\varphi'(a) = f'(a) - {}^b_a\mathcal{K}(f, g) \cdot g'(a) \quad \text{and} \quad \varphi'(b) = f'(b) - {}^b_a\mathcal{K}(f, g) \cdot g'(b).$$

If $\varphi'(a) \geq 0$, then according to assumption we get $\varphi'(b) \geq 0$. So, there exist points $\alpha, \beta \in (a, b)$ such that $\varphi(\alpha) > 0$ and $\varphi(\beta) < 0$. Thus, $\varphi(\alpha)\varphi(\beta) < 0$ and by Bolzano's theorem there exists a point $\xi \in (\alpha, \beta)$ such that $\varphi(\xi) = 0$. The case $\varphi'(a) \leq 0$ and $\varphi'(b) \leq 0$ is analogous. \square

Theorem 3.10 *If $f, g \in \mathcal{D}\langle a, b \rangle$, $g(b) \neq g(a)$ and the inequality (7) holds, then there exists $\eta \in (a, b)$ such that*

$$f'(\eta) - {}^\eta_a\mathcal{K}(f) = {}^b_a\mathcal{K}(f, g) [g'(\eta) - {}^\eta_a\mathcal{K}(g)]. \quad (9)$$

Proof. Let us take the auxiliary function

$$F(x) = \begin{cases} {}^x_a\mathcal{K}(f) - {}^b_a\mathcal{K}(f, g) \cdot {}^x_a\mathcal{K}(g), & x \in (a, b), \\ f'(a) - {}^b_a\mathcal{K}(f, g) \cdot g'(a), & x = a. \end{cases}$$

Observe that $F(x) = {}^x_a\mathcal{K}(\varphi)$ for $x \in (a, b)$, where φ is the auxiliary function from the proof of Lemma 3.9. Thus, by Lemma 3.9 there exists a point $\xi \in (a, b)$ such that $F(\xi) = 0 = F(b)$. Since $F \in \mathcal{C}\langle \xi, b \rangle \cap \mathcal{D}\langle \xi, b \rangle$, then by Rolle's theorem there exists $\eta \in (\xi, b) \subset (a, b)$ such that $F'(\eta) = 0$ which is equivalent to the desired result. \square

In what follows we denote by

$$T_n(f, x_0)(x) := f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

the n -th Taylor's polynomial of a function f at a point x_0 . Rewriting the assertion of Theorem 3.10 in terms of Taylor's polynomial yields

$$f(a) - T_1(f, \eta)(a) = {}^b_a\mathcal{K}(f, g) \cdot (g(a) - T_1(g, \eta)(a)).$$

Geometrical meaning of Theorem 3.10 Realize that $T_1(f, x_0)(x)$ is the tangent to the graph of f at the point x_0 , i.e.,

$$T_1(f, x_0)(x) = f(x_0) + f'(x_0)(x - x_0).$$

Then the equation (9) may be equivalently rewritten as follows

$$(\exists \eta \in (\xi, b) \subset (a, b)) \quad f(a) - T_1(f, \eta)(a) = {}^b_a\mathcal{K}(f, g) \cdot (g(a) - T_1(g, \eta)(a)). \quad (10)$$

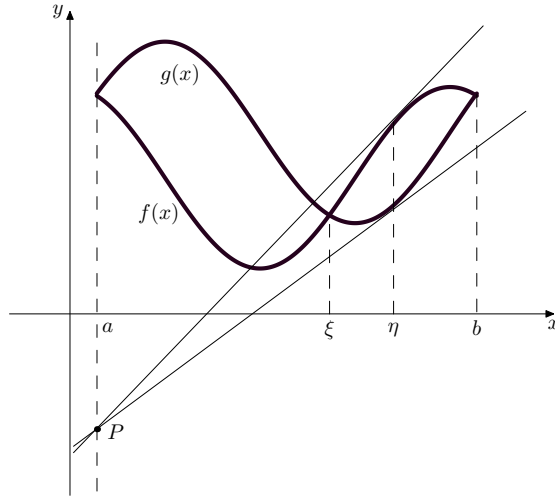


Figure 5: Geometrical interpretation of Theorem 3.10

Since

$$f(b) - f(a) = g(b) - g(a) \Leftrightarrow f(b) - g(b) = f(a) - g(a),$$

then the equation (8) has the form

$$(\exists \xi \in (a, b)) f(\xi) - g(\xi) = f(a) - g(a) = f(b) - g(b), \quad (11)$$

and (10) may be rewritten into

$$(\exists \eta \in (\xi, b)) T_1(f, \eta)(a) - T_1(g, \eta)(a) = f(a) - g(a) = f(b) - g(b). \quad (12)$$

Thus, considering a function g such that $g(a) = f(a)$ and $g(b) = f(b)$ the equation (11) yields

$$(\exists \xi \in (a, b)) f(\xi) = g(\xi)$$

and the equation (12) has the form

$$(\exists \eta \in (\xi, b)) T_1(f, \eta)(a) = T_1(g, \eta)(a).$$

Geometrically it means that tangents at the points $[\eta, f(\eta)]$ and $[\eta, g(\eta)]$ pass through the common point P on the line $x = a$, see Fig. 5.

Remark 3.11 Observe that in the special case of secant joining the end points, i.e., the function

$$g(x) = f(a) + {}_a^b \mathcal{H}(f)(x - a),$$

where f is a function fulfilling assumptions of Theorem 3.10, we get the original Trahan's result of Theorem 3.1 (in fact, a generalization of Flett's theorem) with the explicit geometrical interpretation on Fig. 6.

