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Abstract. The generalized weighted mean operator $\mathbf{M}_w^g f$ is given by

$$[\mathbf{M}_w^g f](x) = g^{-1}\left(\frac{1}{W(x)}\int_0^x w(t)g(f(t))\,\mathrm{d}t\right),$$

with

$$W(x) = \int_0^x w(s)\,\mathrm{d}s, \quad \text{for } x \in (0,+\infty),$$

where w is a positive measurable function on $(0,+\infty)$ and g is a real continuous and strictly monotone function with its inverse g^{-1}.

We give some sufficient conditions on weights u, v on $(0,+\infty)$ for which there exists a positive constant C such that the weighted norm inequality

$$\left(\int_0^\infty u(x)\Big([\mathbf{M}_w^g f](x)\Big)^q\,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty v(x)f(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}},$$

holds for every measurable non-negative functions f, where $p, q \in (0,+\infty)$ satisfy certain restrictions.

Introduction

In recent years the topic of Hardy-type inequalities and their applications seem to become more and more popular. Although the original Hardy's result is dated to 1920-ties, some new versions are stated and old ones are improved almost a century later. One of the reasons of popularity of Hardy-type inequalities are their usefulness in various applications.

Original Hardy's result was discovered in the course of attempts to simplify the proofs of the well-known Hilbert's theorem, see historical part of [9]. Hardy has published his result in 1925 in paper [2] in the following form:

If $1 < p < +\infty$ and $f(x) \ge 0$, then

$$\int_0^\infty \Big([\mathbf{H}f](x)\Big)^p\,\mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p\,\mathrm{d}x, \tag{1}$$

where $[\mathbf{H}f]$ denotes the usual Hardy's averaging operator

$$[\mathbf{H}f](x) := \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t, \quad x > 0. \tag{2}$$

Afterward Hardy proved this inequality for sequences, for functions on the half line, and for functions in Lebesgue's spaces with power weights. From the early 1970-ties onward a great many related results were established under the general heading of Hardy's type inequalities, and a number of papers exist which provide new proofs, improvements, refinements, generalizations and many applications, see e.g. [4], [5], [7], [14], and [16] to name a few. Concerning the history and development of inequality (1) we refer interested reader to the books [9] and [1], [10] devoted to this subject from different viewpoints.

It is also well-known that the classical Pólya-Knopp's (sometimes also denoted as the Carleman-Knopp's) inequality, cf. [3],

$$\int_0^\infty [\mathbf{G}f](x)\,\mathrm{d}x \le \mathrm{e}\int_0^\infty f(x)\,\mathrm{d}x, \tag{3}$$

where $[\mathbf{G}f]$ denotes the geometric mean operator defined as

$$[\mathbf{G}f](x) := \exp\left(\frac{1}{x}\int_0^x \ln f(t)\,\mathrm{d}t\right), \quad x > 0, \tag{4}$$

may be derived as a limiting case of the Hardy's inequality (1) by changing $f \to f^{\frac{1}{p}}$ and tending $p \to \infty$, i.e.,

$$\lim_{p\to\infty}\Big([\mathbf{H}f^{\frac{1}{p}}](x)\Big)^p = [\mathbf{G}f](x), \quad \text{and} \quad \lim_{p\to\infty}\left(\frac{p}{p-1}\right)^p = \mathrm{e}.$$

In this note we propose to concentrate on the following general mean-type inequality problem: Let $p > 0$, $0 < q < +\infty$, and

$$W(x) := \int_0^x w(s)\,\mathrm{d}s, \quad \text{for } x \in (0, +\infty), \tag{5}$$

where w is a positive measurable function on $(0, +\infty)$. Find necessary and/or sufficient conditions on the positive measurable functions u, v (weights) and establish a class of functions g (a real continuous and strictly monotone function with its inverse g^{-1}) so that the following general mean-type inequality

$$\left(\int_0^\infty u(x)\Big([\mathbf{M}_w^g f](x)\Big)^q\,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty v(x)f(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}}, \quad f \ge 0, \tag{6}$$

holds for a positive finite constant C, where

$$[\mathbf{M}_w^g f](x) := g^{-1}\left(\frac{1}{W(x)}\int_0^x w(t)g(f(t))\,\mathrm{d}t\right) \tag{7}$$

is the *generalized weighted mean operator*. In the case of constant weight $w(t)$ we also denote the *generalized mean operator* as

$$[\mathbf{M}^g f](x) = g^{-1}\left(\frac{1}{x}\int_0^x g(f(t))\,\mathrm{d}t\right). \tag{8}$$

