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Strongly unbounded and strongly dominating sets generalized

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Abstract

We generalize the notions of unbounded and strongly dominating subset of the Baire space. We compare the corresponding ideals and tree ideals, in particular we find a condition which implies that some of those ideals are distinct. We also introduce $\mathrm{DU}_{\mathcal{I}}$-property, where $\mathcal{I}$ is an ideal on cardinal κ, to cover these two generalized notions at once. We use two player game defined in Kechris (Trans. Amer. Math. Soc. **229** (1977), 191–207) to show that every λ-Suslin set with $\mathrm{DU}_{\mathcal{I}}$-property contains a perfect subset with $\mathrm{DU}_{\mathcal{I}}$-property, provided that λ is sufficiently small.

Keywords: strongly dominating set, Laver perfect set, strongly unbounded set, superperfect set, ideal, κ-Suslin set

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1 Preliminaries

We use the standard set-theoretic notation, e.g. see [4] or [8]. Therefore ω is the set of all natural numbers, and hence, if X is an arbitrary set, ${}^{\omega}X$ denotes the set of all infinite sequences with the domain ω and with values being elements of the set X. Similarly, ${}^{<\omega}X$ denotes the set of all finite sequences t with $\mathrm{dom}(t) \in \omega$ and $\mathrm{rng}(t) \subseteq X$. Occasionally, we write ${}^{\leq\omega}X$ instead of ${}^{<\omega}X \cup {}^{\omega}X$.

Next, for each sequence t let $|t| = \mathrm{dom}(t) \in \omega + 1$ be the *length* of t. Here $\omega + 1$ is the ordinal successor of ω, the set $\omega \cup \{\omega\}$. Besides, if $n \leq |t|$, then $t{\restriction}n$ denotes the *restriction* of t to the domain n. The *concatenation* of two sequences $s \in {}^{<\omega}X$ and $t \in {}^{\leq}X$ is denoted as $s^\frown t$.

To each finite sequence s assign the set $[s] \subseteq {}^\omega X$ so that $[s] = \{x \in {}^\omega X : x \supseteq s\}$. The family $\{[s] : s \in {}^{<\omega}X\}$ is a base for a topology on ${}^\omega X$. Since it is the only topology considered here on the set ${}^\omega X$, we do not write it explicitly, but from now on we understand that the set ${}^\omega X$ is endowed with this topology. The space ${}^\omega X$ is a complete metric space and, if the set X is countable, it is also separable, thus Polish.

There is a simple and elegant characterization of the closed subsets of ${}^\omega X$. Recall that a set $p \subseteq {}^{<\omega}X$ is a *tree*, if it is closed under the taking initial subsequences, i.e. if $s \in p$ then $s{\restriction}n \in p$, for every $n < |s|$. Denote the set of all infinite branches of the tree p as

$$[p] = \{x \in {}^\omega X : (\forall n \in \omega)\, x{\restriction}n \in p\}.$$

The set $[p]$ is always closed, what is more, for every closed set $A \subseteq {}^\omega X$ there is a tree $q \subseteq {}^{<\omega}X$ such that $A = [q]$. It follows that the set $[p]$ is *perfect* (closed with no isolated points), whenever for every $s \in p$ there is $t \in p$ such that $t \supseteq s$ and $|\{x \in X : t^\frown\langle x\rangle \in p\}| \geq 2$. Such node t is called a *splitting node*.

If $A \subseteq {}^\omega X$ is a given subset, then the two player game $G_X(A)$ is defined as follows (see [8]). There are two players, say I and II, choosing consecutively elements of X, where the first choice is due to player I.

I	x_0		x_2		x_4		$\dots$
II		x_1		x_3		x_5	$\dots$

Player I wins in the game $G_X(A)$, iff the resulting sequence $\langle x_0, x_1, \dots\rangle$ belongs to the set A; otherwise player II wins. Besides, during the one particular run of the game each player knows all the previous moves of both players, thus the game $G_X(A)$ is a game of perfect information.

A *strategy* for player I is any function $\sigma : {}^{<\omega}X \to X$. We say that player I plays according to the strategy σ, if $x_{2n} = \sigma(x_1, x_3, \dots, x_{2n-1})$, for all $n < \omega$. Moreover, the strategy σ is a *winning strategy* for player I in the game $G_X(A)$, if for all $x \in {}^\omega X$ the condition $(\forall n \in \omega)\, x_{2n} = \sigma(x_1, x_3, \dots, x_{2n-1})$ implies $x \in A$. A strategy and a winning strategy for player II are defined similarly.

