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IM Preprint, series A, No. 1/2014
January 2014

Maximum likelihood estimators for extended growth curve model with orthogonal between-individual design matrices

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Abstract

The extended growth curve model is discussed in this paper. There are two versions of the model studied in the literature, which differ in the way how column spaces are nested. The nesting is applied either to the between-individuals or to the within-individuals design matrices. Although both versions are equivalent via reparametrization the properties of estimators cannot be transferred directly because of non-linearity of estimators. Since in many applications the between-individuals matrices are one-way ANOVA matrices, it is reasonable to assume orthogonal column spaces of between-individuals design matrices along with nested column spaces of within-individuals design matrices. We present the maximum likelihood estimators and their basic moments for the model with such orthogonality condition.

Keywords: Extended growth curve model; maximum likelihood estimators; orthogonality; moments

1 Introduction

In this paper, $\mathcal{R}(\mathbf{A})$ denotes the column space of a matrix $\mathbf{A}$, $r(\mathbf{A})$ its rank, and $\mathrm{Tr}(\mathbf{A})$ its trace. $\mathbf{P}_A = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$ and $\mathbf{Q}_A = \mathbf{I} - \mathbf{P}_A$ denote the orthogonal projection matrices onto the column space $\mathcal{R}(\mathbf{A})$ and onto its orthogonal complement, respectively. $\mathbf{P}_A^B = \mathbf{A}(\mathbf{A}'\mathbf{B}\mathbf{A})^-\mathbf{A}'\mathbf{B}$ and $\mathbf{Q}_A^B = \mathbf{I} - \mathbf{P}_A^B$ denote corresponding projection matrices in the metric given by a positive definite matrix $\mathbf{B}$. If $\mathbf{A}$ is a random matrix then the variance-covariance matrix $\mathrm{Var}[\mathbf{A}]$ is defined as $\mathrm{Var}[\mathrm{vec}\,\mathbf{A}]$, where vec is the column-wise vectorization operator. Symbol $\otimes$ denotes the Kronecker product. Symbol $\prod_{l=i}^{j} \mathbf{A}_l$ for $j < i$ denotes decreasing-indices product $\mathbf{A}_i\mathbf{A}_{i-1}\ldots\mathbf{A}_{j+1}\mathbf{A}_j$ of a sequence of matrices.

This paper deals with the model known as the extended growth curve model (EGCM) with fixed effects (also called the sum-of-profiles model):

$$\mathbf{Y} = \sum_{i=1}^{k} \mathbf{X}_i \mathbf{B}_i \mathbf{Z}_i' + \boldsymbol{\mathcal{E}}\,, \quad \operatorname{vec} \boldsymbol{\mathcal{E}} \sim N_{n\times p}\left(\mathbf{0}, \boldsymbol{\Sigma} \otimes \mathbf{I}_n\right). \tag{1}$$

where $\mathbf{Y} \in \mathbb{R}^{n\times p}$ is a matrix of independent p-variate observations, $\mathbf{X}_i \in \mathbb{R}^{n\times r_i}$ and $\mathbf{Z}_i \in \mathbb{R}^{p\times q_i}$, $i = 1, \ldots, k$, are between-individuals and within-individuals design matrices, respectively. $\mathbf{B}_i \in \mathbb{R}^{q_i \times r_i}$, $i = 1, \ldots, k$, are matrices of unknown parameters. The matrix $\boldsymbol{\mathcal{E}}$ is a matrix of random errors whose rows are independently normally distributed with unknown common covariance matrix $\boldsymbol{\Sigma} > 0$.

This model was introduced by Verbyla and Venables [9] as a generalized version of the growth curve model (GCM), which has only one profile, i.e. $\mathbf{Y} = \mathbf{XBZ}' + \boldsymbol{\mathcal{E}}$. They gave several examples how this model may arise. Since the maximum likelihood equations cannot be solved explicitly in this model, they also discussed an iterative algorithm for obtaining the maximum likelihood estimators (MLEs). Von Rosen [6] derived the explicit MLEs of unknown parameters under the additional nested subspace condition of between-individuals matrices

$$\mathcal{R}\left(\mathbf{X}_k\right) \subseteq \cdots \subseteq \mathcal{R}\left(\mathbf{X}_1\right)\,, \tag{2}$$

while nothing is said about different within-individuals matrices $\mathbf{Z}_i$'s. Many results have been presented assuming this condition, such as uniqueness conditions for MLEs or moments of estimators (see e.g. Kollo and Rosen [3]). However, there are situations where the column spaces of between-individuals design matrices should be disjoint or at least the intersection should be as small as possible. Filipiak and Rosen [1] discussed the model with the nested subspace condition of within-individuals design matrices

$$\mathcal{R}\left(\mathbf{Z}_k\right) \subseteq \cdots \subseteq \mathcal{R}\left(\mathbf{Z}_1\right)\,. \tag{3}$$

The conditions (2) and (3) lead to different parametrizations of model (1), however, Filipiak and Rosen [1] showed that via reparametrization one can derive model with condition (3) from model with condition (2) or vice versa, i.e. the two models are equivalent. Let us refer the model with condition (2) as Model I, and with condition (3) as Model II. Because of non-linearity of estimators the properties of estimators from one model cannot be transmitted directly to the other one. In Model II Filipiak and Rosen [1] gave also the MLEs of unknown parameters for the three component model and they discussed the uniqueness conditions and the moments for MLEs.

Hu [2] came up with a modification of Model I, assuming that the column spaces of between-individuals design matrices are orthogonal, i.e.

$$\mathbf{X}_i'\mathbf{X}_j = 0 \quad \forall\, i \neq j, \tag{4}$$

while no ordering among $\mathcal{R}(\mathbf{Z}_i)$'s is assumed. His idea is to separate groups rather then models. The idea is illustrated in Example 1. This example also demonstrates that it is very natural to assume the nested subspace condition of within-individuals design matrices $\mathbf{Z}_i$'s in the case of orthogonal between-individuals design matrices.

Example 1. This is Example 4.1.2, p. 374, given in Kollo and Rosen [3]. Let there be 3 treatment groups of animals, with n_j animals in the j^{th} group, and each group is subjected to different treatment conditions. The aim is to investigate the weight increase of the animals in all groups. All animals have been measured at the same p time points (say t_r, $r = 1, \dots, p$). The expected growth curve for each treatment group is supposed to be a polynomial in time, and the groups differ by the order of the polynomial. Thus, the means of the three treatment groups at time t are

$$\begin{aligned} \boldsymbol{\mu}_1 &= \boldsymbol{\beta}_{11} + \boldsymbol{\beta}_{21} t + \cdots + \boldsymbol{\beta}_{(q-2)1} t^{q-3}, \\ \boldsymbol{\mu}_2 &= \boldsymbol{\beta}_{12} + \boldsymbol{\beta}_{22} t + \cdots + \boldsymbol{\beta}_{(q-2)2} t^{q-3} + \boldsymbol{\beta}_{(q-1)2} t^{q-2}, \\ \boldsymbol{\mu}_3 &= \boldsymbol{\beta}_{13} + \boldsymbol{\beta}_{23} t + \cdots + \boldsymbol{\beta}_{(q-2)3} t^{q-3} + \boldsymbol{\beta}_{(q-1)3} t^{q-2} + \boldsymbol{\beta}_{q3} t^{q-1}. \end{aligned} \tag{5}$$

In order to describe these different responses, we can use the model

$$\mathbf{Y} = \mathbf{X}_1 \mathbf{B}_1 \mathbf{Z}_1' + \mathbf{X}_2 \mathbf{B}_2 \mathbf{Z}_2' + \mathbf{X}_3 \mathbf{B}_3 \mathbf{Z}_3' + \boldsymbol{\mathcal{E}}, \tag{6}$$

where $\mathbf{Y}_{n \times p}$ is the observation matrix, $\boldsymbol{\mathcal{E}}_{n \times p}$ is the matrix of random errors and the remaining matrices are defined as below:

$$\begin{aligned} \mathbf{X}_1 &= \begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{0}_{n_1} & \mathbf{0}_{n_1} \\ \mathbf{0}_{n_2} & \mathbf{1}_{n_2} & \mathbf{0}_{n_2} \\ \mathbf{0}_{n_3} & \mathbf{0}_{n_3} & \mathbf{1}_{n_3} \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} \mathbf{0}_{n_1} & \mathbf{0}_{n_1} \\ \mathbf{1}_{n_2} & \mathbf{0}_{n_2} \\ \mathbf{0}_{n_3} & \mathbf{1}_{n_3} \end{pmatrix}, \quad \mathbf{X}_3 = \begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{n_2} \\ \mathbf{1}_{n_3} \end{pmatrix}, \\ \mathbf{B}_1 &= \begin{pmatrix} \boldsymbol{\beta}_{11} & \boldsymbol{\beta}_{21} & \cdots & \boldsymbol{\beta}_{(q-2)1} \\ \boldsymbol{\beta}_{12} & \boldsymbol{\beta}_{22} & \cdots & \boldsymbol{\beta}_{(q-2)2} \\ \boldsymbol{\beta}_{13} & \boldsymbol{\beta}_{23} & \cdots & \boldsymbol{\beta}_{(q-2)3} \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} \boldsymbol{\beta}_{(q-1)2} \\ \boldsymbol{\beta}_{(q-1)3} \end{pmatrix}, \quad \mathbf{B}_3 = \begin{pmatrix} \boldsymbol{\beta}_{q3} \end{pmatrix}, \\ \mathbf{Z}_1 &= \begin{pmatrix} 1 & t_1 & \cdots & t_1^{q-3} \\ 1 & t_2 & \cdots & t_2^{q-3} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_p & \cdots & t_p^{q-3} \end{pmatrix}, \quad \mathbf{Z}_2 = \begin{pmatrix} t_1^{q-2} \\ t_2^{q-2} \\ \vdots \\ t_p^{q-2} \end{pmatrix}, \quad \mathbf{Z}_3 = \begin{pmatrix} t_1^{q-1} \\ t_2^{q-1} \\ \vdots \\ t_p^{q-1} \end{pmatrix}. \end{aligned}$$

This is clearly Model I, since $\mathcal{R}(\mathbf{X}_3) \subseteq \mathcal{R}(\mathbf{X}_2) \subseteq \mathcal{R}(\mathbf{X}_1)$. Notice that it is not possible to model the mean structure with only one component, i.e. single-profile GCM. We say that Model I separates models, since every component contains polynomial growth of different order.