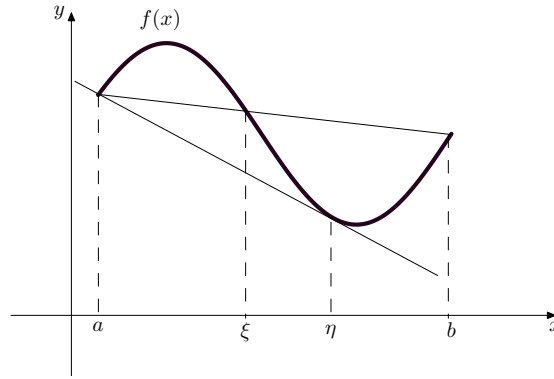


Figure 6: Geometrical interpretation of Theorem 3.10 for secant

Lemma 3.12 *Let $f, g \in \mathcal{D}\langle a, b \rangle$ and f, g be twice differentiable at the point a . If $g(a) \neq g(b)$ and*

$$[f'(a) - {}^b_a\mathcal{K}(f, g) \cdot g'(a)] [f''(a) - {}^b_a\mathcal{K}(f, g) \cdot g''(a)] > 0, \quad (13)$$

then there exists $\xi \in (a, b)$ such that

$$f'(a) - {}^\xi_a\mathcal{K}(f) = {}^b_a\mathcal{K}(f, g) [g'(a) - {}^\xi_a\mathcal{K}(g)].$$

Proof. Consider the function

$$F(x) = \begin{cases} 0, & x = a, \\ f'(a) - {}^x_a\mathcal{K}(f) + {}^b_a\mathcal{K}(f, g) [{}^x_a\mathcal{K}(g) - g'(a)], & x \in (a, b), \\ f'(a) - {}^b_a\mathcal{K}(f, g) \cdot g'(a), & x = b. \end{cases}$$

Then $F \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}\langle a, b \rangle$ with

$$\begin{aligned} F'(a) &= \lim_{x \rightarrow a^+} \frac{f'(a)(x-a) - (f(x) - f(a)) + {}^b_a\mathcal{K}(f, g) [g(x) - g(a) - g'(a)(x-a)]}{(x-a)^2} \\ &= \lim_{x \rightarrow a^+} \frac{f'(a) - f'(x) + {}^b_a\mathcal{K}(f, g) \cdot [g'(x) - g'(a)]}{2(x-a)} \\ &= -\frac{1}{2} \lim_{x \rightarrow a^+} \left(\frac{f'(x) - f'(a)}{x-a} - {}^b_a\mathcal{K}(f, g) \cdot \frac{g'(x) - g'(a)}{x-a} \right) \\ &= -\frac{1}{2} [f''(a) - {}^b_a\mathcal{K}(f, g) \cdot g''(a)], \end{aligned}$$

where L'Hospital rule has been used. Suppose that

$$F(b) = [f'(a) - {}^b_a\mathcal{K}(f, g) \cdot g'(a)] > 0,$$

analogous arguments apply if $F(b) < 0$. Then

$$[f''(a) - {}^b_a\mathcal{K}(f, g) \cdot g''(a)] > 0$$

by assumption of theorem which implies $F'(a) < 0$. Since $F(a) = 0$, then there exists $\alpha \in (a, b)$ such that $F(\alpha) < 0$. According to Bolzano's theorem there exists a point $\xi \in (\alpha, b)$ such that $F(\xi) = 0$, which completes the proof. \square

Theorem 3.13 *Let $f, g \in \mathcal{D}\langle a, b \rangle$ and f, g be twice differentiable at the point a . If $g(a) \neq g(b)$ and the inequality (13) holds, then there exists $\eta \in (a, b)$ such that (9) holds.*

Proof. Consider the function F as in the proof of Lemma 3.12. Then by Lemma 3.12 there exists $\xi \in (a, b)$ such that $F(\xi) = 0 = F(a)$ and by Rolle's theorem there exists $\eta \in (a, \xi)$ such that $F'(\eta) = 0$. \square

Geometrical meaning of Theorem 3.13 Using the Taylor's polynomial we can rewrite the assertion of Lemma 3.12 and Theorem 3.13 as follows

$$(\exists \xi \in (a, b)) f(\xi) - T_1(f, a)(\xi) = {}^b_a\mathcal{K}(f, g) \cdot (g(\xi) - T_1(g, a)(\xi)) \quad (14)$$

and

$$(\exists \eta \in (a, \xi)) f(a) - T_1(f, a)(\eta) = {}^b_a\mathcal{K}(f, g) \cdot (g(a) - T_1(g, a)(\eta)), \quad (15)$$

respectively. Since

$$f(b) - f(a) = g(b) - g(a) \Leftrightarrow f(b) - g(b) = f(a) - g(a),$$

then (14) may be rewritten as

$$(\exists \xi \in (a, b)) f(\xi) - g(\xi) = T_1(f, a)(\xi) - T_1(g, a)(\xi).$$

Similarly (15) may be rewritten as follows

$$(\exists \eta \in (a, \xi)) f(a) - g(a) = T_1(f, \eta)(a) - T_1(g, \eta)(a).$$

If f and g have the same values at the end points, the last equation reduces to

$$(\exists \eta \in (a, b)) T_1(f, \eta)(a) = T_1(g, \eta)(a).$$

Geometrically it means that tangents at points $[\eta, f(\eta)]$ and $[\eta, g(\eta)]$ pass through the common point P on the line $x = a$, see Fig. 7.

Remark 3.14 Again, Theorem 3.13 for the secant

$$g(x) = f(a) + {}^b_a\mathcal{K}(f)(x - a)$$

guarantees the existence of a point $\eta \in (a, b)$ such that $T_1(f, \eta)(a) = f(a)$, i.e., tangent at $[\eta, f(\eta)]$ passes through the point $A = [a, f(a)]$ which is exactly the

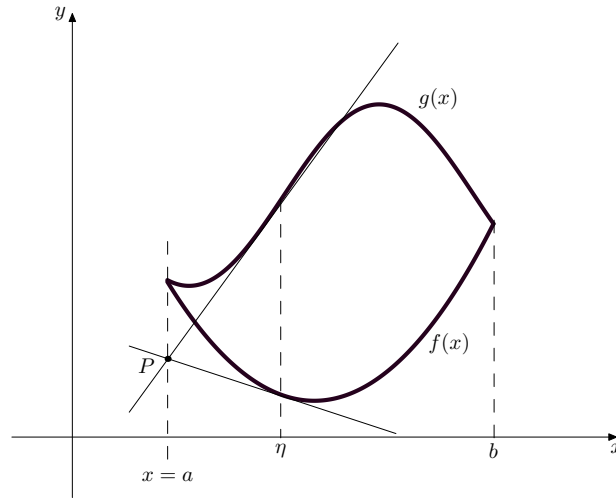


Figure 7: Geometrical interpretation of Theorem 3.13

geometrical interpretation of Flett’s theorem in Fig. 2. The assumption (13) reduces in the secant case to the inequality

$$[f'(a) - {}^b_a\mathcal{K}(f)] f''(a) > 0, \tag{16}$$

i.e.,

$$[f'(a) > {}^b_a\mathcal{K}(f) \wedge f''(a) > 0] \vee [f'(a) < {}^b_a\mathcal{K}(f) \wedge f''(a) < 0].$$