It may be seen that the inequality (6) is a natural generalization of Hardy's inequality. In particular, in the case $g^{-1}(x) = x$ operator $[\mathbf{M}_w^g f]$ reduces to the weighted Hardy's averaging operator $[\mathbf{H}_w f]$ and we have the weighted version of Hardy's inequality (1). Similarly, putting $g^{-1}(x) = \exp(x)$ we get the weighted geometric mean operator $[\mathbf{M}_w^g f] = [\mathbf{G}_w f]$ and then we obtain the weighted form of Pólya-Knopp's inequality (3). Note that the integral operator $[\mathbf{M}_w^g f]$ also generalizes the harmonic mean operator, cf. [13], the power mean operator, cf. [7], and some other integral operators.

This paper consists of several observations on the stated problem. First we will state some preliminary results and prove an equivalency relation between two versions of a general mean-type inequality for generalized weighted mean operator (7) and generalized mean operator (8). Since such a reduction is possible, in last two sections we study only inequalities involving integral operator (8) instead of (7). Using certain known methods we give some sufficient conditions for the inequality (6) to be valid.

1 First observation: Jensen's inequality in action

Jensen's inequality plays an important role when studying some inequalities among different means and operators. In our notation it has the following formulation.

Lemma 1.1 (Jensen's Inequality) *Let w, f be two nonnegative integrable functions on $(0, +\infty)$ such that $a < f(x) < b$ for all $x \in (0, +\infty)$, where $-\infty \le a < b \le +\infty$.*

(i) If g is a convex function on (a, b), then

$$g([\mathbf{H}_w f](x)) \le [\mathbf{H}_w g \circ f](x). \tag{9}$$

(ii) If g is a concave function on (a, b), then

$$g([\mathbf{H}_w f](x)) \ge [\mathbf{H}_w g \circ f](x).$$

As a direct consequence of Jensen's inequality we obtain the following useful result.

Corollary 1.2 *Let w, f be two nonnegative integrable functions on $(0, +\infty)$ such that $a < f(x) < b$ for all $x \in (0, +\infty)$, where $-\infty \le a < b \le +\infty$.*

(i) If g is a convex increasing or concave decreasing function on (a,b), then

$$[\mathbf{H}_w f](x) \le [\mathbf{M}_w^g f](x). \tag{10}$$

(ii) If g is a convex decreasing or concave increasing function on (a,b), then

$$[\mathbf{H}_w f](x) \ge [\mathbf{M}_w^g f](x).$$

Proof. Let g be a convex increasing function. Applying the inverse of g to both sides of Jensen's inequality (9) we obtain the desired result (10). Proofs of remaining parts are similar. □

Immediately, as an easy consequence of Jensen's inequality we get the following result.

Theorem 1.3 *Let u be a weight function on $(0,+\infty)$ and let $w(x) \ge 0$ for each $x \in (0,+\infty)$. Assume that $\frac{w(t)u(x)}{W(x)}$ is locally integrable on $(0,+\infty)$ for each fixed $t \in (0,+\infty)$, and define the function v by*

$$v(t) = w(t) \int_t^\infty \frac{u(x)}{W(x)}\,\mathrm{d}x < +\infty, \quad t \in (0,+\infty).$$

If $g : (0,+\infty) \to (a,b)$, where $-\infty \le a < b \le +\infty$, is either convex decreasing or concave increasing, then

$$\int_0^\infty u(x)[\mathbf{M}_w^g f](x)\,\mathrm{d}x \le \int_0^\infty v(x) f(x)\,\mathrm{d}x,$$

for all f such that $a < f(x) < b$ with $x \in [0,+\infty)$.