The same mean structure (5) could be modeled by model (6) with the following matrices:

$$\mathbf{X}_1 = \begin{pmatrix} \mathbf{1}_{n_1} \\ \mathbf{0}_{n_2} \\ \mathbf{0}_{n_3} \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{1}_{n_2} \\ \mathbf{0}_{n_3} \end{pmatrix}, \quad \mathbf{X}_3 = \begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{n_2} \\ \mathbf{1}_{n_3} \end{pmatrix},$$

$$
\begin{aligned}
\mathbf{B}_1 &= \begin{pmatrix} \boldsymbol{\beta}_{11} & \boldsymbol{\beta}_{21} & \cdots & \boldsymbol{\beta}_{(q-2)1} \end{pmatrix}, \\
\mathbf{B}_2 &= \begin{pmatrix} \boldsymbol{\beta}_{12} & \boldsymbol{\beta}_{22} & \cdots & \boldsymbol{\beta}_{(q-2)2} & \boldsymbol{\beta}_{(q-1)2} \end{pmatrix}, \\
\mathbf{B}_3 &= \begin{pmatrix} \boldsymbol{\beta}_{13} & \boldsymbol{\beta}_{23} & \cdots & \boldsymbol{\beta}_{(q-2)3} & \boldsymbol{\beta}_{(q-1)3} & \boldsymbol{\beta}_{q3} \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Z}_1 &= \begin{pmatrix} 1 & t_1 & \cdots & t_1^{q-3} \\ 1 & t_2 & \cdots & t_2^{q-3} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_p & \cdots & t_p^{q-3} \end{pmatrix}, \quad \mathbf{Z}_2 = \begin{pmatrix} 1 & t_1 & \cdots & t_1^{q-3} & t_1^{q-2} \\ 1 & t_2 & \cdots & t_2^{q-3} & t_2^{q-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & t_p & \cdots & t_p^{q-3} & t_p^{q-2} \end{pmatrix}, \\
\mathbf{Z}_3 &= \begin{pmatrix} 1 & t_1 & \cdots & t_1^{q-3} & t_1^{q-2} & t_1^{q-1} \\ 1 & t_2 & \cdots & t_2^{q-3} & t_2^{q-2} & t_2^{q-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & t_p & \cdots & t_p^{q-3} & t_p^{q-2} & t_p^{q-1} \end{pmatrix}.
\end{aligned}
$$

Here we have a model similar to Model II but with opposite order of spaces, since $\mathcal{R}(\mathbf{Z}_1) \subseteq \mathcal{R}(\mathbf{Z}_2) \subseteq \mathcal{R}(\mathbf{Z}_3)$. We say that this model separates groups (or treatments), since each component models growth of only one group of observations. Moreover, the column spaces of between-individuals matrices in this model are orthogonal, i.e. $\mathbf{X}_i'\mathbf{X}_j = 0$ for $i \neq j$.

The model with assumption of nested column spaces of within-individuals design matrices and with an additional assumption of orthogonal between-individuals design matrices will be referred as Model III:

$$
\begin{gathered}
\mathbf{Y} = \sum_{i=1}^{k} \mathbf{X}_i \mathbf{B}_i \mathbf{Z}_i' + \boldsymbol{\mathcal{E}}, \quad \operatorname{vec}(\boldsymbol{\mathcal{E}}) \sim N_{n \times p}\left(\mathbf{0}, \boldsymbol{\Sigma} \otimes \mathbf{I}_n\right), \\
\mathbf{X}_i'\mathbf{X}_j = 0 \ \ \forall\, i \neq j, \\
\mathcal{R}(\mathbf{Z}_1) \subseteq \cdots \subseteq \mathcal{R}(\mathbf{Z}_k).
\end{gathered} \tag{7}
$$

In order to have all parameters of interest estimable, we also assume that $n > \sum_{i=1}^{k} r(X_i) + p$.

ECGM with Hu's condition, i.e. Model III, is much easier to handle in the case of known variance-covariance matrix $\boldsymbol{\Sigma}$. To illustrate the situation, let us look at two components Model I. If $\mathbf{X}_1, \mathbf{X}_2, \mathbf{Z}_1, \mathbf{Z}_2$ are of full rank and $\mathcal{R}(\mathbf{Z}_1) \cap \mathcal{R}(\mathbf{Z}_2) = \emptyset$, then closed form of unbiased least-squares estimators of $\mathbf{B}_1, \mathbf{B}_2$ is known to be

$$
\begin{aligned}
\widehat{\mathbf{B}}_1 &= \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1} \mathbf{X}_1'\mathbf{Y}\boldsymbol{\Sigma}^{-1}\mathbf{Z}_1 \left(\mathbf{Z}_1'\boldsymbol{\Sigma}^{-1}\mathbf{Z}_1\right)^{-1} - \\
&\quad - \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1} \mathbf{X}_1'\mathbf{P}_{X_2}\mathbf{Y} \left(\mathbf{P}_{Z_2}^{\Sigma^{-1}\mathbf{Q}_{Z_1}^{\Sigma^{-1}}}\right)' \boldsymbol{\Sigma}^{-1}\mathbf{Z}_1 \left(\mathbf{Z}_1'\boldsymbol{\Sigma}^{-1}\mathbf{Z}_1\right)^{-1}, \\
\widehat{\mathbf{B}}_2 &= \left(\mathbf{X}_2'\mathbf{X}_2\right)^{-1} \mathbf{X}_2'\mathbf{Y}\boldsymbol{\Sigma}^{-1}\mathbf{Z}_2 \left(\mathbf{Z}_2'\boldsymbol{\Sigma}^{-1}\mathbf{Q}_{Z_1}^{\Sigma^{-1}}\mathbf{Z}_2\right)^{-1},
\end{aligned}
$$

see Žežula [10]. The situation is much more complicated for more than two components. On the other hand, explicit form of the least-squares estimator of all $\mathbf{B}_i$'s in Model III with arbitrary number of components is known to be

$$\widehat{\mathbf{B}}_i = \left(\mathbf{X}_i'\mathbf{X}_i\right)^{-1}\mathbf{X}_i'\mathbf{Y}\boldsymbol{\Sigma}^{-1}\mathbf{Z}_i\left(\mathbf{Z}_i'\boldsymbol{\Sigma}^{-1}\mathbf{Z}_i\right)^{-1},$$

see [2].

The aim of this paper is to present results for Model III. We derive the maximum likelihood estimators of unknown parameters as well as basic moments of these estimators. The benefit of Hu's condition appears mainly in deriving the variances of estimators of location parameters, since the maximum likelihood estimators do not depend on each other (as is the case in Model I and Model II). Also, these variances in Model I and Model II are known only for three component model, while we are able to solve this task for general k-component model in Model III.

2 Maximum likelihood estimators

The maximum likelihood estimators of unknown parameters $\mathbf{B}_i$, $i = 1, \dots, k$, and $\boldsymbol{\Sigma}$ for Model I are given in Kollo and Rosen [3], Theorem 4.1.7. For Model II Filipiak and Rosen [1] derived the MLEs only for the three component model. The next theorem shows the MLEs in Model III.

Theorem 1. *Let us consider Model III, and denote* $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k)$. *The maximum likelihood estimators of unknown parameters in this model are*

$$\begin{aligned}\widehat{\mathbf{B}}_i &= \left(\mathbf{X}_i'\mathbf{X}_i\right)^{-}\mathbf{X}_i'\mathbf{Y}\mathbf{S}_i^{-1}\mathbf{Z}_i\left(\mathbf{Z}_i'\mathbf{S}_i^{-1}\mathbf{Z}_i\right)^{-} + \\ &\quad + \mathbf{G}_i - (\mathbf{X}_i'\mathbf{X}_i)^{-}\mathbf{X}_i'\mathbf{X}_i\mathbf{G}_i\mathbf{Z}_i'\mathbf{S}_i^{-1}\mathbf{Z}_i(\mathbf{Z}_i'\mathbf{S}_i^{-1}\mathbf{Z}_i)^{-}, \\ n\widehat{\boldsymbol{\Sigma}} &= \left(\mathbf{Y} - \sum_{i=1}^{k}\mathbf{X}_i\widehat{\mathbf{B}}_i\mathbf{Z}_i'\right)'\left(\mathbf{Y} - \sum_{i=1}^{k}\mathbf{X}_i\widehat{\mathbf{B}}_i\mathbf{Z}_i'\right) = \\ &= \mathbf{S}_1 + \sum_{i=1}^{k}\mathbf{Q}_{Z_i}^{S_i^{-1}}\mathbf{Y}'\mathbf{P}_{X_i}\mathbf{Y}\left(\mathbf{Q}_{Z_i}^{S_i^{-1}}\right)',\end{aligned}$$

where $\mathbf{G}_i$, $i = 1, \dots, k$, *are arbitrary* $r_i \times q_i$ *matrices,* $\mathbf{S}_1 = \mathbf{Y}'\mathbf{Q}_X\mathbf{Y}$ *and* $\mathbf{S}_i = \mathbf{S}_{i-1} + \mathbf{Q}_{Z_{i-1}}^{S_{i-1}^{-1}}\mathbf{Y}'\mathbf{P}_{X_{i-1}}\mathbf{Y}\left(\mathbf{Q}_{Z_{i-1}}^{S_{i-1}^{-1}}\right)'$ *for* $i = 2, \dots, k$.