Considering the first case yields

$$\begin{aligned} f'(a) > {}^b_a\mathcal{K}(f) \wedge f''(a) > 0 &\Leftrightarrow f(b) < f(a) + f'(a)(b - a) \wedge f''(a) > 0 \\ &\Leftrightarrow f(b) < T_1(f, a)(b) \wedge f''(a) > 0. \end{aligned}$$

This means that there exists a point $X = [\xi, f(\xi)]$ such that the line AX is a tangent to the graph of f at $A = [a, f(a)]$. Then from the assertion of Theorem 3.13 we have the existence of a point $E = [\eta, f(\eta)]$, where $\eta \in (a, \xi)$, such that the tangent to the graph of f at E passes through the point $A = [a, f(a)]$, see Fig. 8. Similarly for the second case.

Remark 3.15 Observe that if $f'(a) = {}^b_a\mathcal{K}(f)$, then the condition (16) is not fulfilled, but the assertion (2) of Flett’s theorem still holds by Trahan’s condition. On the other hand, if $f'(a) \neq {}^b_a\mathcal{K}(f)$ and $f''(a) = 0$, then the assertion (2) does not need to hold, e.g., for $f(x) = \sin x$ on the interval $\langle 0, \pi \rangle$ we have $(f'(0) - {}^{\pi}_0\mathcal{K}(f)) \cdot f''(0) = 1 \cdot 0 = 0$, but there is no such a point $\eta \in (0, \pi)$ which is a solution of the equation $\eta \cos \eta = \sin \eta$.

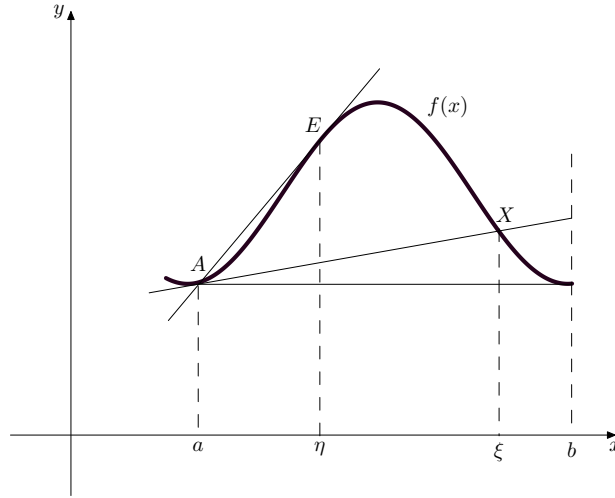


Figure 8: Geometrical interpretation of Theorem 3.13 for secant

We have to point out that the inequality (16) was observed as a sufficient condition for validity of (2) in [10], but starting from a different point, therefore our general result of Theorem 3.13 seems to be new. Indeed, Malešević in [10] considers some "iterations" of Flett's auxiliary function in terms of an infinitesimal function, i.e., for $f \in \mathcal{D}\langle a, b \rangle$ which is differentiable arbitrary number of times in a right neighbourhood of the point a he defines the following functions

$$\alpha_1(x) = \begin{cases} {}_a^x \mathcal{K}(f) - f'(a), & x \in (a, b) \\ 0, & x = a \end{cases}, \quad \dots \quad \alpha_{k+1}(x) = \begin{cases} {}_a^x \mathcal{K}(\alpha_k) - \alpha'_k(a), & x \in (a, b) \\ 0, & x = a, \end{cases}$$

for $k = 1, 2, \dots$. Then he proves the following result.

Theorem 3.16 (Malešević, 1999) *Let $f \in \mathcal{D}\langle a, b \rangle$ and f be $(n + 1)$ -times differentiable in a right neighbourhood of the point a . If*

$$\text{either } \alpha'_n(b)\alpha_n(b) < 0, \quad \text{or } \alpha'_n(a)\alpha_n(b) < 0,$$

then there exists $\eta \in (a, b)$ such that $\alpha'_n(\eta) = 0$.

For $n = 1$ Malešević's condition $\alpha'_1(b)\alpha_1(b) < 0$ is equivalent to Trahan's condition (5) where the second differentiability of f in a right neighbourhood of a is a superfluous constraint. The second Malešević's condition $\alpha'_1(a)\alpha_1(b) < 0$ is equivalent to our condition (16), because

$$\alpha'_1(a) = \lim_{x \rightarrow a^+} \frac{f(x) - f(a) - f'(a)(x - a)}{(x - a)^2} = \lim_{x \rightarrow a^+} \frac{f'(x) - f'(a)}{2(x - a)} = \frac{1}{2}f''(a)$$

and then the inequality

$$0 > \alpha'_1(a)\alpha_1(b) = \frac{1}{2}f''(a) ({}_a^b \mathcal{K}(f) - f'(a))$$

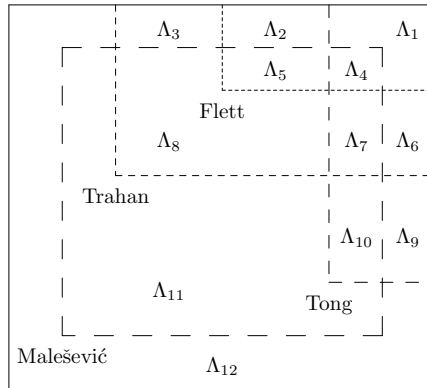


Figure 9: The diagram of four families of functions

holds if and only if (16) holds. However, we require only the existence of $f''(a)$ in (16). Note that for $n > 1$ Malešević's result does not correspond to Pawlikowska's theorem (a generalization of Flett's theorem for higher-order derivatives), see Section 5, but it goes a different way.