The game $G_X(A)$ is *determined* if one of the players has a winning strategy. It is known as *Borel determinacy* [7] that the game $G_X(A)$ is determined, whenever the set $A \subseteq {}^\omega X$ is Borel. We say that the game $G_X(A)$ is Borel, if the set A is Borel. Hence every Borel game is determined.

Last but not least, a set $\mathcal{I} \subseteq \mathcal{P}(X)$ is an *ideal* on X, iff

1. $X \notin \mathcal{I}$,
2. $(\forall I, J \in \mathcal{I})\, I \cup J \in \mathcal{I}$,
3. $(\forall I \in \mathcal{I})(\forall A \in \mathcal{P}(X))(A \subseteq I \to A \in \mathcal{I})$.

Then the set of $\mathcal{I}$-positive sets is the set $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$ and $\mathrm{add}(\mathcal{I})$ and $\mathrm{non}(\mathcal{I})$ denotes the additivity and the uniformity of the ideal $\mathcal{I}$ respectively.

From now on we follow the next convention in this paper. Unless otherwise is written down, letters i, j, k and n denote natural numbers, letters s, t, u and v denote finite sequences and letters x, y and z denote infinite sequences.

2 Introduction

In the following text κ always denotes an infinite cardinal and $\mathcal{I}$ always denotes an ideal on κ which covers κ, i.e. $\bigcup \mathcal{I} = \kappa$. Since the cardinal κ is determined by the ideal $\mathcal{I}$, we do not write κ as variable (or parameter) in formulas and in function symbols as well.

The notion of strongly dominating set was firstly introduced in [3]. Since the original definition is not very suitable for generalization, especially when you are looking for perfect set theorem like results, we rather use definition from [2] and [10].

We say that a set $A \subseteq {}^{\omega}\kappa$ is *strongly $\mathcal{I}$-dominating*, whenever the following condition is satisfied:

$$(\forall f : {}^{<\omega}\kappa \to \mathcal{I})(\exists x \in A)(\forall^{\infty} n)\, x(n) \notin f(x{\restriction}n).$$

Clearly the strongly dominating sets are just strongly $[\omega]^{<\omega}$-dominating subsets of ${}^{\omega}\omega$, where $[\omega]^{<\omega}$ is the ideal of finite subsets of ω.

Clearly every nonempty open subset of ${}^{\omega}\kappa$ is strongly $\mathcal{I}$-dominating and, on the other hand, not every closed set is. However, an important example of a strongly dominating closed set (with respect to the original definition) is a Laver perfect set. We adapt this notion as follows.

We say that a tree $q \subseteq {}^{<\omega}\kappa$ is an *$\mathcal{I}$-Laver tree*, iff there is $s \in q$ such that for every $t \in q$

1. either $t \subseteq s$ or $t \supseteq s$,
2. if $t \supseteq s$, then $\mathrm{br}_q(t) = \{\alpha \in \kappa : t^\frown\langle\alpha\rangle \in q\} \in \mathcal{I}^+$.

In such a case the node s is unique and is called the *stem*. Clearly the set $[q]$ is a strongly $\mathcal{I}$-dominating closed set. What is more, it is also perfect, since there is no singleton in $\mathcal{I}^+$. Therefore it is justified to call a set of this form an *$\mathcal{I}$-Laver perfect set*.

Let us recall that a set $A \subseteq {}^{\omega}\kappa$ is called *λ-Suslin*, if it is a projection of a closed subset of ${}^{\omega}\kappa \times {}^{\omega}\lambda$, i.e. there is a closed set $B \subseteq {}^{\omega}\kappa \times {}^{\omega}\lambda$ with $\pi_0[B] = A$. Here π_0 is the projection to the first coordinate. Clearly analytic set is an ω-Suslin set.