Proof. The likelihood function equals

$$\begin{aligned}L(\mathbf{B}_1, \dots, \mathbf{B}_k, \boldsymbol{\Sigma}; \mathbf{Y}) &= (2\pi)^{-\frac{np}{2}}\left|\boldsymbol{\Sigma}\right|^{-\frac{n}{2}} \times \\ &\quad \times \mathbf{e}^{-\frac{1}{2}\operatorname{Tr}\left[\boldsymbol{\Sigma}^{-1}\left(\mathbf{Y}-\sum_{i=1}^{k}\mathbf{X}_i\mathbf{B}_i\mathbf{Z}_i'\right)'\left(\mathbf{Y}-\sum_{i=1}^{k}\mathbf{X}_i\mathbf{B}_i\mathbf{Z}_i'\right)\right]}.\end{aligned}$$

Differentiating the log-likelihood $\ln L$ with respect to unknown parameters and setting this derivatives to zero, we get MLE's as the solutions of the following likelihood equations:

$$\begin{aligned} \mathbf{0} &= \mathbf{X}_i' \left(\mathbf{Y} - \mathbf{X}_i \mathbf{B}_i \mathbf{Z}_i'\right) \mathbf{\Sigma}^{-1} \mathbf{Z}_i, \quad i = 1, \ldots, k, & (8) \\ n\mathbf{\Sigma} &= \left(\mathbf{Y} - \sum_{i=1}^{k} \mathbf{X}_i \mathbf{B}_i \mathbf{Z}_i'\right)' \left(\mathbf{Y} - \sum_{i=1}^{k} \mathbf{X}_i \mathbf{B}_i \mathbf{Z}_i'\right). & (9) \end{aligned}$$

According to Puntanen et al. [5], Theorem 11, p. 267, the general solution of equation (8) is:

$$\begin{aligned} \mathbf{B}_i &= \left(\mathbf{X}_i'\mathbf{X}_i\right)^{-} \mathbf{X}_i'\mathbf{Y}\mathbf{\Sigma}^{-1}\mathbf{Z}_i \left(\mathbf{Z}_i'\mathbf{\Sigma}^{-1}\mathbf{Z}_i\right)^{-} + & (10) \\ &\quad + \mathbf{G}_i - (\mathbf{X}_i'\mathbf{X}_i)^{-}\mathbf{X}_i'\mathbf{X}_i\mathbf{G}_i\mathbf{Z}_i'\mathbf{\Sigma}^{-1}\mathbf{Z}_i(\mathbf{Z}_i'\mathbf{\Sigma}^{-1}\mathbf{Z}_i)^{-}, \end{aligned}$$

where $\mathbf{G}_i$ is arbitrary matrix of appropriate order.

Since the column spaces of matrices $\mathbf{X}_i$ are orthogonal, it holds $\mathbf{P}_X = \sum_{i=1}^{k} \mathbf{P}_{X_i}$. Then we can rewrite the equation (9) as

$$n\mathbf{\Sigma} = \mathbf{S}_1 + \sum_{i=1}^{k} \left(\mathbf{P}_{X_i}\mathbf{Y} - \mathbf{X}_i\mathbf{B}_i\mathbf{Z}_i'\right)' \left(\mathbf{P}_{X_i}\mathbf{Y} - \mathbf{X}_i\mathbf{B}_i\mathbf{Z}_i'\right),$$

where $\mathbf{S}_1 = \mathbf{Y}'\mathbf{Q}_X\mathbf{Y}$, and replacing $\mathbf{B}_i$ by formula (10) we obtain

$$n\mathbf{\Sigma} = \mathbf{S}_1 + \sum_{i=1}^{k} \mathbf{Q}_{Z_i}^{\Sigma^{-1}} \mathbf{Y}'\mathbf{P}_{X_i}\mathbf{Y} \left(\mathbf{Q}_{Z_i}^{\Sigma^{-1}}\right)'. \tag{11}$$

Because of nested subspace condition $\mathcal{R}(\mathbf{Z}_1) \subseteq \cdots \subseteq \mathcal{R}(\mathbf{Z}_k)$ it is easy to see that $\left(\mathbf{Q}_{Z_i}^{\Sigma^{-1}}\right)' \mathbf{\Sigma}^{-1}\mathbf{Z}_j = \mathbf{0}$ for $j \leq i$. Therefore we first multiply (11) by $\mathbf{\Sigma}^{-1}\mathbf{Z}_1$ and we obtain $n\mathbf{Z}_1 = \mathbf{S}_1\mathbf{\Sigma}^{-1}\mathbf{Z}_1$ which is equivalent to

$$\mathbf{\Sigma}^{-1}\mathbf{Z}_1 = n\mathbf{S}_1^{-1}\mathbf{Z}_1.$$

Replacing this expression into (11) we have

$$n\mathbf{\Sigma} = \mathbf{S}_2 + \sum_{i=2}^{k} \mathbf{Q}_{Z_i}^{\Sigma^{-1}} \mathbf{Y}'\mathbf{P}_{X_i}\mathbf{Y} \left(\mathbf{Q}_{Z_i}^{\Sigma^{-1}}\right)',$$

where $\mathbf{S}_2 = \mathbf{S}_1 + \mathbf{Q}_{Z_1}^{S_1^{-1}} \mathbf{Y}'\mathbf{P}_{X_1}\mathbf{Y} \left(\mathbf{Q}_{Z_1}^{S_1^{-1}}\right)'$. Now we multiply last expression by $\mathbf{\Sigma}^{-1}\mathbf{Z}_2$ and obtain $\mathbf{\Sigma}^{-1}\mathbf{Z}_2 = n\mathbf{S}_2^{-1}\mathbf{Z}_2$. Continuing this process we finally obtain that

$$\mathbf{\Sigma}^{-1}\mathbf{Z}_i = n\mathbf{S}_i^{-1}\mathbf{Z}_i, \quad i = 1, 2, \ldots, k, \tag{12}$$

where $\mathbf{S}_i = \mathbf{S}_{i-1} + \mathbf{Q}_{Z_{i-1}}^{S_{i-1}^{-1}} \mathbf{Y}'\mathbf{P}_{X_{i-1}}\mathbf{Y} \left(\mathbf{Q}_{Z_{i-1}}^{S_{i-1}^{-1}}\right)'$ for $i = 2, \ldots, k$, and $\mathbf{S}_1 = \mathbf{Y}'\mathbf{Q}_X\mathbf{Y}$. Finally replacing (12) into (10) and (11) we obtain the desired result for $\widehat{\mathbf{B}}_i$ and $n\widehat{\mathbf{\Sigma}}$. □

It is obvious that $\widehat{\mathbf{\Sigma}}$ is unique. However, estimators $\widehat{\mathbf{B}}_i$ are not. For the uniqueness conditions in Model I see e.g. [8, 3] and for Model II they are given only for three-component model in Filipiak and Rosen [1]. For Model III the uniqueness conditions are given in the next theorem.

Theorem 2. *The maximum likelihood estimator $\widehat{\mathbf{B}}_i$, $i = 1, \dots, k$, is unique if and only if $\mathbf{X}_i$ and $\mathbf{Z}_i$ are of full rank, i.e. $r(\mathbf{X}_i) = r_i$ and $r(\mathbf{Z}_i) = q_i$. In that case*

$$\widehat{\mathbf{B}}_i = \left(\mathbf{X}_i'\mathbf{X}_i\right)^{-1} \mathbf{X}_i'\mathbf{Y}\mathbf{S}_i^{-1}\mathbf{Z}_i \left(\mathbf{Z}_i'\mathbf{S}_i^{-1}\mathbf{Z}_i\right)^{-1},$$

where $\mathbf{S}_i$ is given in Theorem 1.

Proof. It follows from Theorem 1 that $\widehat{\mathbf{B}}_i$ is unique if and only if

$$\mathbf{G}_i = (\mathbf{X}_i'\mathbf{X}_i)^{-}\mathbf{X}_i'\mathbf{X}_i\mathbf{G}_i\mathbf{Z}_i'\mathbf{S}_i^{-1}\mathbf{Z}_i(\mathbf{Z}_i'\mathbf{S}_i^{-1}\mathbf{Z}_i)^{-}$$

for all g-inverses, which holds if and only if $\mathbf{X}_i$ and $\mathbf{Z}_i$ are of full rank. □

3 Basic moments of estimators

In the following we will assume that design matrices $\mathbf{X}_i$ and $\mathbf{Z}_i$ are of full rank. The moments of maximum likelihood estimators in Model I are given Kollo and Rosen [3] and for Model II in Filipiak and Rosen [1]. However, the variance of estimators were found only for three-component model, since for k components the calculations are very tedious in general. In Model III the moments can be derived for general k components model.

Lemma 3. *Let $\mathbf{A}$ and $\mathbf{V} > 0$ be any $p \times q$ and $p \times p$ matrices, respectively. Then*

(i) $\mathbf{P}_A^{V^{-1}} = \left(\mathbf{Q}_{Q_A}^{V}\right)'$.

(ii) $\mathbf{P}_A^{V} = \left(\mathbf{Q}_{Q_A}^{V^{-1}}\right)'$.

(iii) $\mathbf{V}^{-1} = \mathbf{A}\left(\mathbf{A}'\mathbf{V}\mathbf{A}\right)^{-}\mathbf{A}' + \mathbf{V}^{-1}\mathbf{Q}_A\left(\mathbf{Q}_A\mathbf{V}^{-1}\mathbf{Q}_A\right)^{-}\mathbf{Q}_A\mathbf{V}^{-1}$

Proof. Let start with (i). Following Markiewicz [4], proof of Lemma 1, it is easy to see that $\mathbf{P}_{V^{-1/2}A}\mathbf{V}^{1/2}\mathbf{Q}_A = 0$. Then

$$\begin{aligned}\mathbf{P}_{V^{1/2}Q_A} &= \mathbf{V}^{1/2}\mathbf{Q}_A\left(\mathbf{Q}_A\mathbf{V}\mathbf{Q}_A\right)^{-}\mathbf{Q}_A\mathbf{V}^{1/2} = \\ &= \left(\mathbf{I} - \mathbf{P}_{V^{-1/2}A}\right)\mathbf{V}^{1/2}\mathbf{Q}_A\left(\mathbf{Q}_A\mathbf{V}\mathbf{Q}_A\right)^{-}\mathbf{Q}_A\mathbf{V}^{1/2} = \\ &= \left(\mathbf{I} - \mathbf{P}_{V^{-1/2}A}\right)\mathbf{P}_{V^{1/2}Q_A}\end{aligned}$$