Fig. 9 shows all the possible cases of relations of classes of functions satisfying assumptions of Flett, Trahan, Tong and Malešević, respectively. Some examples of functions belonging to sets $\Lambda_1, \dots, \Lambda_{12}$ were already mentioned (e.g. $y = x^3$, $x \in \langle -1, 1 \rangle$, belongs to Λ_1 ; $y = \sin x$, $x \in \langle -\frac{\pi}{2}, \frac{5}{2}\pi \rangle$, belongs to Λ_2 ; $y = x^3$, $x \in \langle -\frac{2}{3}, 1 \rangle$, belongs to Λ_3 ; $y = \arcsin x$, $x \in \langle -1, 1 \rangle$, belongs to Λ_9 , and $y = \operatorname{sgn} x$, $x \in \langle -2, 1 \rangle$, belongs to Λ_{12}), other (and more sophisticated) examples is not so difficult to find.

Remark 3.17 Again, if we strengthen our assumption and consider only the functions $f \in \mathcal{D}\langle a, b \rangle$ which are twice differentiable at a , we may ask the legitimate question: *Is each of the sets Λ_i , $i = 1, \dots, 12$, in Fig. 9 non-empty?* In the positive case, it would be interesting to provide a complete characterization of all the classes of functions.

Problem 3.18 *All the presented conditions are only sufficient for the assertion of Flett's theorem to hold. Provide necessary condition(s) for the validity of (2).*

4 Integral Flett's mean value theorem

Naturally as in the case of Lagrange's theorem we may ask whether Flett's theorem has its analogical form in integral calculus. Consider therefore a function $f \in \mathcal{C}\langle a, b \rangle$. Putting

$$F(x) = \int_a^x f(t) dt, \quad x \in \langle a, b \rangle,$$

the fundamental theorem of integral calculus yields that $F \in \mathcal{D}\langle a, b \rangle$ with $F'(a) = f(a)$ and $F'(b) = f(b)$. If $f(a) = f(b)$, then the function F on the interval $\langle a, b \rangle$ fulfils the assumptions of Flett's theorem and we get the following result. It was proved by STANLEY G. WAYMENT in 1970 and it is nothing but the integral version of Flett's theorem. Although our presented reflection is a trivial proof of this result, we add here the original Wayment's proof adopted from [25] which does not use the original Flett's theorem.

Theorem 4.1 (Wayment, 1970) *If $f \in \mathcal{C}\langle a, b \rangle$ with $f(a) = f(b)$, then there exists $\eta \in (a, b)$ such that*

$$f(\eta) = I_f(a, \eta). \quad (17)$$

Proof. Consider the function

$$F(t) = \begin{cases} (t - a)[f(t) - I_f(a, t)], & t \in (a, b), \\ 0, & t = a. \end{cases}$$

If f is a constant on $\langle a, b \rangle$, then $F \equiv 0$ and the assertion of theorem holds trivially. Thus, suppose that f is non-constant. Since $f \in \mathcal{C}\langle a, b \rangle$, then by Weierstrass' theorem on the existence of extrema there exist points $t_1, t_2 \in \langle a, b \rangle$ such that

$$(\forall t \in \langle a, b \rangle) f(t_1) \leq f(t) \leq f(t_2).$$

From $f(a) = f(b)$ we deduce that f cannot achieve both extrema at the end points a and b .

If $t_2 \neq a$, then $F(b) < 0 < F(t_2)$ and by Bolzano's theorem there exists $\eta \in (t_2, b)$ such that $F(\eta) = 0$. If $t_1 \neq a$, then $F(t_1) < 0 < F(b)$ and analogously as above we conclude that there exists $\eta \in (t_1, b)$ such that $F(\eta) = 0$. Finally, consider the case when none of t_1 and t_2 is equal to a . Then

$$a < \min\{t_1, t_2\} < \max\{t_1, t_2\} < b,$$

and so $F(t_1) \leq 0 \leq F(t_2)$. From Bolzano's theorem applied to function F on the interval $\langle t_1, t_2 \rangle$ we have that there exists a point $\eta \in (a, b)$ such that $F(\eta) = 0$.

□

Geometrical meaning of Wayment's theorem Geometrically Wayment's theorem says that the area under the curve f on the interval $\langle a, \eta \rangle$ is equal to $(\eta - a)f(\eta)$, i.e., volume of rectangle with sides $\eta - a$ and $f(\eta)$, see Fig. 10.

Removing the condition $f(a) = f(b)$ yields the following integral version of Riedel-Sahoo's theorem. Its proof is based on using Riedel-Sahoo's theorem for function

$$F(x) = \int_a^x f(t) dt, \quad x \in \langle a, b \rangle.$$

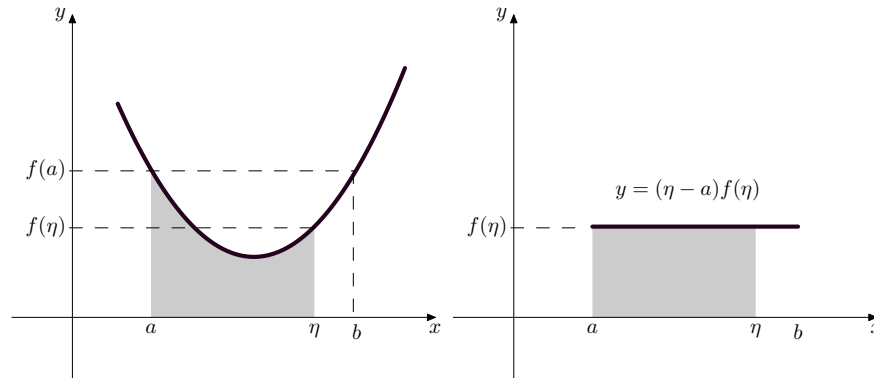


Figure 10: Geometrical interpretation of Wayment's theorem

Theorem 4.2 *If $f \in \mathcal{C}\langle a, b \rangle$, then there exists $\eta \in (a, b)$ such that*

$$f(\eta) = I_f(a, \eta) + \frac{\eta - a}{2} \cdot {}^b_a\mathcal{K}(f).$$

In what follows we present some results from Section 3 in their integral form to show some sufficient conditions for validity of (17) with a short idea of their proofs. The first one is Trahan's result.