Proof. By Corollary 1.2 we get

$$\int_0^\infty u(x)[\mathbf{M}_w^g f](x)\,\mathrm{d}x \le \int_0^\infty u(x)[\mathbf{H}_w f](x)\,\mathrm{d}x.$$

Applying Fubini theorem we find that

$$\begin{aligned}\int_0^\infty u(x)\left(\frac{1}{W(x)}\int_0^x w(t)f(t)\,\mathrm{d}t\right)\mathrm{d}x &= \int_0^\infty f(t)\left(w(t)\int_t^\infty \frac{1}{W(x)}u(x)\,\mathrm{d}x\right)\mathrm{d}t \\ &= \int_0^\infty v(x)f(x)\,\mathrm{d}x.\end{aligned}$$

Hence the result. □

For the special case $w(t) \equiv 1$ we have

Corollary 1.4 *Let u be a weight function on $(0,+\infty)$ and let v be defined as*

$$v(t) = t\int_t^\infty u(x)\,\frac{\mathrm{d}x}{x}, \quad t \in (0,+\infty).$$

If $g : (0,+\infty) \to (a,b)$, where $-\infty \le a < b \le +\infty$, is either convex decreasing or concave increasing, then

$$\int_0^\infty u(x)[\mathbf{M}^g f](x)\,\mathrm{d}x \le \int_0^\infty v(x)f(x)\,\mathrm{d}x, \tag{11}$$

for all f such that $a < f(x) < b$ with $x \in [0,+\infty)$.

Observe that under the conditions of Corollary 1.4 the inequality (11) is equivalent to

$$\int_0^\infty u(x)\Phi\Big([\mathbf{H}h](x)\Big)\,\frac{\mathrm{d}x}{x} \le \int_0^\infty v(x)\Phi(f(x))\,\frac{\mathrm{d}x}{x},$$

when replacing $g \circ f$ by h, g^{-1} by Φ, $u(x)$ by $\frac{u(x)}{x}$ and $v(x)$ by $\frac{v(x)}{x}$.

Example 1.5 Choosing the function $g(x) = \ln x$ and replacing f by f^p with $p > 0$ in (11) we get the following Pólya-Knopp's type inequality

$$\int_0^\infty u(x)\Big([\mathbf{G}f](x)\Big)^p\,\mathrm{d}x \le \int_0^\infty v(x)f(x)^p\,\mathrm{d}x,$$

where $u(x)$ and $v(x)$ are defined as in Corollary 1.4.

2 Second observation: the reduction lemma

If we want to consider the general mean-type inequality (6) with weights we propose to reduce the weighted operator (7) into (8) and then solve this inequality in a reduced form.

Lemma 2.1 *Let $0 < p$ and $q < +\infty$. Let u and v be two weight functions defined on $(0,+\infty)$, f be a positive function defined on $(0,+\infty)$ and g be a real continuous and strictly monotone function. Moreover, let w be a strictly positive function on $(0,+\infty)$ and W be defined as in (5) such that $W(+\infty) = +\infty$. Then the inequality*

$$\left(\int_0^\infty u(x)\Big([\mathbf{M}^g_w f](x)\Big)^q\,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty v(x)f(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{12}$$

holds if and only if the inequality

$$\left(\int_0^\infty U(x)\Big([\mathbf{M}^g f](x)\Big)^q\,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty V(x)f(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{13}$$

holds with the same positive finite constant C and weights

$$U(x) := \frac{u\left(W^{-1}(x)\right)}{W'\left(W^{-1}(x)\right)}, \quad V(x) := \frac{v\left(W^{-1}(x)\right)}{W'\left(W^{-1}(x)\right)}. \tag{14}$$

Proof. Considering the generalized mean operators (7) and (8), we have that

$$[\mathbf{M}_w^g f](x) = [\mathbf{M}^g h](W(x)),$$

where

$$h(y) = f\left(W^{-1}(y)\right).$$

Now, the inequality (12) reads

$$\left(\int_0^\infty u(x)\Big([\mathbf{M}^g h](W(x))\Big)^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty v(x)\Big(h(W(x))\Big)^p \,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{15}$$

Using the substitution $y = W(x)$ we obtain that the inequality (15) is equivalent to the inequality (13). □

Remark 2.2 Lemma 2.1 was used for the first time in paper [7] in the context of the weighted Pólya-Knopp's inequality, i.e., $g^{-1}(x) = \exp(x)$. In principle, lemma says that the inequality (12) is not more general than the inequality (13). Also, for the case when w is a continuous and strictly positive function, then the inequalities of the type (12) may be obtained by only studying the basic inequality (13). So for this reason we will study some inequalities related to the inequality (13) only.