However, since $r\left(\mathbf{V}^{1/2}\mathbf{Q}_A\right) = n - r\left(\mathbf{V}^{-1/2}\mathbf{A}\right)$ it follows that

$$\mathbf{P}_{V^{1/2}Q_A} = \left(\mathbf{I} - \mathbf{P}_{V^{-1/2}A}\right) = \mathbf{Q}_{V^{-1/2}A}$$

which is equivalent with $\mathbf{P}_{V^{-1/2}A} = \mathbf{Q}_{V^{1/2}Q_A}$. Then

$$\begin{aligned}
\mathbf{P}_A^{V^{-1}} &= \mathbf{A}\left(\mathbf{A}'\mathbf{V}^{-1}\mathbf{A}\right)^{-}\mathbf{A}'\mathbf{V}^{-1} = \\
&= \mathbf{V}^{1/2}\mathbf{V}^{-1/2}\mathbf{A}\left(\mathbf{A}'\mathbf{V}^{-1/2}\mathbf{V}^{-1/2}\mathbf{A}\right)^{-}\mathbf{A}'\mathbf{V}^{-1/2}\mathbf{V}^{-1/2} = \\
&= \mathbf{V}^{1/2}\mathbf{P}_{V^{-1/2}A}\mathbf{V}^{-1/2} = \mathbf{V}^{1/2}\mathbf{Q}_{V^{1/2}Q_A}\mathbf{V}^{-1/2} = \\
&= \mathbf{I} - \mathbf{V}^{1/2}\mathbf{V}^{1/2}\mathbf{Q}_A\left(\mathbf{Q}_A\mathbf{V}^{1/2}\mathbf{V}^{1/2}\mathbf{Q}_A\right)^{-}\mathbf{Q}_A\mathbf{V}^{1/2}\mathbf{V}^{-1/2} = \\
&= \mathbf{I} - \mathbf{V}\mathbf{Q}_A\left(\mathbf{Q}_A\mathbf{V}\mathbf{Q}_A\right)^{-}\mathbf{Q}_A = \left(\mathbf{Q}_{Q_A}^{V}\right)'.
\end{aligned}$$

Similarly, (ii) can be proved. For (iii) observe that $\mathbf{A}\left(\mathbf{A}'\mathbf{V}\mathbf{A}\right)^{-}\mathbf{A}' = \mathbf{P}_A^V\mathbf{V}^{-1}$. Then using (ii) we have

$$\begin{aligned}
\mathbf{A}\left(\mathbf{A}'\mathbf{V}\mathbf{A}\right)^{-}\mathbf{A}' &= \left(\mathbf{Q}_{Q_A}^{V^{-1}}\right)'\mathbf{V}^{-1} = \mathbf{V}^{-1} - \left(\mathbf{P}_{Q_A}^{V^{-1}}\right)'\mathbf{V}^{-1} = \\
&= \mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{Q}_A\left(\mathbf{Q}_A\mathbf{V}^{-1}\mathbf{Q}_A\right)^{-}\mathbf{Q}_A\mathbf{V}^{-1}.
\end{aligned}$$

□

Lemma 4. *Let us consider Model III, and define* $\mathbf{W}_1 = \mathbf{\Sigma}^{-1/2}\mathbf{S}_1\mathbf{\Sigma}^{-1/2}$, $\mathbf{W}_i = \mathbf{W}_{i-1} + \mathbf{N}_{i-1}$ *for* $i = 2, \ldots, k$, *where* $\mathbf{S}_1 = \boldsymbol{\mathcal{E}}'\mathbf{Q}_X\boldsymbol{\mathcal{E}}$ *and* $\mathbf{N}_i = \mathbf{\Sigma}^{-1/2}\boldsymbol{\mathcal{E}}'\mathbf{P}_{X_i}\boldsymbol{\mathcal{E}}\mathbf{\Sigma}^{-1/2}$. *Then,*

$$\mathbf{W}_i \sim \mathcal{W}_p\left(n - \sum_{j=i}^{k} r(\mathbf{X}_j), \mathbf{I}_p\right) \qquad \forall\, i = 1, \ldots, k.$$

Proof. Denoting $\mathbf{H} = \boldsymbol{\mathcal{E}}\mathbf{\Sigma}^{-1/2}$ it is easy to see that $\mathbf{H} \sim N_{n\times p}(\mathbf{0}, \mathbf{I}_n, \mathbf{I}_p)$, i.e. $\operatorname{vec}\mathbf{H} \sim N(\mathbf{0}, \mathbf{I}_p \otimes \mathbf{I}_n)$. Then it is obvious that

$$\mathbf{W}_1 = \mathbf{H}'\mathbf{Q}_X\mathbf{H} \sim \mathcal{W}_p\left(n - \sum_{j=1}^{k} r(\mathbf{X}_j), \mathbf{I}_p\right).$$

Observe that denoting $\mathbf{T}_i = \mathbf{Q}_X + \sum_{j=1}^{i-1}\mathbf{P}_{X_j} = \mathbf{I}_n - \sum_{j=i}^{k}\mathbf{P}_{X_j}$ we can write

$$\mathbf{W}_i = \mathbf{H}'\mathbf{Q}_X\mathbf{H} + \sum_{j=1}^{i-1}\mathbf{H}'\mathbf{P}_{X_j}\mathbf{H} = \mathbf{H}'\mathbf{T}_i\mathbf{H} \tag{13}$$

for $i = 2, \ldots, k$. Since matrix $\mathbf{T}_i$ is symmetric and idempotent with $r(\mathbf{T}_i) = n - \sum_{j=i}^{k} r(\mathbf{X}_j)$, the result follows. □

Lemma 5. *Let us consider Model III with* $\mathbf{W}_1, \ldots, \mathbf{W}_k$ *defined as in Lemma 4. Let* $\mathbf{D}_i = \mathbf{\Sigma}^{1/2}\mathbf{Q}_{Z_i}$, $i = 1, \ldots, k$. *Then for any* i, j, *such that* $1 \le j < i < k$, *it holds*

$$\begin{aligned}
\operatorname{Tr}\left(\mathrm{E}\left[\mathbf{P}_{D_i}^{W_i}\cdots\mathbf{P}_{D_{j+1}}^{W_{j+1}}\mathbf{D}_j\left(\mathbf{D}_j'\mathbf{W}_j\mathbf{D}_j\right)^{-}\mathbf{D}_j'\left(\mathbf{P}_{D_{j+1}}^{W_{j+1}}\right)'\cdots\left(\mathbf{P}_{D_i}^{W_i}\right)'\right]\right) = \\
= d_j d_{j+1}\ldots d_{i-1}c_i,
\end{aligned}$$

where $d_j = \frac{n-\sum_{l=j+1}^{k} r(\mathbf{X}_l)-r(\mathbf{D}_j)-1}{n-\sum_{l=j}^{k} r(\mathbf{X}_l)-r(\mathbf{D}_j)-1}$ *and* $c_i = \frac{r(\mathbf{D}_i)}{n-\sum_{l=i}^{k} r(\mathbf{X}_l)-r(\mathbf{D}_i)-1}$.

Proof. It is clear that $\mathbf{D}_i$ is a $p \times p$ matrix of rank $r(\mathbf{D}_i) = p - r(\mathbf{Z}_i)$, thus it is not of full rank. However, $\mathbf{D}_i \left(\mathbf{D}_i'\mathbf{W}_i\mathbf{D}_i\right)^{-} \mathbf{D}_i'\mathbf{W}_i$ is a projection matrix and such a matrix depends only on the column space, therefore matrix $\mathbf{D}_i$ can be replaced by any $\mathbf{\Delta}_i \in \mathbb{R}^{p\times r(D_i)}$ such that $\mathcal{R}(\mathbf{D}_i) = \mathcal{R}(\mathbf{\Delta}_i)$ and

$$\mathbf{D}_i \left(\mathbf{D}_i'\mathbf{W}_i\mathbf{D}_i\right)^{-} \mathbf{D}_i' = \mathbf{\Delta}_i \left(\mathbf{\Delta}_i'\mathbf{W}_i\mathbf{\Delta}_i\right)^{-1} \mathbf{\Delta}_i'.$$

This implies

$$\begin{aligned}\operatorname{Tr}&\left(\mathrm{E}\left[\mathbf{P}_{D_i}^{W_i}\cdots\mathbf{P}_{D_{j+1}}^{W_{j+1}}\mathbf{D}_j\left(\mathbf{D}_j'\mathbf{W}_j\mathbf{D}_j\right)^{-}\mathbf{D}_j'\left(\mathbf{P}_{D_{j+1}}^{W_{j+1}}\right)'\cdots\left(\mathbf{P}_{D_i}^{W_i}\right)'\right]\right) = \\ &= \operatorname{Tr}\left(\mathrm{E}\left[\mathbf{P}_{\Delta_i}^{W_i}\cdots\mathbf{P}_{\Delta_{j+1}}^{W_{j+1}}\mathbf{\Delta}_j\left(\mathbf{\Delta}_j'\mathbf{W}_j\mathbf{\Delta}_j\right)^{-1}\mathbf{\Delta}_j'\left(\mathbf{P}_{\Delta_{j+1}}^{W_{j+1}}\right)'\cdots\left(\mathbf{P}_{\Delta_i}^{W_i}\right)'\right]\right). \quad (14)\end{aligned}$$

Since the column spaces of $\mathbf{Z}_i$'s are nested, we have $\mathcal{R}(\mathbf{\Delta}_k) \subseteq \cdots \subseteq \mathcal{R}(\mathbf{\Delta}_1)$. Therefore for any $1 \le j < k$ there exists matrix $\mathbf{U}_j$ such that $\mathbf{\Delta}_{j+1} = \mathbf{\Delta}_j\mathbf{U}_j$. Let us denote

$$\mathbf{K}_j = \left(\mathbf{\Delta}_j'\mathbf{W}_{j+1}\mathbf{\Delta}_j\right)^{-1/2}\left(\mathbf{\Delta}_j'\mathbf{W}_j\mathbf{\Delta}_j\right)\left(\mathbf{\Delta}_j'\mathbf{W}_{j+1}\mathbf{\Delta}_j\right)^{-1/2}.$$