Proposition 4.3 *If $f \in \mathcal{C}\langle a, b \rangle$ and*

$$[f(a) - I_f(a, b)] \cdot [f(b) - I_f(a, b)] \geq 0,$$

then there exists $\eta \in (a, b)$ such that (17) holds.

For the proof it is enough to consider the function

$$g(x) = \begin{cases} I_f(a, x), & x \in (a, b) \\ f(a), & x = a, \end{cases}$$

and apply Trahan's lemma [24, Lemma 1]. To show an analogy with Tong's result consider the following means

$$B_f(a, b) = \frac{b-a}{2} I_f(a, b), \quad J_f(a, b) = b I_f(a, b) - \frac{1}{b-a} \int_a^b t f(t) dt.$$

Proposition 4.4 *Let $f \in \mathcal{C}\langle a, b \rangle$. If $B_f(a, b) = J_f(a, b)$, then there exists $\eta \in (a, b)$ such that $f(\eta) = I_f(a, \eta)$.*

In the proof we consider the auxiliary function

$$h(x) = \begin{cases} (x-a)[B_f(a, x) - J_f(a, x)], & x \in (a, b), \\ 0, & x = a, \end{cases}$$

and the further steps coincide with the original Tong's proof of Theorem 3.4.

Removing the condition $B_f(a, b) = J_f(a, b)$ we obtain the following result.

Proposition 4.5 *Let $f \in \mathcal{C}\langle a, b \rangle$. Then there exists $\eta \in (a, b)$ such that*

$$f(\eta) = I_f(a, \eta) + \frac{6[B_f(a, b) - J_f(a, b)]}{(b - a)^2}(\eta - a).$$

Proof, in which we use the following auxiliary function

$$H(x) = f(x) - \frac{6[B_f(a, b) - J_f(a, b)]}{(b - a)^2}(2x - a - b), \quad x \in \langle a, b \rangle,$$

is again analogous to the proof of Tong's Theorem 3.7.

In the end of this chapter we formally present integral analogies of new sufficient conditions of validity of Flett's theorem.

Lemma 4.6 *Let $f, g \in \mathcal{C}\langle a, b \rangle$ and*

$$[f(a)I_g(a, b) - g(a)I_f(a, b)] \cdot [f(b)I_g(a, b) - g(b)I_f(a, b)] \geq 0. \quad (18)$$

Then there exists $\xi \in (a, b)$ such that

$$I_f(a, \xi) I_g(a, b) = I_g(a, \xi) I_f(a, b).$$

Proof. Considering the function

$$\varphi(x) = \begin{cases} I_f(a, x)I_g(a, b) - I_g(a, x)I_f(a, b), & x \in (a, b), \\ 0, & x = a, \end{cases}$$

we have

$$\varphi'(a) = f(a)I_g(a, b) - g(a)I_f(a, b), \quad \varphi'(b) = f(b)I_g(a, b) - g(b)I_f(a, b).$$

Further steps are analogous to the proof of Lemma 3.9. \square

Theorem 4.7 *Let $f, g \in \mathcal{C}\langle a, b \rangle$ and the inequality (18) holds. Then there exists $\eta \in (a, b)$ such that*

$$I_g(a, b) \cdot (f(\eta) - I_f(a, \eta)) = I_f(a, b) \cdot (g(\eta) - I_g(a, \eta)). \quad (19)$$

Proof. Take the auxiliary function

$$F(x) = \begin{cases} I_g(a, b) I_f(a, x) - I_f(a, b) I_g(a, x), & x \in (a, b), \\ f(a)I_g(a, b) - g(a)I_f(a, b), & x = a. \end{cases}$$

By Lemma 4.6 there exists $\xi \in (a, b)$ such that $F(\xi) = 0 = F(b)$. Then by Rolle's theorem for F on the interval $\langle \xi, b \rangle$ there exists $\eta \in (\xi, b)$ such that $F'(\eta) = 0$. \square

Similarly we may prove the following integral versions of Lemma 3.12 and Theorem 3.13.

Lemma 4.8 *Let $f, g \in \mathcal{C}\langle a, b \rangle$ and f, g be differentiable at a . If*

$$[f(a)I_g(a, b) - g(a)I_f(a, b)] \cdot [f'(a)I_g(a, b) - g'(a)I_f(a, b)] > 0, \quad (20)$$

then there exists $\xi \in (a, b)$ such that

$$I_g(a, b) \cdot (f(a) - I_f(a, \xi)) = I_f(a, b) \cdot (g(a) - I_g(a, \xi)).$$

Theorem 4.9 *Let $f, g \in \mathcal{C}\langle a, b \rangle$ and f, g are differentiable at a . If (20) holds, then there exists $\eta \in (a, b)$ such that (19) holds.*

5 Flett's theorem for higher-order derivatives

The previous sections dealt with the question of replacing the condition $f(a) = f(b)$ in Rolle's theorem by $f'(a) = f'(b)$. In this section we will consider a natural question of generalizing Flett's theorem for higher-order derivatives. We will provide the original solution of Pawlikowska and present a new proof of her result together with some other observations.

The problem of generalizing Flett's theorem to higher-order derivatives was posed first time by ZSOLT PALES in 1997 at the 35th international symposium of functional equations in Graz. Solution has already appeared two years later by polish mathematician IWONA PAWLIKOWSKA in her paper [13] and it has the following form.