Example 2.3 (cf. [17]) Let $0 < p \le q < +\infty$, $\lambda > 0$, and $u(x) = x^r$, $v(x) = x^s$, where $r, s \in \mathbb{R}$ such that $\frac{r+1}{q} = \frac{s+1}{p}$. When $g^{-1}(x) = \exp(x)$ and $w(t) = t^{\lambda-1}$, we have the Cochran-Lee's type inequality, cf. [7] and [8],

$$\left(\int_0^\infty x^r \left[\exp\left(\frac{\lambda}{x^\lambda}\int_0^x t^{\lambda-1}\ln f(t)\,\mathrm{d}t\right)\right]^q \mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty x^s f(x)^p \,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{16}$$

for a positive finite constant C. If we use the reduction lemma and notation (4), then the inequality (16) is equivalent to

$$\left(\int_0^\infty U(x)\Big([\mathbf{G}h](x)\Big)^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty V(x)h(x)^p \,\mathrm{d}x\right)^{\frac{1}{p}},$$

where

$$h(x) = f\left((\lambda x)^{\frac{1}{\lambda}}\right), \quad U(x) = (\lambda x)^{\frac{r+1}{\lambda}-1}, \quad V(x) = (\lambda x)^{\frac{s+1}{\lambda}-1}.$$

Then we may rewrite the inequality (16) in the following form:

$$\left(\int_0^\infty (\lambda x)^{\frac{r+1}{\lambda}-1}\left[\exp\left(\frac{1}{x}\int_0^x \ln h(t)\,\mathrm{d}t\right)\right]^q \mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (\lambda x)^{\frac{s+1}{\lambda}-1} h(x)^p \,\mathrm{d}x\right)^{\frac{1}{p}}.$$

3 Third observation: Levinson's approach

As already mentioned, an elementary approach to the stated problem consists in using the notion of convexity. Indeed, according to Corollary 1.2 we have that if g is a convex increasing or concave decreasing function, then

$$[\mathbf{H}_w f](x) \leq [\mathbf{M}_w^g f](x)$$

for all admissible functions f on $(0, +\infty)$, and the inequality (6) reduces to the weighted Hardy's inequality for which the problem is solved by many authors. However, this is quite a rough approach.

Note that if g is a strictly monotone function then replacing g^{-1} by φ and $g(f)$ by h and using Lemma 2.1, the inequality (6) may be rewritten as follows:

$$\left(\int_0^\infty U(x)\Big(\varphi([\mathbf{H}h](x))\Big)^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty V(x)\Big(\varphi(h(x))\Big)^p\right)^{\frac{1}{p}}, \qquad (17)$$

where $U(x), V(x)$ are given in (14) for w a strictly positive function on $(0, +\infty)$. Also, if g is either a convex decreasing or concave increasing function, then φ is a convex function.

Observe that the inequality (17) is the weighted extension of Levinson's modular inequality which was studied in non-weighted form for N-functions in [11]. Its weighted form for the case $p = q = 1$ was proved by Hans P. Heinig in [6]. In this section we will study inequality (17) in a general case for $1 < p \leq q < +\infty$ and we will prove some its modifications. For this purpose we will consider the following class of functions, cf. [11].

Definition 3.1 *A function $\varphi : (a, b) \to (0, +\infty)$, where $0 \leq a < b \leq +\infty$, belongs to the class Φ_r, $r > 1$, if*

$$\varphi(x)\varphi''(x) \geq \left(1 - \frac{1}{r}\right)[\varphi'(x)]^2 \qquad (18)$$

holds for all $x > 0$. If $r = +\infty$, we write $\Phi_\infty = \Phi$.