Lemma 4 implies that $\mathbf{\Delta}_j'\mathbf{W}_j\mathbf{\Delta}_j \sim \mathcal{W}_{r(\mathbf{\Delta}_j)}\left(n - \sum_{l=j}^{k} r(\mathbf{X}_l), \mathbf{\Delta}_j'\mathbf{\Delta}_j\right)$ and it is independent of $\mathbf{\Delta}_j'\mathbf{H}'\mathbf{P}_{X_j}\mathbf{H}\mathbf{\Delta}_j \sim \mathcal{W}_{r(\mathbf{\Delta}_j)}\left(r(\mathbf{X}_j), \mathbf{\Delta}_j'\mathbf{\Delta}_j\right)$ due to the orthogonality of column spaces of $\mathbf{X}_i$'s. Since $\mathbf{W}_{j+1} = \mathbf{W}_j + \mathbf{H}'\mathbf{P}_{X_j}\mathbf{H}$, it follows that $\mathbf{K}_j$ has a multivariate beta distribution of type I (see e.g. Kollo and Rosen [3], Theorem 2.4.8 and Definition 2.4.2, p. 248–249). Then $\mathrm{E}\left[\mathbf{K}_j^{-1}\right] = d_j\mathbf{I}$, where $d_j = \frac{n-\sum_{l=j+1}^{k} r(\mathbf{X}_l)-r(\mathbf{\Delta}_j)-1}{n-\sum_{l=j}^{k} r(\mathbf{X}_l)-r(\mathbf{\Delta}_j)-1}$ (see e.g. von Rosen [7], Lemma 2.3 (vi)). Taking only the middle part of the right hand side of (14) we may write

$$\begin{aligned}\mathbf{P}_{\Delta_{j+1}}^{W_{j+1}}&\mathbf{\Delta}_j\left(\mathbf{\Delta}_j'\mathbf{W}_j\mathbf{\Delta}_j\right)^{-1}\mathbf{\Delta}_j'\left(\mathbf{P}_{\Delta_{j+1}}^{W_{j+1}}\right)' = \\ &= \mathbf{\Delta}_{j+1}\left(\mathbf{\Delta}_{j+1}'\mathbf{W}_{j+1}\mathbf{\Delta}_{j+1}\right)^{-1}\mathbf{U}_j'\left(\mathbf{\Delta}_j'\mathbf{W}_{j+1}\mathbf{\Delta}_j\right)^{1/2}\mathbf{K}_j^{-1}\times \\ &\quad\times\left(\mathbf{\Delta}_j'\mathbf{W}_{j+1}\mathbf{\Delta}_j\right)^{1/2}\mathbf{U}_j\left(\mathbf{\Delta}_{j+1}'\mathbf{W}_{j+1}\mathbf{\Delta}_{j+1}\right)^{-1}\mathbf{\Delta}_{j+1}'.\end{aligned}$$

According to Kollo and Rosen [3], Corollary 2.4.8.1, p. 250, $\mathbf{K}_j$ is independent of $\mathbf{\Delta}_j'\mathbf{W}_{j+1}\mathbf{\Delta}_j$. Then $\mathbf{K}_j$ is also independent of $\mathbf{\Delta}_l'\mathbf{W}_l\mathbf{\Delta}_l$ for $l \ge j+1$, since there exist matrix $\mathbf{U}_{l,j+1}$ such that

$$\mathbf{\Delta}_l'\mathbf{W}_l\mathbf{\Delta}_l = \mathbf{U}_{l,j+1}'\left(\mathbf{\Delta}_j'\mathbf{W}_{j+1}\mathbf{\Delta}_j + \mathbf{\Delta}_j\mathbf{H}'\sum_{s=j+1}^{l-1}\mathbf{P}_{X_s}\mathbf{H}\mathbf{\Delta}_j\right)\mathbf{U}_{l,j+1},$$

and $\mathbf{W}_{j+1}$ and $\mathbf{H}'\sum_{s=j+1}^{l-1}\mathbf{P}_{X_s}\mathbf{H}$ are independent. Thus, denoting

$$\mathbf{C} = \mathbf{P}_{\Delta_i}^{W_i}\cdots\mathbf{P}_{\Delta_{j+2}}^{W_{j+2}}\boldsymbol{\Delta}_{j+1}\left(\boldsymbol{\Delta}_{j+1}'\mathbf{W}_{j+1}\boldsymbol{\Delta}_{j+1}\right)^{-1}\mathbf{U}_j'\left(\boldsymbol{\Delta}_j'\mathbf{W}_{j+1}\boldsymbol{\Delta}_j\right)^{1/2},$$

$\mathbf{C}$ is independent of $\mathbf{K}_j$ and therefore

$$\begin{aligned}
&\mathrm{Tr}\left(\mathrm{E}\left[\mathbf{P}_{\Delta_i}^{W_i}\cdots\mathbf{P}_{\Delta_{j+1}}^{W_{j+1}}\boldsymbol{\Delta}_j\left(\boldsymbol{\Delta}_j'\mathbf{W}_j\boldsymbol{\Delta}_j\right)^{-1}\boldsymbol{\Delta}_j'\left(\mathbf{P}_{\Delta_{j+1}}^{W_{j+1}}\right)'\cdots\left(\mathbf{P}_{\Delta_i}^{W_i}\right)'\right]\right) = \\
&= \mathrm{Tr}\left(\mathrm{E}\left[\mathbf{C}\mathbf{K}_j^{-1}\mathbf{C}'\right]\right) = \mathrm{Tr}\left(\mathrm{E}\left[\mathbf{K}_j^{-1}\mathbf{C}'\mathbf{C}\right]\right) = \mathrm{Tr}\left(\mathrm{E}\left[\mathbf{K}_j^{-1}\right]\mathrm{E}\left[\mathbf{C}'\mathbf{C}\right]\right) = \\
&= d_j\,\mathrm{Tr}\left(\mathrm{E}\left[\mathbf{C}'\mathbf{C}\right]\right) = d_j\,\mathrm{Tr}\left(\mathrm{E}\left[\mathbf{C}\mathbf{C}'\right]\right) = \\
&= d_j\,\mathrm{Tr}\left(\mathrm{E}\left[\mathbf{P}_{\Delta_i}^{W_i}\cdots\mathbf{P}_{\Delta_{j+2}}^{W_{j+2}}\boldsymbol{\Delta}_{j+1}\left(\boldsymbol{\Delta}_{j+1}'\mathbf{W}_{j+1}\boldsymbol{\Delta}_{j+1}\right)^{-1}\boldsymbol{\Delta}_{j+1}'\times\right.\right. \\
&\quad\left.\left.\times\left(\mathbf{P}_{\Delta_{j+2}}^{W_{j+1}}\right)'\cdots\left(\mathbf{P}_{\Delta_i}^{W_i}\right)'\right]\right).
\end{aligned}$$

We can repeatedly apply the same technique, and at the end we obtain

$$\begin{aligned}
&\mathrm{Tr}\left(\mathrm{E}\left[\mathbf{P}_{\Delta_i}^{W_i}\cdots\mathbf{P}_{\Delta_{j+1}}^{W_{j+1}}\boldsymbol{\Delta}_j\left(\boldsymbol{\Delta}_j'\mathbf{W}_j\boldsymbol{\Delta}_j\right)^{-1}\boldsymbol{\Delta}_j'\left(\mathbf{P}_{\Delta_{j+1}}^{W_{j+1}}\right)'\cdots\left(\mathbf{P}_{\Delta_i}^{W_i}\right)'\right]\right) = \\
&\qquad = d_jd_{j+1}\ldots d_{i-1}\,\mathrm{Tr}\left(\mathrm{E}\left[\boldsymbol{\Delta}_i\left(\boldsymbol{\Delta}_i'\mathbf{W}_i\boldsymbol{\Delta}_i\right)^{-1}\boldsymbol{\Delta}_i'\right]\right) = \\
&\qquad = d_jd_{j+1}\ldots d_{i-1}\frac{r\left(\boldsymbol{\Delta}_i\right)}{n-\sum_{l=i}^{k}r\left(\mathbf{X}_l\right)-r\left(\boldsymbol{\Delta}_i\right)-1}.
\end{aligned}$$

where the last expectation is due to Theorem 2.4.14 (iii) from Kollo and Rosen [3], p. 257. The result follows since $r\left(\boldsymbol{\Delta}_i\right) = r\left(\mathbf{D}_i\right)$ for all i. □

Theorem 6. $\widehat{\mathbf{B}}_i$ *is unbiased estimator of* $\mathbf{B}_i$, $i = 1,\ldots,k$, *i.e.* $\mathrm{E}[\widehat{\mathbf{B}}_i] = \mathbf{B}_i$. *The variance of the estimator is*

$$\begin{aligned}
\mathrm{Var}\left[\widehat{\mathbf{B}}_i\right] = &\left((1+c_i)\left(\mathbf{Z}_i'\boldsymbol{\Sigma}^{-1}\mathbf{Z}_i\right)^{-1} + \sum_{j=1}^{i-1}(d_j-1)d_{j+1}\ldots d_{i-1}c_i\times\right. \\
&\left.\times\left(\mathbf{Z}_i'\mathbf{Z}_i\right)^{-1}\mathbf{Z}_i'\mathbf{Z}_j\left(\mathbf{Z}_j'\boldsymbol{\Sigma}^{-1}\mathbf{Z}_j\right)^{-1}\mathbf{Z}_j'\mathbf{Z}_i\left(\mathbf{Z}_i'\mathbf{Z}_i\right)^{-1}\right)\otimes\left(\mathbf{X}_i'\mathbf{X}_i\right)^{-1},
\end{aligned}$$

where $c_i = \frac{p-r(\mathbf{Z}_i)}{n-\sum_{l=i}^{k}r(\mathbf{X}_l)-p+r(\mathbf{Z}_i)-1}$ *and* $d_j = \frac{n-\sum_{l=j+1}^{k}r(\mathbf{X}_l)-p+r(\mathbf{Z}_j)-1}{n-\sum_{l=j}^{k}r(\mathbf{X}_l)-p+r(\mathbf{Z}_j)-1}$ *for* $1 \leq j < i$.