Theorem 5.1 (Pawlikowska, 1999) *If $f \in \mathcal{D}^n\langle a, b \rangle$ with $f^{(n)}(a) = f^{(n)}(b)$, then there exists $\eta \in (a, b)$ such that*

$${}_a^\eta \mathcal{K}(f) = \sum_{i=1}^n \frac{(-1)^{i+1}}{i!} (\eta - a)^{i-1} f^{(i)}(\eta). \quad (21)$$

Pawlikowska in her paper [13] generalized original Flett's proof in such a way that she uses $(n - 1)$ -th derivative of Flett's auxiliary function g given by (3) and Rolle's theorem. More precisely, the function

$$G_f(x) = \begin{cases} g^{(n-1)}(x), & x \in (a, b) \\ \frac{1}{n} f^{(n)}(a), & x = a. \end{cases} \quad (22)$$

plays here an important role. Indeed, $G_f \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$ and

$$\begin{aligned} g^{(n)}(x) &= \frac{(-1)^n n!}{(x - a)^n} \left({}_a^x \mathcal{K}(f) + \sum_{i=1}^n \frac{(-1)^i}{i!} (x - a)^{i-1} f^{(i)}(x) \right) \\ &= \frac{1}{x - a} (f^{(n)}(x) - n g^{(n-1)}(x)) \end{aligned}$$

for $x \in (a, b)$ which can be verified by induction. Moreover, if $f^{(n+1)}(a)$ exists, then

$$\lim_{x \rightarrow a^+} g^{(n)}(x) = \frac{1}{n+1} f^{(n+1)}(a).$$

Further steps of Pawlikowska's proof is analogous to the original proof of Flett's theorem using Rolle's theorem. Similarly we may proceed using Fermat's theorem.

We have found a new proof of Pawlikowska theorem (it was not published yet, it exists only in the form of preprint [11]) which deals only with Flett's theorem. The basic idea consists in iteration of Flett's theorem using an appropriate auxiliary function.

New proof of Pawlikowska's theorem. For $k = 1, 2, \dots, n$ consider the function

$$\varphi_k(x) = \sum_{i=0}^k \frac{(-1)^{i+1}}{i!} (k-i)(x-a)^i f^{(n-k+i)}(x) + x f^{(n-k+1)}(a), \quad x \in \langle a, b \rangle.$$

Running through all indices $k = 1, 2, \dots, n$ we show that its derivative fulfills assumptions of Flett's mean value theorem and it implies the validity of Flett's mean value theorem for l -th derivative of f , where $l = n-1, n-2, \dots, 1$.

Indeed, for $k = 1$ we have

$$\varphi_1(x) = -f^{(n-1)}(x) + x f^{(n)}(a) \quad \text{and} \quad \varphi_1'(x) = -f^{(n)}(x) + f^{(n)}(a)$$

for each $x \in \langle a, b \rangle$. Clearly, $\varphi_1'(a) = 0 = \varphi_1'(b)$, so applying Flett's theorem for φ_1 on $\langle a, b \rangle$ there exists $u_1 \in (a, b)$ such that

$$\varphi_1'(u_1) = {}_a^{u_1} \mathcal{K}(\varphi_1) \quad \Leftrightarrow \quad {}_a^{u_1} \mathcal{K}(f^{(n-1)}) = f^{(n)}(u_1). \quad (23)$$

Then for $\varphi_2(x) = -2f^{(n-2)}(x) + (x-a)f^{(n-1)}(x) + x f^{(n-1)}(a)$ we get

$$\varphi_2'(x) = -f^{(n-1)}(x) + (x-a)f^{(n)}(x) + f^{(n-1)}(a)$$

and $\varphi_2'(a) = 0 = \varphi_2'(u_1)$ by (23). So, by Flett's theorem for φ_2 on $\langle a, u_1 \rangle$ there exists $u_2 \in (a, u_1) \subset (a, b)$ such that

$$\varphi_2'(u_2) = {}_a^{u_2} \mathcal{K}(\varphi_2) \quad \Leftrightarrow \quad {}_a^{u_2} \mathcal{K}(f^{(n-2)}) = f^{(n-1)}(u_2) - \frac{1}{2}(u_2-a)f^{(n)}(u_2).$$

Continuing this way after $n-1$ steps, $n \geq 2$, there exists $u_{n-1} \in (a, b)$ such that

$${}_a^{u_{n-1}} \mathcal{K}(f') = \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i!} (u_{n-1}-a)^{i-1} f^{(i+1)}(u_{n-1}). \quad (24)$$

Considering the function φ_n we get

$$\begin{aligned} \varphi'_n(x) &= -f'(x) + f'(a) + \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i!} (x-a)^i f^{(i)}(x) \\ &= f'(a) + \sum_{i=0}^{n-1} \frac{(-1)^{i+1}}{i!} (x-a)^i f^{(i+1)}(x). \end{aligned}$$

Clearly, $\varphi'_n(a) = 0$ and then

$${}_{a}^{u_{n-1}}\mathcal{K}(\varphi'_n) = -{}_{a}^{u_{n-1}}\mathcal{K}(f') + \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i!} (u_{n-1}-a)^{i-1} f^{(i+1)}(u_{n-1}) = 0$$

by (24). From it follows that $\varphi'_n(u_{n-1}) = 0$ and by Flett's theorem for φ_n on $\langle a, u_{n-1} \rangle$ there exists $\eta \in (a, u_{n-1}) \subset (a, b)$ such that

$$\varphi'_n(\eta) = {}_{a}^{\eta}\mathcal{K}(\varphi). \tag{25}$$

Since

$$\varphi'_n(\eta) = f'(a) + \sum_{i=1}^n \frac{(-1)^i}{(i-1)!} (\eta-a)^{i-1} f^{(i)}(\eta)$$

and

$${}_{a}^{\eta}\mathcal{K}(\varphi) = f'(a) - n \cdot {}_{a}^{\eta}\mathcal{K}(f) + \sum_{i=1}^n \frac{(-1)^{i+1}}{i!} (n-i)(\eta-a)^{i-1} f^{(i)}(\eta),$$

the equality (25) yields

$$\begin{aligned} -n \cdot {}_{a}^{\eta}\mathcal{K}(f) &= \sum_{i=1}^n \frac{(-1)^i}{(i-1)!} (\eta-a)^{i-1} f^{(i)}(\eta) \left(1 + \frac{n-i}{i}\right) \\ &= n \sum_{i=1}^n \frac{(-1)^i}{i!} (\eta-a)^{i-1} f^{(i)}(\eta), \end{aligned}$$

which corresponds to (21). □

Remark 5.2 Recall that the assertion of Flett's theorem has an equivalent form $f(a) = T_1(f, \eta)(a)$. Now, in the assertion of Pawlikowska's theorem we can observe a deeper (and very natural) relation with Taylor's polynomial. Indeed, $f(a) = T_n(f, \eta)(a)$ is an equivalent form of (21). Geometrically it means that Taylor's polynomial $T_n(f, \eta)(x)$ intersects the graph of f at the point $A = [a, f(a)]$.