The usual examples of functions belonging to the class Φ are the Euler gamma function Γ as well as functions $\varphi_1(x) = x^{-a}$ for $a > 0$, and $\varphi_2(x) = \mathrm{e}^{x^b}$ for $b \geq 1$. The inclusion $\Phi \subset \Phi_r$ is strict, because when choosing $\varphi(x) = x^s$ for $s \geq r$, then $\varphi \in \Phi_r$, but $\varphi \notin \Phi$. Thus, φ_1 and φ_2 are in the class Φ_r for each $r > 1$. However, for $b \in (0, 1)$ we have $\varphi_2 \notin \Phi_r$ for any $r > 1$.

It is also easy to verify that for $r > 1$ we have that if $\varphi \in \Phi_r$, then the function $\psi = \varphi^{\frac{1}{r}}$ is convex, whereas if $\varphi \in \Phi$, then the function $\psi = \ln \varphi$ is convex. This enables us to state the following theorem which is a generalization of weighted extension of Levinson's result [11]. Observe that the well-known weight condition of Muckenhoupt, cf. [12], is used.

Theorem 3.2 *Let $1 < p \le q < +\infty$, and let p' denote the dual index of p, i.e., $p' = \frac{p}{p-1}$.*

(a) Let $\varphi \in \Phi_q$ and

$$\sup_{\tau>0} \left(\int_\tau^\infty \frac{U(x)}{x^q} \, \mathrm{d}x \right)^{\frac{1}{q}} \left(\int_0^\tau V(x)^{1-p'} \right)^{\frac{1}{p'}} < +\infty. \tag{19}$$

Then

$$\left(\int_0^\infty U(x) \varphi([\mathbf{H}h](x)) \, \mathrm{d}x \right)^{\frac{1}{q}} \le C \left(\int_0^\infty V(x) \varphi(h(x)) \, \mathrm{d}x \right)^{\frac{1}{p}}. \tag{20}$$

(b) Let $s > 0$, and $\varphi \in \Phi$. If

$$V(x) = x^\lambda \int_x^\infty \frac{U(t)}{t^{\lambda+1}} \, \mathrm{d}t, \quad \textit{for } \lambda > 0,$$

then

$$\int_0^\infty U(x) \Big(\varphi([\mathbf{H}h](x)) \Big)^s \mathrm{d}x \le C \int_0^\infty V(x) \Big(\varphi(h(x)) \Big)^s \mathrm{d}x$$

with constant $C = \mathrm{e}^\lambda$.

Proof. First we prove the part (a). Since $\psi = \varphi^{\frac{1}{q}}$ is a convex function, then by Jensen's inequality we have

$$\begin{aligned} \left(\int_0^\infty U(x) \varphi([\mathbf{H}h](x)) \, \mathrm{d}x \right)^{\frac{1}{q}} &= \left(\int_0^\infty U(x) \Big(\psi([\mathbf{H}h](x)) \Big)^q \mathrm{d}x \right)^{\frac{1}{q}} \\ &\le \left(\int_0^\infty U(x) \Big([\mathbf{H}\psi(h)](x) \Big)^q \mathrm{d}x \right)^{\frac{1}{q}}. \end{aligned}$$

By the well-known Muckenhoupt's weight condition (19) we obtain

$$\left(\int_0^\infty U(x) \Big([\mathbf{H}\psi(h)](x) \Big)^q \mathrm{d}x \right)^{\frac{1}{q}} \le C \left(\int_0^\infty V(x) \Big(\psi(h(x)) \Big)^p \mathrm{d}x \right)^{\frac{1}{p}}. \tag{21}$$

For $1 < p \le q < +\infty$ and $\varphi \in \Phi_q$ we have $\varphi \in \Phi_p$, because

$$\varphi(x)\varphi''(x) \ge \left(1 - \frac{1}{q}\right) [\varphi'(x)]^2 \ge \left(1 - \frac{1}{p}\right) [\varphi'(x)]^2$$

holds for all $x > 0$. Therefore $\psi(h)^p = \varphi(h)$ and the inequality (20) is valid.