Proof. First observe that $\mathbf{S}_1,\ldots,\mathbf{S}_k$ could be written by means of residuals, i.e.

$$\begin{aligned}
\mathbf{S}_1 &= \boldsymbol{\mathcal{E}}'\mathbf{Q}_X\boldsymbol{\mathcal{E}}, \\
\mathbf{S}_i &= \mathbf{S}_{i-1} + \mathbf{Q}_{Z_{i-1}}^{S_{i-1}^{-1}}\boldsymbol{\mathcal{E}}'\mathbf{P}_{X_{i-1}}\boldsymbol{\mathcal{E}}\left(\mathbf{Q}_{Z_{i-1}}^{S_{i-1}^{-1}}\right)', \quad i = 2,\ldots,k.
\end{aligned}$$

Then it is easy to see that

$$\widehat{\mathbf{B}}_i - \mathbf{B}_i = \left(\mathbf{X}_i'\mathbf{X}_i\right)^{-1}\mathbf{X}_i'\boldsymbol{\mathcal{E}}\mathbf{S}_i^{-1}\mathbf{Z}_i\left(\mathbf{Z}_i'\mathbf{S}_i^{-1}\mathbf{Z}_i\right)^{-1}. \tag{15}$$

From Kollo and Rosen [3], Theorem 2.2.4. (iv), p. 196, it follows that $\mathbf{X}_i'\boldsymbol{\mathcal{E}}$ and $\mathbf{S}_i$ are independent, therefore

$$\mathrm{E}\left[\widehat{\mathbf{B}}_i - \mathbf{B}_i\right] = \mathrm{E}\left[\left(\mathbf{X}_i'\mathbf{X}_i\right)^{-1}\mathbf{X}_i'\boldsymbol{\mathcal{E}}\right]\cdot\mathrm{E}\left[\mathbf{S}_i^{-1}\mathbf{Z}_i\left(\mathbf{Z}_i'\mathbf{S}_i^{-1}\mathbf{Z}_i\right)^{-1}\right] = \mathbf{0}.$$

Now, let us derive the variance of $\widehat{\mathbf{B}}_i$. Using (15), Lemma 3 and the properties of vec operator we have

$$\begin{aligned}
\mathrm{Var}\left[\widehat{\mathbf{B}}_i\right] &= \mathrm{E}\left[\left(\mathbf{Z}_i'\mathbf{S}_i^{-1}\mathbf{Z}_i\right)^{-1}\mathbf{Z}_i'\mathbf{S}_i^{-1}\boldsymbol{\Sigma}\mathbf{S}_i^{-1}\mathbf{Z}_i\left(\mathbf{Z}_i'\mathbf{S}_i^{-1}\mathbf{Z}_i\right)^{-1}\right]\otimes\left(\mathbf{X}_i'\mathbf{X}_i\right)^{-1} = \\
&= \left(\mathbf{Z}_i'\mathbf{Z}_i\right)^{-1}\mathbf{Z}_i'\mathrm{E}\left[\mathbf{P}_{Z_i}^{S_i^{-1}}\boldsymbol{\Sigma}\left(\mathbf{P}_{Z_i}^{S_i^{-1}}\right)'\right]\mathbf{Z}_i\left(\mathbf{Z}_i'\mathbf{Z}_i\right)^{-1}\otimes\left(\mathbf{X}_i'\mathbf{X}_i\right)^{-1} = \\
&= \left(\mathbf{Z}_i'\mathbf{Z}_i\right)^{-1}\mathbf{Z}_i'\mathrm{E}\left[\left(\mathbf{Q}_{Q_{Z_i}}^{S_i}\right)'\boldsymbol{\Sigma}\mathbf{Q}_{Q_{Z_i}}^{S_i}\right]\mathbf{Z}_i\left(\mathbf{Z}_i'\mathbf{Z}_i\right)^{-1}\otimes\left(\mathbf{X}_i'\mathbf{X}_i\right)^{-1} = \\
&\overset{df}{=} \left(\mathbf{Z}_i'\mathbf{Z}_i\right)^{-1}\mathbf{Z}_i'\mathbf{F}_i\mathbf{Z}_i\left(\mathbf{Z}_i'\mathbf{Z}_i\right)^{-1}\otimes\left(\mathbf{X}_i'\mathbf{X}_i\right)^{-1}.
\end{aligned} \tag{16}$$

First let $i = 1$. Since $\mathbf{S}_1 \sim \mathcal{W}_p\left(n - \sum_{j=1}^k r(\mathbf{X}_j), \boldsymbol{\Sigma}\right)$, via calculations similar to Kollo and Rosen [3][(4.2.13)-(4.2.23), p. 412-413] we obtain

$$\begin{aligned}
\mathrm{Var}\left[\widehat{\mathbf{B}}_1\right] &= \mathrm{E}\left[\left(\mathbf{Z}_1'\mathbf{S}_1^{-1}\mathbf{Z}_1\right)^{-1}\mathbf{Z}_1'\mathbf{S}_1^{-1}\boldsymbol{\Sigma}\mathbf{S}_1^{-1}\mathbf{Z}_1\left(\mathbf{Z}_1'\mathbf{S}_1^{-1}\mathbf{Z}_1\right)^{-1}\right]\otimes\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1} = \\
&= \frac{n - \sum_{j=1}^k r(\mathbf{X}_j) - 1}{n - \sum_{j=1}^k r(\mathbf{X}_j) - p + r(\mathbf{Z}_1) - 1}\left(\mathbf{Z}_1'\boldsymbol{\Sigma}^{-1}\mathbf{Z}_1\right)^{-1}\otimes\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}.
\end{aligned}$$

Now let $i > 1$. Let us define

$$\mathbf{V}_1 = \boldsymbol{\Sigma}^{-1/2}\mathbf{S}_1\boldsymbol{\Sigma}^{-1/2},$$
$$\text{and } \mathbf{V}_i = \mathbf{V}_{i-1} + \left(\mathbf{P}_{D_{i-1}}^{V_{i-1}}\right)'\mathbf{N}_{i-1}\mathbf{P}_{D_{i-1}}^{V_{i-1}}, \quad \text{for } i = 2,\dots,k,$$

where $\mathbf{N}_i = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mathcal{E}}'\mathbf{P}_{X_i}\boldsymbol{\mathcal{E}}\boldsymbol{\Sigma}^{-1/2}$ $\forall i$ as in Lemma 4, and $\mathbf{D}_i = \boldsymbol{\Sigma}^{1/2}\mathbf{Q}_{Z_i}$ $\forall i$ as in Lemma 5. Then, we can re-write the expression for $\mathbf{F}_i$ as

$$\begin{aligned}
\mathbf{F}_i &= \boldsymbol{\Sigma} - \mathrm{E}\left[\left(\mathbf{P}_{Q_{Z_i}}^{S_i}\right)'\boldsymbol{\Sigma}\right] - \mathrm{E}\left[\boldsymbol{\Sigma}\mathbf{P}_{Q_{Z_i}}^{S_i}\right] + \mathrm{E}\left[\left(\mathbf{P}_{Q_{Z_i}}^{S_i}\right)'\boldsymbol{\Sigma}\mathbf{P}_{Q_{Z_i}}^{S_i}\right] = \\
&= \boldsymbol{\Sigma} - \boldsymbol{\Sigma}^{1/2}\left(\mathrm{E}\left[\mathbf{P}_{D_i}^{V_i}\right]' + \mathrm{E}\left[\mathbf{P}_{D_i}^{V_i}\right] - \mathrm{E}\left[\left(\mathbf{P}_{D_i}^{V_i}\right)'\mathbf{P}_{D_i}^{V_i}\right]\right)\boldsymbol{\Sigma}^{1/2}
\end{aligned}$$

Since $\mathcal{R}(\mathbf{D}_k) \subseteq \cdots \subseteq \mathcal{R}(\mathbf{D}_1)$, for any regular matrix $\mathbf{A}$ we obtain

$$\begin{aligned}
\mathbf{D}_i'\mathbf{V}_i\mathbf{D}_i &= \mathbf{D}_i'\mathbf{W}_i\mathbf{D}_i, \\
\mathbf{D}_i'\mathbf{V}_i\mathbf{D}_{i-1} &= \mathbf{D}_i'\mathbf{W}_i\mathbf{D}_{i-1}, \\
\mathbf{P}_{D_i}^{A}\mathbf{D}_j &= \mathbf{D}_j, \quad \forall j > i.
\end{aligned} \tag{17}$$

Using these relations we can write

$$\begin{aligned}
\mathbf{P}_{D_i}^{V_i} &= \mathbf{D}_i \left(\mathbf{D}_i'\mathbf{V}_i\mathbf{D}_i\right)^{-} \mathbf{D}_i'\mathbf{V}_i = \mathbf{D}_i \left(\mathbf{D}_i'\mathbf{W}_i\mathbf{D}_i\right)^{-} \mathbf{D}_i' \times \\
&\quad \times \Big(\mathbf{V}_{i-1} + \mathbf{N}_{i-1}\mathbf{D}_{i-1}\left(\mathbf{D}_{i-1}'\mathbf{W}_{i-1}\mathbf{D}_{i-1}\right)^{-}\mathbf{D}_{i-1}'\mathbf{V}_{i-1}\Big) = \\
&= \mathbf{D}_i \left(\mathbf{D}_i'\mathbf{W}_i\mathbf{D}_i\right)^{-} \mathbf{D}_i' \times \\
&\quad \times \Big(\mathbf{I} + \mathbf{N}_{i-1}\mathbf{D}_{i-1}\left(\mathbf{D}_{i-1}'\mathbf{W}_{i-1}\mathbf{D}_{i-1}\right)^{-}\mathbf{D}_{i-1}'\Big)\mathbf{V}_{i-1} = \\
&= \mathbf{D}_i \left(\mathbf{D}_i'\mathbf{W}_i\mathbf{D}_i\right)^{-} \mathbf{D}_i' \times \\
&\quad \times \Big(\mathbf{I} + \mathbf{N}_{i-1}\mathbf{D}_{i-1}\left(\mathbf{D}_{i-1}'\mathbf{W}_{i-1}\mathbf{D}_{i-1}\right)^{-}\mathbf{D}_{i-1}'\Big)\Big(\mathbf{P}_{D_{i-1}}^{V_{i-1}}\Big)'\mathbf{V}_{i-1} = \\
&= \mathbf{D}_i \left(\mathbf{D}_i'\mathbf{W}_i\mathbf{D}_i\right)^{-} \mathbf{D}_i' \times \\
&\quad \times \Big(\mathbf{I} + \mathbf{N}_{i-1}\mathbf{D}_{i-1}\left(\mathbf{D}_{i-1}'\mathbf{W}_{i-1}\mathbf{D}_{i-1}\right)^{-}\mathbf{D}_{i-1}'\Big)\mathbf{V}_{i-1}\mathbf{P}_{D_{i-1}}^{V_{i-1}} = \\
&= \mathbf{D}_i \left(\mathbf{D}_i'\mathbf{W}_i\mathbf{D}_i\right)^{-} \mathbf{D}_i'\mathbf{V}_i\mathbf{D}_{i-1}\left(\mathbf{D}_{i-1}'\mathbf{V}_{i-1}\mathbf{D}_{i-1}\right)^{-}\mathbf{D}_{i-1}'\mathbf{V}_{i-1} = \\
&= \mathbf{D}_i \left(\mathbf{D}_i'\mathbf{W}_i\mathbf{D}_i\right)^{-} \mathbf{D}_i'\mathbf{W}_i\mathbf{D}_{i-1}\left(\mathbf{D}_{i-1}'\mathbf{V}_{i-1}\mathbf{D}_{i-1}\right)^{-}\mathbf{D}_{i-1}'\mathbf{V}_{i-1} = \\
&= \mathbf{P}_{D_i}^{W_i}\mathbf{P}_{D_{i-1}}^{V_{i-1}}. \qquad (18)
\end{aligned}$$