Denote as $\mathcal{D}_{\mathcal{I}}$ the family of subsets of ${}^{\omega}\kappa$ which are not strongly $\mathcal{I}$-dominating. It is not difficult to see that $\mathcal{D}_{\mathcal{I}}$ is a σ-ideal. If λ is a cardinal smaller than $\mathrm{add}(\mathcal{D}_{\mathcal{I}})$, then it is possible to characterize λ-Suslin strongly $\mathcal{I}$-dominating sets with the aid of $\mathcal{I}$-Laver trees. This was done in [2] just for strongly dominating subsets of ${}^{\omega}\omega$. Hence the motivation for this paper and consequently one of our goals is to present a proof of the following theorem.

Theorem 2.1. *Suppose that $A \subseteq {}^{\omega}\kappa$ is a λ-Suslin set, for some $\lambda < \mathrm{add}(\mathcal{D}_{\mathcal{I}})$. Then the following conditions are equivalent:*

1. *The set A is strongly $\mathcal{I}$-dominating.*
2. *There is an $\mathcal{I}$-Laver tree $p \subseteq {}^{<\omega}\kappa$ such that $[p] \subseteq A$.*

Similarly we may define a strongly $\mathcal{I}$-unbounded set. We say that a set $A \subseteq {}^\omega\kappa$ is *strongly $\mathcal{I}$-unbounded*, iff the following condition is satisfied:

$$(\forall f : {}^{<\omega}\kappa \to \mathcal{I})(\exists x \in A)(\exists^\infty n)\, x(n) \notin f(x{\restriction}n).$$

Clearly every strongly $\mathcal{I}$-dominating set is strongly $\mathcal{I}$-unbounded. Denote as $\mathcal{U}_\mathcal{I}$ the σ-ideal of subsets of ${}^\omega\kappa$ which are not strongly $\mathcal{I}$-unbounded. In [6] Kechris proved a variant of Theorem 2.1 for strongly unbounded subsets of ${}^\omega\omega$, even it is not explicitly written there. This is because in the Baire space unbounded and strongly unbounded mean the same. The characteristic closed set in this case is the so called superperfect set. We say that a tree $q \subseteq {}^{<\omega}\kappa$ is an *$\mathcal{I}$-superperfect tree*, iff there is $s \in q$ such that $\mathrm{br}_q(s) \in \mathcal{I}^+$ and for every $t \in q$

1. either $t \subseteq s$ or $t \supseteq s$,
2. there is $u \supseteq t$ such that $\mathrm{br}_q(u) \in \mathcal{I}^+$,
3. if $\mathrm{br}_q(t) \in \mathcal{I}$, then $|\,\mathrm{br}_q(t)| = 1$.

Corresponding perfect set $[q]$ is therefore called an *$\mathcal{I}$-superperfect set.*

Kechris's approach includes a particular two player game, which we generalize in the next section in order to prove Theorem 2.1 together with the following theorem.

Theorem 2.2. *Suppose that $A \subseteq {}^\omega\kappa$ is a λ-Suslin set, for some $\lambda < \mathrm{add}(\mathcal{U}_\mathcal{I})$. Then the following conditions are equivalent:*

1. *The set A is strongly $\mathcal{I}$-unbounded.*
2. *There is an $\mathcal{I}$-superperfect tree $p \subseteq {}^{<\omega}\kappa$ such that $[p] \subseteq A$.*

To capture the notions of strongly $\mathcal{I}$-dominating set and strongly $\mathcal{I}$-unbounded set at once we introduce $\mathrm{DU}_\mathcal{I}$-property in the fourth section. However, beforehand we take a look at quite opposite problem.

Let us recall that a set $A \subseteq {}^\omega\kappa$ is said to be a Bernstein set, if the set A and its complement intersect every uncountable closed subset of ${}^\omega\kappa$. If $2^\kappa = \mathfrak{c}$ one can easily construct by transfinite induction a strongly $\mathcal{I}$-dominating Bernstein set. Such a set cannot contain any $\mathcal{I}$-superperfect subset. Provided that the ideal $\mathcal{I}$ satisfies certain condition we can go even further.

In [2] authors constructed a strongly dominating set $A \subseteq {}^\omega\omega$ such that for every Laver tree $p \subseteq {}^{<\omega}\omega$ there is a Laver subtree $q \subseteq p$ with $[q] \cap A = \emptyset$. In the next section we present a sufficient condition on the ideal $\mathcal{I}$ that grants us the existence of such a strongly $\mathcal{I}$-dominating set. Assuming CH this condition is also necessary. Besides, we look whether similar results hold true for strongly $\mathcal{I}$-unbounded sets.