For the part (b) we apply Jensen's inequality for a convex function $\psi = \ln\varphi$ to get

$$\begin{aligned}\int_0^\infty U(x)\Big(\varphi([\mathbf{H}h](x))\Big)^s\,\mathrm{d}x &= \int_0^\infty U(x)\left[\exp\Big(\psi([\mathbf{H}h](x))\Big)\right]^s\,\mathrm{d}x\\ &\le \int_0^\infty U(x)\left[\exp\Big([\mathbf{H}\psi(h)](x)\Big)\right]^s\,\mathrm{d}x\\ &= \int_0^\infty U(x)\Big([\mathbf{G}\varphi(h)](x)\Big)^s\,\mathrm{d}x.\end{aligned}$$

Using the fact $[\mathbf{G}f]^s = [\mathbf{G}f^s]$ and substitution $t = xy$, we have

$$\begin{aligned}\int_0^\infty U(x)\Big([\mathbf{G}\varphi(h)](x)\Big)^s\,\mathrm{d}x &= \int_0^\infty U(x)[\mathbf{G}(\varphi(h))^s](x)\,\mathrm{d}x\\ &= \int_0^\infty U(x)\exp\left(\int_0^1 \ln\Big(\varphi(h(xy))\Big)^s\,\mathrm{d}y\right)\,\mathrm{d}x.\end{aligned}$$

Since

$$-\lambda = \int_0^1 \ln y^\lambda\,\mathrm{d}y, \quad \text{for } \lambda > 0,$$

we obtain

$$\begin{aligned}&\int_0^\infty U(x)\exp\left(\int_0^1 \ln\Big(\varphi(h(xy))\Big)^s\,\mathrm{d}y\right)\,\mathrm{d}x\\ &= \mathrm{e}^\lambda\int_0^\infty U(x)\exp\left(\int_0^1 \ln\left[y^\lambda\cdot\Big(\varphi(h(xy))\Big)^s\right]\,\mathrm{d}y\right)\,\mathrm{d}x.\end{aligned}$$

Applying Jensen's inequality again and interchanging the order of integration we get

$$\begin{aligned}&\mathrm{e}^\lambda\int_0^\infty U(x)\exp\left(\int_0^1 \ln y^\lambda\cdot\Big(\varphi(h(xy))\Big)^s\,\mathrm{d}y\right)\,\mathrm{d}x\\ &\le \mathrm{e}^\lambda\int_0^\infty U(x)\left(\int_0^1 y^\lambda\Big(\varphi(h(xy))\Big)^s\,\mathrm{d}y\right)\,\mathrm{d}x\\ &= \mathrm{e}^\lambda\int_0^1 y^\lambda\left(\int_0^\infty U(x)\Big(\varphi(h(xy))\Big)^s\,\mathrm{d}x\right)\,\mathrm{d}y.\end{aligned}$$

Substituting $t = xy$ and using Fubini's theorem one has

$$\begin{aligned}&\mathrm{e}^\lambda\int_0^1 y^{\lambda-1}\left(\int_0^\infty U\left(\frac{t}{y}\right)\Big(\varphi(h(t))\Big)^s\,\mathrm{d}t\right)\,\mathrm{d}y\\ &= \mathrm{e}^\lambda\int_0^\infty \Big(\varphi(h(t))\Big)^s\left(\int_0^1 y^{\lambda-1}U\left(\frac{t}{y}\right)\,\mathrm{d}y\right)\,\mathrm{d}t\\ &= \mathrm{e}^\lambda\int_0^\infty \Big(\varphi(h(t))\Big)^s\left(t^\lambda\int_t^\infty \frac{U(x)}{x^{\lambda+1}}\,\mathrm{d}x\right)\,\mathrm{d}t.\end{aligned}$$

Hence the result. □

Recall that the condition (19) was first established in B. Muckenhoupt's paper [12] for $p = q$. Moreover, the condition (19) is necessary and sufficient for (21).