Then, using (13) we get for any $2 \leq j \leq i$

$$\begin{aligned}
&\mathbf{P}_{D_i}^{V_i}\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right) = \mathbf{P}_{D_i}^{W_i}\mathbf{P}_{D_{i-1}}^{W_{i-1}}\cdots\mathbf{P}_{D_{j+1}}^{W_{j+1}}\mathbf{P}_{D_j}^{V_j}\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right) = \\
&\quad = \mathbf{P}_{D_i}^{W_i}\mathbf{P}_{D_{i-1}}^{W_{i-1}}\cdots\mathbf{P}_{D_{j+1}}^{W_{j+1}}\mathbf{P}_{D_j}^{W_j}\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right) = \prod_{l=i}^{j}\mathbf{P}_{D_l}^{W_l}\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right) = \\
&\quad = \prod_{l=i}^{j}\mathbf{D}_l(\mathbf{D}_l'\mathbf{H}'\mathbf{T}_l\mathbf{H}\mathbf{D}_l)^{-}\mathbf{D}_l'\mathbf{H}'\mathbf{T}_l\mathbf{H}\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right).
\end{aligned}$$

Since $\mathbf{H}\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right)$ is independent of $\mathbf{H}\mathbf{D}_l$ for any $l \geq j$, we have

$$\begin{aligned}
&\mathrm{E}\left[\mathbf{P}_{D_i}^{V_i}\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right)\right] = \\
&\qquad = \mathrm{E}\left[\prod_{l=i}^{j}\mathbf{D}_l(\mathbf{D}_l'\mathbf{H}'\mathbf{T}_l\mathbf{H}\mathbf{D}_l)^{-}\mathbf{D}_l'\mathbf{H}'\mathbf{T}_l\right]\underbrace{\mathrm{E}\left[\mathbf{H}\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right)\right]}_{=0} = 0. \qquad (19)
\end{aligned}$$

Similarly since $\mathbf{V}_1 = \mathbf{W}_1$ and $\mathbf{HQ}_{D_1}$ is independent of $\mathbf{HD}_i$ for all $i \geq 1$ we obtain

$$\begin{aligned}
\mathrm{E}\left[\mathbf{P}_{D_i}^{V_i}\mathbf{Q}_{D_1}\right] &= \mathrm{E}\left[\prod_{j=i}^{1}\mathbf{P}_{D_j}^{W_j}\mathbf{Q}_{D_1}\right] = \\
&= \mathrm{E}\left[\prod_{j=i}^{1}\mathbf{D}_j(\mathbf{D}_j'\mathbf{H}'\mathbf{T}_j\mathbf{HD}_j)^{-}\mathbf{D}_j'\mathbf{H}'\mathbf{T}_j\mathbf{HQ}_{D_1}\right] = \\
&= \mathrm{E}\left[\prod_{j=i}^{1}\mathbf{D}_j(\mathbf{D}_j'\mathbf{H}'\mathbf{T}_j\mathbf{HD}_j)^{-}\mathbf{D}_j'\mathbf{H}'\mathbf{T}_j\right]\underbrace{\mathrm{E}\left[\mathbf{HQ}_{D_1}\right]}_{=0} = 0.
\end{aligned} \tag{20}$$

It is easy to see that $\mathbf{P}_{D_i}^{V_i}\mathbf{P}_{D_i} = \mathbf{P}_{D_i}$, therefore

$$\mathrm{E}\left[\mathbf{P}_{D_i}^{V_i}\mathbf{P}_{D_i}\right] = \mathbf{P}_{D_i}. \tag{21}$$

Observe that $\mathbf{I} = \mathbf{P}_{D_i} + \sum_{j=2}^{i}\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right) + \mathbf{Q}_{D_1}$. Therefore, using (19), (20) and (21) we obtain

$$\begin{aligned}
\mathrm{E}\left[\mathbf{P}_{D_i}^{V_i}\right] &= \mathrm{E}\left[\mathbf{P}_{D_i}^{V_i}\mathbf{P}_{D_i}\right] + \sum_{j=2}^{i}\mathrm{E}\left[\mathbf{P}_{D_i}^{V_i}\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right)\right] + \mathrm{E}\left[\mathbf{P}_{D_i}^{V_i}\mathbf{Q}_{D_1}\right] = \\
&= \mathbf{P}_{D_i}.
\end{aligned}$$

Let us now look at $\left(\mathbf{P}_{D_i}^{V_i}\right)'\mathbf{P}_{D_i}^{V_i}$. Since $\mathbf{P}_{D_i}^{V_i}\mathbf{P}_{D_i} = \mathbf{P}_{D_i}$ and $\mathbf{P}_{D_i}\mathbf{P}_{D_i}^{V_i} = \mathbf{P}_{D_i}^{V_i}$, it holds

$$\mathrm{E}\left[\mathbf{P}_{D_i}\left(\mathbf{P}_{D_i}^{V_i}\right)'\mathbf{P}_{D_i}^{V_i}\mathbf{Q}_{D_1}\right] = \mathrm{E}\left[\mathbf{P}_{D_i}^{V_i}\mathbf{Q}_{D_1}\right] = 0.$$

Using similar principle as in (19) and (20), we obtain

$$\begin{aligned}
\mathrm{E}\left[\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right)\left(\mathbf{P}_{D_i}^{V_i}\right)'\mathbf{P}_{D_i}^{V_i}\mathbf{Q}_{D_1}\right] &= 0, \quad j = 2,\ldots,i, \\
\mathrm{E}\left[\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right)\left(\mathbf{P}_{D_i}^{V_i}\right)'\mathbf{P}_{D_i}^{V_i}\mathbf{P}_{D_i}\right] &= 0, \quad j = 2,\ldots,i, \\
\mathrm{E}\left[\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right)\left(\mathbf{P}_{D_i}^{V_i}\right)'\mathbf{P}_{D_i}^{V_i}\left(\mathbf{P}_{D_{k-1}} - \mathbf{P}_{D_k}\right)\right] &= 0, \quad k,j = 2,\ldots,i, k \neq j.
\end{aligned}$$

Therefore we can write

$$\begin{aligned}
\mathrm{E}\left[\left(\mathbf{P}_{D_i}^{V_i}\right)'\mathbf{P}_{D_i}^{V_i}\right] = \mathrm{E}&\left[\mathbf{P}_{D_i}\left(\mathbf{P}_{D_i}^{V_i}\right)'\mathbf{P}_{D_i}^{V_i}\mathbf{P}_{D_i}\right] + \\
&+ \sum_{j=2}^{i}\mathrm{E}\left[\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right)\left(\mathbf{P}_{D_i}^{V_i}\right)'\mathbf{P}_{D_i}^{V_i}\left(\mathbf{P}_{D_{j-1}} - \mathbf{P}_{D_j}\right)\right] + \\
&+ \mathrm{E}\left[\mathbf{Q}_{D_1}\left(\mathbf{P}_{D_i}^{V_i}\right)'\mathbf{P}_{D_i}^{V_i}\mathbf{Q}_{D_1}\right].
\end{aligned} \tag{22}$$

Trivially, $\mathrm{E}\left[\mathbf{P}_{D_i}\left(\mathbf{P}^{V_i}_{D_i}\right)'\mathbf{P}^{V_i}_{D_i}\mathbf{P}_{D_i}\right]=\mathbf{P}_{D_i}$. Using (18), we express the last term as

$$\begin{aligned}\mathbf{Q}_{D_1}\left(\mathbf{P}^{V_i}_{D_i}\right)'\mathbf{P}^{V_i}_{D_i}\mathbf{Q}_{D_1} &= \mathbf{Q}_{D_1}\prod_{j=1}^{i}\mathbf{H}'\mathbf{T}_j\mathbf{H}\mathbf{D}_j(\mathbf{D}_j'\mathbf{H}'\mathbf{T}_j\mathbf{H}\mathbf{D}_j)^{-}\mathbf{D}_j'\times\\ &\times\prod_{j=i}^{1}\mathbf{D}_j(\mathbf{D}_j'\mathbf{H}'\mathbf{T}_j\mathbf{H}\mathbf{D}_j)^{-}\mathbf{D}_j'\mathbf{H}'\mathbf{T}_j\mathbf{H}\mathbf{Q}_{D_1}.\end{aligned}$$

Since $\mathbf{H}\mathbf{Q}_{D_1}$ is independent of $\mathbf{H}\mathbf{D}_j$, $j=1,\ldots,k$, and $\mathbf{T}_1=\mathbf{Q}_X$, using Lemma 5 and Theorem 2.2.9 (i) of Kollo and Rosen [3], we may write

$$\begin{aligned}&\mathbf{Q}_{D_1}\mathrm{E}\left[\left(\mathbf{P}^{V_i}_{D_i}\right)'\mathbf{P}^{V_i}_{D_i}\right]\mathbf{Q}_{D_1}=\mathrm{E}\Bigg[\mathrm{Tr}\Bigg(\mathbf{D}_1(\mathbf{D}_1'\mathbf{H}'\mathbf{T}_1\mathbf{H}\mathbf{D}_1)^{-}\mathbf{D}_1'\mathbf{H}'\mathbf{T}_1\mathbf{T}_1\mathbf{H}\times\\ &\quad\times\mathbf{D}_1(\mathbf{D}_1'\mathbf{H}'\mathbf{T}_1\mathbf{H}\mathbf{D}_1)^{-}\mathbf{D}_1'\prod_{j=2}^{i}\mathbf{H}'\mathbf{T}_j\mathbf{H}\mathbf{D}_j(\mathbf{D}_j'\mathbf{H}'\mathbf{T}_j\mathbf{H}\mathbf{D}_j)^{-}\mathbf{D}_j'\times\\ &\quad\times\prod_{j=i}^{2}\mathbf{D}_j(\mathbf{D}_j'\mathbf{H}'\mathbf{T}_j\mathbf{H}\mathbf{D}_j)^{-}\mathbf{D}_j'\mathbf{H}'\mathbf{T}_j\mathbf{H}\Bigg)\Bigg]\mathbf{Q}_{D_1}=\\ &=\mathrm{E}\Bigg[\mathrm{Tr}\Bigg(\mathbf{D}_1(\mathbf{D}_1'\mathbf{W}_1\mathbf{D}_1)^{-}\mathbf{D}_1'\mathbf{W}_1\mathbf{D}_1(\mathbf{D}_1'\mathbf{W}_1\mathbf{D}_1)^{-}\mathbf{D}_1'\times\\ &\quad\times\prod_{j=2}^{i}\left(\mathbf{P}^{W_j}_{D_j}\right)'\prod_{j=i}^{2}\mathbf{P}^{W_j}_{D_j}\Bigg)\Bigg]\mathbf{Q}_{D_1}=\\ &=\mathrm{E}\left[\mathrm{Tr}\left(\prod_{j=i}^{2}\mathbf{P}^{W_j}_{D_j}\mathbf{D}_1(\mathbf{D}_1'\mathbf{W}_1\mathbf{D}_1)^{-}\mathbf{D}_1'\prod_{j=2}^{i}\left(\mathbf{P}^{W_j}_{D_j}\right)'\right)\right]\mathbf{Q}_{D_1}=\\ &=d_1\ldots d_{i-1}c_i\mathbf{Q}_{D_1}.\end{aligned}\tag{23}$$