3 Tree ideals

Let us denote by $\mathbb{S}_{\mathcal{I}}$ the set of all $\mathcal{I}$-superperfect subtrees of ${}^{<\omega}\kappa$ and by $\mathbb{L}_{\mathcal{I}}$ the set of all $\mathcal{I}$-Laver subtrees of ${}^{<\omega}\kappa$. Next let $p \in \mathbb{S}_{\mathcal{I}}$ be a tree with stem s. For $x \in [p]$ denote as $z^p_x \in {}^{\omega}\omega$ the infinite sequence with $z^p_x(n)$ being the length of the nth splitting node in p along x. Clearly $z^p_x(0) = |s|$ and the set $Z_p = \{(x, z^p_x) : x \in [p]\}$ is perfect. Now we modify definitions of ideals studied in [5] and [3] as follows:

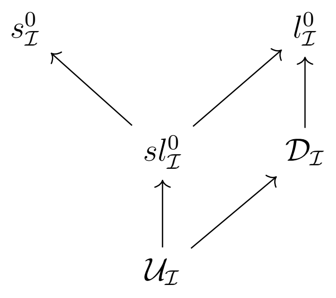
$$\begin{aligned} s^0_{\mathcal{I}} &= \{A \subseteq {}^{\omega}\kappa : (\forall p \in \mathbb{S}_{\mathcal{I}})(\exists q \in \mathbb{S}_{\mathcal{I}})(q \subseteq p \text{ and } [q] \cap A = \emptyset)\},\\ l^0_{\mathcal{I}} &= \{A \subseteq {}^{\omega}\kappa : (\forall p \in \mathbb{L}_{\mathcal{I}})(\exists q \in \mathbb{L}_{\mathcal{I}})(q \subseteq p \text{ and } [q] \cap A = \emptyset)\},\\ sl^0_{\mathcal{I}} &= \{A \subseteq {}^{\omega}\kappa : (\forall p \in \mathbb{S}_{\mathcal{I}})(\exists q \in \mathbb{S}_{\mathcal{I}})(Z_q \subseteq Z_p \text{ and } [q] \cap A = \emptyset)\}. \end{aligned}$$

Let us note that if the inclusion $Z_q \subseteq Z_p$ occurs, for $p, q \in \mathbb{S}_{\mathcal{I}}$, then $q \subseteq p$ and what is more, both trees p and q are of the same type, e.g. if p is an $\mathcal{I}$-Laver tree, then q is an $\mathcal{I}$-Laver tree with the same stem as p has. Consequently $sl^0_{\mathcal{I}} \subseteq s^0_{\mathcal{I}} \cap l^0_{\mathcal{I}}$.

Lemma 3.1. *Families $s^0_{\mathcal{I}}$, $l^0_{\mathcal{I}}$ and $sl^0_{\mathcal{I}}$ are σ-ideals on ${}^{\omega}\kappa$ with $\mathcal{U}_{\mathcal{I}} \subseteq sl^0_{\mathcal{I}}$ and $\mathcal{D}_{\mathcal{I}} \subseteq l^0_{\mathcal{I}}$.*

Proof. A standard fusion argument shows that each of $s^0_{\mathcal{I}}$, $l^0_{\mathcal{I}}$ and $sl^0_{\mathcal{I}}$ is a σ-ideal. To see $\mathcal{U}_{\mathcal{I}} \subseteq sl^0_{\mathcal{I}}$, let $A \in \mathcal{U}_{\mathcal{I}}$ and $p \in \mathbb{S}_{\mathcal{I}}$ be given. It follows that there is $f : {}^{<\omega}\kappa \to \mathcal{I}$ so that $(\forall x \in A)(\forall^\infty n)\, x(n) \in f(x\restriction n)$. Find $q \in \mathbb{S}_{\mathcal{I}}$, $q \subseteq p$ with $\mathrm{br}_q(t) = \mathrm{br}_p(t)$, if $\mathrm{br}_p(t) \in \mathcal{I}$, and $\mathrm{br}_q(t) = \mathrm{br}_p(t) \setminus f(t)$, otherwise. Clearly $Z_q \subseteq Z_p$ and $[q] \cap A = \emptyset$. The same argument shows that $\mathcal{D}_{\mathcal{I}} \subseteq l^0_{\mathcal{I}}$. □

Next diagram shows all easy to see inclusions between ideals from the previous lemma. What is more, for every tree $p \in \mathbb{S}_{\mathcal{I}}$ with $[p] \in \mathcal{D}_{\mathcal{I}}$ we have $[p] \in l^0_{\mathcal{I}} \setminus s^0_{\mathcal{I}}$, thus $l^0_{\mathcal{I}} \neq s^0_{\mathcal{I}}$ and $l^0_{\mathcal{I}} \neq sl^0_{\mathcal{I}}$, too.