From the proof of Theorem 3.2 (b) we immediately have that for $\varphi \in \Phi$ the inequality

$$\left(\int_0^\infty U(x)\Big(\varphi([\mathbf{H}h](x))\Big)^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le \left(\int_0^\infty U(x)\Big([\mathbf{G}(\varphi(h)](x)\Big)^q \,\mathrm{d}x\right)^{\frac{1}{q}}$$

holds which means that we may deal with the classical non-weighted geometric mean operator and therefore we may use the following well-known result for $[\mathbf{G}\varphi(h)]$ under the condition

$$\sup_{x>0} x^{-\frac{1}{p}} \left(\int_0^x U(x)\left(\left[\mathbf{G}\left(\frac{1}{V}\right)\right](x)\right)^{\frac{q}{p}} \mathrm{d}t\right)^{\frac{1}{q}} < +\infty, \tag{22}$$

whenever $0 < p \le q < +\infty$, cf. [15], which implies the inequality

$$\left(\int_0^\infty U(x)\Big([\mathbf{G}(\varphi(h)](x)\Big)^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty V(x)\Big(\varphi(h(x))\Big)^p \,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{23}$$

Summarizing the above we get

Theorem 3.3 *Let $0 < p \le q < +\infty$, and w be a strictly positive function on $(0, +\infty)$ with $W(+\infty) = +\infty$. Let f, g be nonnegative functions on $(0, +\infty)$ and, moreover, let g be real continuous and strictly monotone on $(0, +\infty)$ such that $g^{-1} \in \Phi$. If u, v are positive measurable functions on $(0, +\infty)$ and U, V given by (14) satisfy the condition (22), then the inequality (6) holds with a positive finite constant C.*

4 Fourth observation: Wedestig's approach

Using the approach from [18] we have the following similar result as Theorem 2.1 therein (observe only the sufficient condition in our case).

Theorem 4.1 *Let $1 < p \le q < +\infty$, $s \in (1, p)$, and g be a real continuous either convex decreasing or concave increasing function on (a, b), where $-\infty \le a < b \le +\infty$. Put*

$$A(s) = \sup_{t>0} \widetilde{V}(t)^{\frac{q(s-1)}{p}} \left(\int_t^\infty U(x)\widetilde{V}(x)^{\frac{q(p-s)}{p}} \frac{\mathrm{d}x}{x^q}\right)^{1/q}, \tag{24}$$

where U, V are given by (14) for u, v positive measurable functions on $(0, +\infty)$, w is a strictly positive function on $(0, +\infty)$ with $W(+\infty) = +\infty$, and $\widetilde{V}(t) =$

$\int_0^t V(y)^{1-p'}\,\mathrm{d}y$. *If* $A(s) < +\infty$, *then the inequality (6) holds for all* f *such that* $a < f(x) < b$ *with* $x \in [0, +\infty)$. *Moreover, if* C *is the best possible constant in (6), then*

$$C \le \inf_{1<s<p} \left(\frac{p-1}{p-s}\right)^{1/p'} A(s). \tag{25}$$

Proof. Using Lemma 2.1, replacing $g(f)$ by h and g^{-1} by φ we have the inequality (17). Then applying Jensen's inequality to the left-hand side of (17) we get

$$\left(\int_0^\infty U(x)\Big(\varphi([\mathbf{H}h](x))\Big)^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le \left(\int_0^\infty U(x)\Big([\mathbf{H}\varphi(h)](x)\Big)^q \,\mathrm{d}x\right)^{\frac{1}{q}}.$$

It is clear that if we prove the estimate

$$\left(\int_0^\infty U(x)\left(\frac{1}{x}\int_0^x \varphi(h(t))\,\mathrm{d}t\right)^q \mathrm{d}x\right)^{\frac{1}{q}} \le \left(\int_0^\infty V(x)\Big(\varphi(h(x))\Big)^p \mathrm{d}x\right)^{\frac{1}{p}},$$

then we get the upper estimate in (17) and consequently in (6). For this purpose put

$$V(x)\Big(\varphi(h(x))\Big)^p = \varphi(k(x)).$$

Now the inequality (17) takes the form

$$\left(\int_0^\infty U(x)\left(\frac{1}{x}\int_0^x V(t)^{-\frac{1}{p}}\Big(\varphi(k(t))\Big)^{\frac{1}{p}} \mathrm{d}t\right)^q \mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty \varphi(k(x))\,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{26}$$