It can be obtained in a similar way for $j=2,\ldots,i$

$$\begin{aligned}&\left(\mathbf{P}_{D_{j-1}}-\mathbf{P}_{D_j}\right)\mathrm{E}\left[\left(\mathbf{P}^{V_i}_{D_i}\right)'\mathbf{P}^{V_i}_{D_i}\right]\left(\mathbf{P}_{D_{j-1}}-\mathbf{P}_{D_j}\right)=\\ &=\mathrm{E}\left[\mathrm{Tr}\left(\mathbf{P}^{W_i}_{D_i}\cdots\mathbf{P}^{W_{j+1}}_{D_{j+1}}\mathbf{D}_j\left(\mathbf{D}_j'\mathbf{W}_j\mathbf{D}_j\right)^{-}\mathbf{D}_j'\left(\mathbf{P}^{W_{j+1}}_{D_{j+1}}\right)'\cdots\left(\mathbf{P}^{W_i}_{D_i}\right)'\right)\right]\times\\ &\quad\times\left(\mathbf{P}_{D_{j-1}}-\mathbf{P}_{D_j}\right)=d_jd_{j+1}\ldots d_{i-1}c_i\left(\mathbf{P}_{D_{j-1}}-\mathbf{P}_{D_j}\right).\end{aligned}\tag{24}$$

Combining (22), (23) and (24), we have

$$
\begin{aligned}
&\mathrm{E}\left[\left(\mathbf{P}_{D_i}^{V_i}\right)'\mathbf{P}_{D_i}^{V_i}\right]=\\
&=\mathbf{P}_{D_i}+\sum_{j=2}^{i} d_j d_{j+1}\dots d_{i-1}c_i\left(\mathbf{P}_{D_{j-1}}-\mathbf{P}_{D_j}\right)+d_1\dots d_{i-1}c_i\mathbf{Q}_{D_1}=\\
&=\mathbf{P}_{D_i}+\sum_{j=2}^{i} d_j d_{j+1}\dots d_{i-1}c_i\left(\mathbf{Q}_{D_j}-\mathbf{Q}_{D_{j-1}}\right)+d_1\dots d_{i-1}c_i\mathbf{Q}_{D_1}=\\
&=\mathbf{P}_{D_i}+c_i\mathbf{Q}_{D_i}+\sum_{j=1}^{i-1}\left(d_j-1\right)d_{j+1}\dots d_{i-1}c_i\mathbf{Q}_{D_j}. \qquad (25)
\end{aligned}
$$

Finally, due to (22) and (25) and since $\mathbf{Q}_{D_i}=\mathbf{P}_{\Sigma^{-1/2}Z_i}$ and $r\left(\mathbf{D}_i\right)=p-r\left(\mathbf{Z}_i\right)$ for any $i=1,\dots,k$

$$
\begin{aligned}
\mathbf{F}_i&=\mathbf{\Sigma}^{1/2}\left[\mathbf{Q}_{D_i}+c_i\mathbf{Q}_{D_i}+\sum_{j=1}^{i-1}\left(d_j-1\right)d_{j+1}\dots d_{i-1}c_i\mathbf{Q}_{D_j}\right]\mathbf{\Sigma}^{1/2}=\\
&=\mathbf{\Sigma}^{1/2}\left[\left(1+c_i\right)\mathbf{P}_{\Sigma^{-1/2}Z_i}+\sum_{j=1}^{i-1}\left(d_j-1\right)d_{j+1}\dots d_{i-1}c_i\mathbf{P}_{\Sigma^{-1/2}Z_j}\right]\mathbf{\Sigma}^{1/2}=\\
&=\left(1+c_i\right)\mathbf{Z}_i\left(\mathbf{Z}_i'\mathbf{\Sigma}^{-1}\mathbf{Z}_i\right)^{-1}\mathbf{Z}_i'+\\
&\quad+\sum_{j=1}^{i-1}\left(d_j-1\right)d_{j+1}\dots d_{i-1}c_i\mathbf{Z}_j\left(\mathbf{Z}_j'\mathbf{\Sigma}^{-1}\mathbf{Z}_j\right)^{-1}\mathbf{Z}_j', \qquad (26)
\end{aligned}
$$

where in d_j and c_i are $r\left(\mathbf{D}_l\right)$ replaced by $p-r\left(\mathbf{Z}_l\right)$. Finally, from (16) and (26) the variance of $\widehat{\mathbf{B}}_i$, $i=2,\dots,k$, is obtained. □

Observe that $c_i>0$ and $d_i>1$ $\forall i$.

The following theorem presents $\mathrm{E}\left[\widehat{\mathbf{\Sigma}}\right]$ in Model III. Expectation of corresponding estimators for Model I and Model II can be found in Kollo and Rosen [3], Theorem 4.2.9, p. 435, and Filipiak and Rosen [1], respectively.

Theorem 7. *Let estimator $\widehat{\mathbf{\Sigma}}$ be given in Theorem 1. Then*

$$
\mathrm{E}\left[\widehat{\mathbf{\Sigma}}\right]=\left(1+\sum_{i=1}^{k}\frac{r\left(\mathbf{X}_i\right)}{n}\left[(c_i-1)\mathbf{P}_{Z_i}^{\Sigma^{-1}}+\sum_{j=1}^{i-1}(d_j-1)d_{j+1}\dots d_{i-1}c_i\mathbf{P}_{Z_j}^{\Sigma^{-1}}\right]\right)\mathbf{\Sigma}, \quad (27)
$$

where c_i and d_j for all i and $j<i$ are given in Theorem 6.

Proof. Observe that we can write

$$\begin{aligned} n\widehat{\mathbf{\Sigma}} &= \mathbf{S}_1 + \sum_{i=1}^{k} \mathbf{Q}_{Z_i}^{S_i^{-1}} \mathbf{Y}'\mathbf{P}_{X_i}\mathbf{Y} \left(\mathbf{Q}_{Z_i}^{S_i^{-1}}\right)' = \\ &= \mathbf{S}_1 + \sum_{i=1}^{k} \left(\mathbf{P}_{Q_{Z_i}}^{S_i}\right)' \boldsymbol{\mathcal{E}}'\mathbf{P}_{X_i}\boldsymbol{\mathcal{E}}\mathbf{P}_{Q_{Z_i}}^{S_i}, \end{aligned}$$

where $\mathbf{S}_1 = \boldsymbol{\mathcal{E}}'\mathbf{Q}_X\boldsymbol{\mathcal{E}}$. Since $\mathbf{P}_{X_i}\boldsymbol{\mathcal{E}}$ and $\mathbf{S}_i$ are independent, utilizing the first moment of a Wishart matrix we obtain

$$\begin{aligned} \mathrm{E}\left[n\widehat{\mathbf{\Sigma}}\right] &= \mathrm{E}\left[\mathbf{S}_1\right] + \sum_{i=1}^{k} \mathrm{E}\left[\left(\mathbf{P}_{Q_{Z_i}}^{S_i}\right)' \mathrm{E}\left[\boldsymbol{\mathcal{E}}'\mathbf{P}_{X_i}\boldsymbol{\mathcal{E}}\right] \mathbf{P}_{Q_{Z_i}}^{S_i}\right] = \\ &= \left(n - \sum_{i=1}^{k} r\left(\mathbf{X}_i\right)\right)\mathbf{\Sigma} + \sum_{i=1}^{k} r\left(\mathbf{X}_i\right) \mathrm{E}\left[\left(\mathbf{P}_{Q_{Z_i}}^{S_i}\right)' \mathbf{\Sigma}\mathbf{P}_{Q_{Z_i}}^{S_i}\right] = \\ &= \left(n - \sum_{i=1}^{k} r\left(\mathbf{X}_i\right)\right)\mathbf{\Sigma} + \sum_{i=1}^{k} r\left(\mathbf{X}_i\right) \mathbf{\Sigma}^{1/2}\mathrm{E}\left[\left(\mathbf{P}_{D_i}^{V_i}\right)' \mathbf{\Sigma}\mathbf{P}_{D_i}^{V_i}\right] \mathbf{\Sigma}^{1/2}. \end{aligned}$$

The result now follows from relation (25). □

4 Conclusions

To conclude, the authors would like to stress that orthogonal decompositions are very useful in all areas of statistics and mathematics. If modelling of a real world phenomenon allows creating an orthogonal structure, the resulting model can be expected to have much nicer properties than all others. In our case it is ECGM Model III, where we assume orthogonal structure in the ANOVA part (which is always the case in simple one-way model). Under such assumptions, we have derived explicit form of MLEs of both the first- and the second-order parameters, and also their basic moments, in general k profile setting. This is very useful especially for the comparison of small sample properties of MLEs with other competitive estimators. As a drawback of MLE estimation in this setting can be viewed possible big bias of $\widehat{\Sigma}$.

Acknowledgement

The research was supported by grants VEGA MŠ SR VEGA 1/0832/12 and VEGA MŠ SR 1/0344/14.

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