Remark 5.3 Equivalent form of Pawlikowska's theorem in the form of determinant is as follows

$$\begin{vmatrix} f^{(n)}(\eta) & h_n^{(n)}(\eta) & h_{n-1}^{(n)}(\eta) & \cdots & h_0^{(n)}(\eta) \\ f^{(n-1)}(\eta) & h_n^{(n-1)}(\eta) & h_{n-1}^{(n-1)}(\eta) & \cdots & h_0^{(n-1)}(\eta) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ f'(\eta) & h_n'(\eta) & h_{n-1}'(\eta) & \cdots & h_0'(\eta) \\ f(\eta) & h_n(\eta) & h_{n-1}(\eta) & \cdots & h_0(\eta) \\ f(a) & h_n(a) & h_{n-1}(a) & \cdots & h_0(a) \end{vmatrix} = 0, \quad h_i(x) = \frac{x^i}{i!}.$$

Verification of this fact is not as complicated as it is rather long. It is based on n -times application of Laplace's formula according to the last column.

Again we may ask whether it is possible to remove the condition $f^{(n)}(a) = f^{(n)}(b)$ to obtain Lagrange's type result. The first proof of this fact was given in Pawlikowska's paper [13]. Here we present two other new proofs.

Theorem 5.4 (Pawlikowska, 1999) *If $f \in \mathcal{D}^n \langle a, b \rangle$, then there exists $\eta \in (a, b)$ such that*

$$f(a) = T_n(f, \eta)(a) + \frac{(a - \eta)^{n+1}}{(n + 1)!} \cdot {}^b_a \mathcal{K} (f^{(n)}).$$

Proof I. Consider the auxiliary function

$$\psi_k(x) = \varphi_k(x) + \frac{(-1)^{k+1}(x - a)^{k+1}}{(k + 1)!} \cdot {}^b_a \mathcal{K} (f^{(n)}), \quad k = 1, 2, \dots, n,$$

where φ_k is the function from our proof of Pawlikowska's theorem. Then

$$\psi_1(x) = \varphi_1(x) + \frac{(x - a)^2}{2} \cdot {}^b_a \mathcal{K} (f^{(n)}), \quad x \in \langle a, b \rangle,$$

and so

$$\psi_1'(x) = \varphi_1'(x) + (x - a) \cdot {}^b_a \mathcal{K} (f^{(n)}).$$

Thus, $\psi_1'(a) = 0 = \psi_1'(b)$ and by Flett's theorem for function ψ_1 on $\langle a, b \rangle$ there exists $u_1 \in (a, b)$ such that

$${}^{u_1}_a \mathcal{K} (f^{(n-1)}) = f^{(n)}(u_1) + \frac{a - u_1}{2} \cdot {}^b_a \mathcal{K} (f^{(n)}).$$

After $n - 1$ steps we conclude that there exists $u_{n-1} \in (a, b)$ such that

$${}^{u_{n-1}}_a \mathcal{K} (f') = \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i!} (u_{n-1} - a)^{i-1} f^{(i+1)}(u_{n-1}) - \frac{(a - u_{n-1})^{n-1}}{n!} \cdot {}^b_a \mathcal{K} (f^{(n)}).$$

By Flett’s theorem for ψ_n on the interval $\langle a, u_{n-1} \rangle$ we get the desired result (all the steps are identical with the steps of previous proof). \square

Proof II. Applying Pawlikowska’s theorem to the function

$$F(x) = \begin{vmatrix} f(x) & x^{n+1} & x^n & 1 \\ f(a) & a^{n+1} & a^n & 1 \\ f^{(n)}(a) & (n+1)!a & n! & 0 \\ f^{(n)}(b) & (n+1)!b & n! & 0 \end{vmatrix}, \quad x \in \langle a, b \rangle$$

we get the result. \square

Remark 5.5 By Remark 5.3 we may rewrite the assertion of Theorem 5.4 in the form of determinant as follows

$$\begin{vmatrix} f^{(n)}(\eta) & h_n^{(n)}(\eta) & h_{n-1}^{(n)}(\eta) & \dots & h_0^{(n)}(\eta) \\ f^{(n-1)}(\eta) & h_n^{(n-1)}(\eta) & h_{n-1}^{(n-1)}(\eta) & \dots & h_0^{(n-1)}(\eta) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ f'(\eta) & h'_n(\eta) & h'_{n-1}(\eta) & \dots & h'_0(\eta) \\ f(\eta) & h_n(\eta) & h_{n-1}(\eta) & \dots & h_0(\eta) \\ f(a) & h_n(a) & h_{n-1}(a) & \dots & h_0(a) \end{vmatrix} = {}^b_a\mathcal{K}(f^{(n)}) \cdot \frac{(\eta - a)^n}{(n+1)!},$$

where $h_i(x) = \frac{x^i}{i!}$ for $i = 0, 1, \dots, n$.

A relatively easy generalization of Pawlikowska’s theorem may be obtained for two functions (as a kind of “Cauchy” version of it).

Theorem 5.6 Let $f, g \in \mathcal{D}^n \langle a, b \rangle$ and $g^{(n)}(a) \neq g^{(n)}(b)$. Then there exists $\eta \in (a, b)$ such that

$$f(a) - T_n(f, \eta)(a) = {}^b_a\mathcal{K}(f^{(n)}, g^{(n)}) \cdot [g(a) - T_n(g, \eta)(a)].$$

Proof I. Considering the function

$$h(x) = f(x) - {}^b_a\mathcal{K}(f^{(n)}, g^{(n)}) \cdot g(x), \quad x \in \langle a, b \rangle,$$

we have $h \in \mathcal{D}^n \langle a, b \rangle$ and $h^{(n)}(a) = h^{(n)}(b)$. By Pawlikowska’s theorem there exists $\eta \in (a, b)$ such that

$${}^{\eta}_a\mathcal{K}(h) = \sum_{i=1}^n \frac{(-1)^{i+1}}{i!} (\eta - a)^{i-1} h^{(i)}(\eta),$$

which is equivalent to the stated result. \square

Naturally, as in the case of Flett’s theorem we would like to generalize e.g. Trahan’s result for higher-order derivatives. Our idea of this generalization is based on application of Trahan’s approach to Pawlikowska’s auxiliary function (22).