Using Hölder's inequality with indices p and p' for the left-hand side integral in (26), we obtain

$$\begin{aligned}
&\left(\int_0^\infty U(x)\left(\frac{1}{x}\int_0^x V(t)^{-\frac{1}{p}}\Big(\varphi(k(t))\Big)^{\frac{1}{p}} \mathrm{d}t\right)^q \mathrm{d}x\right)^{\frac{1}{q}}\\
&= \left(\int_0^\infty U(x)\left(\frac{1}{x}\int_0^x \Big(\varphi(k(t))\Big)^{\frac{1}{p}} \widetilde{V}(t)^{\frac{s-1}{p}}\widetilde{V}(t)^{-\frac{s-1}{p}} V(t)^{-\frac{1}{p}}\,\mathrm{d}t\right)^q \mathrm{d}x\right)^{\frac{1}{q}}\\
&\le \left(\int_0^\infty U(x)\left(\int_0^x \varphi(k(t))\widetilde{V}(t)^{s-1}\,\mathrm{d}t\right)^{\frac{q}{p}} \left(\int_0^x \widetilde{V}(t)^{-\frac{p'(s-1)}{p}} V(t)^{-\frac{p'}{p}}\,\mathrm{d}t\right)^{\frac{q}{p'}} \frac{\mathrm{d}x}{x^q}\right)^{\frac{1}{q}}.
\end{aligned}$$

Using the fact that $V(t)^{-p'/p} = V(t)^{1-p'}$, the last integral is equivalent to

$$\begin{aligned}
&\left(\frac{p}{p-(s-1)p'}\right)^{\frac{1}{p'}} \left(\int_0^\infty U(x)\widetilde{V}(x)^{\frac{p-(s-1)p'}{p}\cdot\frac{q}{p'}} \left(\int_0^x \varphi(k(t))\widetilde{V}(t)^{s-1}\,\mathrm{d}t\right)^{\frac{q}{p}} \frac{\mathrm{d}x}{x^q}\right)^{\frac{1}{q}}\\
&= \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} \left(\int_0^\infty U(x)\widetilde{V}(x)^{\frac{q(p-s)}{p}} \left(\int_0^x \varphi(k(t))\widetilde{V}(t)^{s-1}\,\mathrm{d}t\right)^{\frac{q}{p}} \frac{\mathrm{d}x}{x^q}\right)^{\frac{1}{q}}.
\end{aligned} \tag{27}$$

By Minkowski's inequality the integral (27) is dominated by

$$\left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}}\left(\int_0^\infty \varphi(k(t))\widetilde{V}(t)^{s-1}\left(\int_t^\infty U(x)\widetilde{V}(x)^{\frac{q(p-s)}{p}}\frac{\mathrm{d}x}{x^q}\right)^{\frac{p}{q}}\mathrm{d}t\right)^{\frac{1}{p}}$$
$$\leq \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} A(s)\left(\int_0^\infty \varphi(k(t))\,\mathrm{d}t\right)^{\frac{1}{p}}.$$

Hence (26), thus (17) and consequently by Lemma 2.1 the inequality (6) holds with a constant satisfying the right hand side inequality in (25). □

As the following example shows the condition (24) is not necessary for the inequality (6) to hold.

Example 4.2 Let $g(x) = \sqrt{x}$, $U(x) = V(x) = x$ on $[0, +\infty)$ and $p = q = 2$. Then the inequality (6) holds, but the condition (24) is not satisfied.

Concluding Remarks

In this paper we discussed some direct methods to find sufficient conditions for the general mean-type inequality (6) to be valid. However, the obtained results do not take the "mean function" g into account, i.e., Muckenhoupt's as well as Wedestig's condition is, in fact, independent on a particular choice of g. We suppose that the "ideal" conditions should contain this function. This motivates us to develop new methods to deal with this problem to find sufficient and/or necessary conditions for the validity of inequality (6) in its general form.

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