Theorem 5.7 *Let $f \in \mathcal{D}^n\langle a, b \rangle$ and*

$$\left(\frac{f^{(n)}(a)(a-b)^n}{n!} + \mathfrak{T}_f(a) \right) \left(\frac{f^{(n)}(b)(a-b)^n}{n!} + \mathfrak{T}_f(a) \right) \geq 0,$$

where $\mathfrak{T}_f(a) := T_{n-1}(f, b)(a) - f(a)$. Then there exists $\eta \in (a, b)$ such that (21) holds.

Proof. Consider the function g given by (3) and function G_f given by (22). Clearly, $G_f \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$ and

$$g^{(n)}(x) = \frac{(-1)^n n!}{(x-a)^{n+1}} (T_n(f, x)(a) - f(a)), \quad x \in (a, b).$$

To apply Trahan's lemma [24, Lemma 1] we need to know signum of

$$[G_f(b) - G_f(a)] G'_f(b) = \left(g^{(n-1)}(b) - \frac{1}{n} f^{(n)}(a) \right) g^{(n)}(b),$$

i.e.,

$$-\frac{n!(n-1)!}{(b-a)^{2n+1}} \left(\frac{f^{(n)}(a)(a-b)^n}{n!} + \mathfrak{T}_f(a) \right) \cdot \left(\frac{f^{(n)}(b)(a-b)^n}{n!} + \mathfrak{T}_f(a) \right) \leq 0$$

by assumption. Then by Trahan's Lemma [24, Lemma 1] there exists $\eta \in (a, b)$ such that $G'_f(\eta) = 0$, which is equivalent to the assertion of theorem. \square

Here we present another proof of "Cauchy" type Theorem 5.6 which is independent on Flett's theorem, but uses again Trahan's lemma [24, Lemma 1].

Proof of Theorem 5.6 II. For $x \in (a, b)$ put $\varphi(x) = {}^x_a \mathcal{K}(f)$ and $\psi(x) = {}^x_a \mathcal{K}(g)$. Consider the auxiliary function

$$F(x) = \begin{cases} \varphi^{(n-1)}(x) - {}^b_a \mathcal{K}(f^{(n)}, g^{(n)}) \cdot \psi^{(n-1)}(x), & x \in (a, b) \\ \frac{1}{n} [f^{(n)}(a) - {}^b_a \mathcal{K}(f^{(n)}, g^{(n)}) \cdot g^{(n)}(a)], & x = a. \end{cases}$$

Then $F \in \mathcal{C}\langle a, b \rangle \cap \mathcal{D}(a, b)$ and for $x \in (a, b)$ we have

$$\begin{aligned} F'(x) &= \varphi^{(n)}(x) - {}^b_a \mathcal{K}(f^{(n)}, g^{(n)}) \cdot \psi^{(n)}(x) \\ &= \frac{(-1)^n n!}{(x-a)^{n+1}} (T_n(f, x)(a) - f(a) - {}^b_a \mathcal{K}(f^{(n)}, g^{(n)}) \cdot (T_n(g, x)(a) - g(a))). \end{aligned}$$

Then it is easy to verify that

$$[F(b) - F(a)] F'(b) = -\frac{1}{b-a} (F(b) - F(a))^2 \leq 0,$$

thus by Trahan's lemma [24, Lemma 1] there exists $\eta \in (a, b)$ such that $F'(\eta) = 0$, i.e.,

$$f(a) - T_n(f, \eta)(a) = {}^b_a \mathcal{K}(f^{(n)}, g^{(n)}) \cdot [g(a) - T_n(g, \eta)(a)]. \quad \square$$

Remark 5.8 Similarly as in the case of Flett's and Riedel-Sahoo's points it is possible to give the stability results for the so called n th order Flett's and Riedel-Sahoo's points. These results were proved by Pawlikowska in her paper [15], which is recommended to interested reader. Also, some results which concern the connection between polynomials and the set of (n th order) Flett's points are proven in [14].

6 Concluding remarks

In this paper we provided results related to Flett's mean value theorem of differential and integral calculus of a real-valued function of one real variable. Indeed, we showed that for $f \in \mathcal{D}\langle a, b \rangle$ the assertion of Flett's theorem holds in each of the following cases:

- (i) $f'(a) = f'(b)$ (Flett's condition);
- (ii) $(f'(a) - {}^b_a\mathcal{K}(f)) \cdot (f'(b) - {}^b_a\mathcal{K}(f)) \geq 0$ (Trahan's condition);
- (iii) $A_f(a, b) = I_f(a, b)$ (Tong's condition);
- (iv) $(f'(a) - {}^b_a\mathcal{K}(f)) \cdot f''(a) > 0$ provided $f''(a)$ exists (Malešević's condition).

Then we discussed possible generalization of Flett's theorem to higher-order derivatives and provided a new proof of Pawlikowska's theorem and related results. Up to a few questions and open problems explicitly formulated in this paper, there are several problems and directions for the future research.

The survey of results related to Flett's mean value theorem should be continued in [7], because we did not mention here any known and/or new generalizations and extensions of Flett's theorem made at least in two directions: to move from the real line to more general spaces (e.g. vector-valued functions of vector argument [22], holomorphic functions [2], etc.), and/or to consider other types of differentiability of considered functions (e.g. Dini's derivatives [19], symmetric derivatives [21], v -derivatives [12], etc.). Also, a characterization of all the functions that attain their Flett's mean value at a particular point between the endpoints of the interval [20], other functional equations and means related to Flett's theorem should be mentioned in the future.

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