To proceed further in the investigation of the above diagram we make use of the following lemma.

Lemma 3.2.

$$l^0_{\mathcal{I}} = \{A \subseteq {}^{\omega}\kappa : (\forall p \in \mathbb{L}_{\mathcal{I}})(\exists q \in \mathbb{L}_{\mathcal{I}})(Z_q \subseteq Z_p \textit{ and } [q] \cap X = \emptyset)\}.$$

Proof. For a tree $p \subseteq {}^{<\omega}\kappa$ and $t \in p$ denote the tree $(p)_t = \{s \in p : s \subseteq t \text{ or } t \subseteq s\}$. Let $A \in l^0_{\mathcal{I}}$, $s \in {}^{<\omega}\kappa$ and $p \in \mathbb{L}_{\mathcal{I}}$ be arbitrary and set

$$r = \{t \in p : (\forall q \in \mathbb{L}_{\mathcal{I}})(Z_q \subseteq Z_{(p)_t} \to [q] \cap A \neq \emptyset)\}.$$

Assume that $s \in r$. It follows that r is an $\mathcal{I}$-Laver tree with stem s. Since $A \in l^0_{\mathcal{I}}$ there is a tree $q \in \mathbb{L}_{\mathcal{I}}$ with stem t such that $q \subseteq r$ and $[q] \cap A = \emptyset$. Clearly $t \in r$, but on the other hand $t \notin r$, due to $Z_q \subseteq Z_{(p)_t}$ and $[q] \cap A = \emptyset$, a contradiction. Therefore $s \notin r$, what proves the lemma. □

The above lemma suggests a question whether $s^0_{\mathcal{I}} = sl^0_{\mathcal{I}}$. However, it turns out that this equality does not hold, at least consistently. We prove it assuming CH.

Let $\{p_\alpha : \alpha < \mathfrak{c}\}$ and $\{q_\alpha : \alpha < \mathfrak{c}\}$ be enumerations of the set $\{p \in \mathbb{S}_{\mathcal{I}} : [p] \in \mathcal{D}_{\mathcal{I}}\}$ and the set $\mathbb{L}_{\mathcal{I}}$ respectively. Next for each $\alpha < \mathfrak{c}$ pick $x_\alpha \in [q_\alpha] \setminus \bigcup_{\beta<\alpha}[p_\beta]$ and let $A = \{x_\alpha : \alpha < \mathfrak{c}\}$. Obviously $A \notin l^0_{\mathcal{I}}$. On the other hand, the set $\{p_\alpha : \alpha < \mathfrak{c}\}$ is dense in $\mathbb{S}_{\mathcal{I}}$, $\mathrm{non}(s^0_{\mathcal{I}}) = \mathfrak{c}$, due to CH, and $|A \cap [p_\alpha]| < \mathfrak{c}$, for every $\alpha < \mathfrak{c}$. Hence $A \in s^0_{\mathcal{I}}$ and consequently $s^0_{\mathcal{I}} \setminus l^0_{\mathcal{I}} \neq \emptyset$ and $sl^0_{\mathcal{I}} \subseteq s^0_{\mathcal{I}} \cap l^0_{\mathcal{I}} \subsetneq s^0_{\mathcal{I}}$. This result motivates another question.

Question 3.3. Is it true that $sl^0_{\mathcal{I}} = s^0_{\mathcal{I}} \cap l^0_{\mathcal{I}}$?

At this time we do not know even partial answer for the above question. However, the situation is completely different in case of ideals $\mathcal{D}_{\mathcal{I}}$ and $l^0_{\mathcal{I}}$. Let us recall that an ideal $\mathcal{I}$ is *prime*, if $a \in \mathcal{I}$ or $\kappa \setminus a \in \mathcal{I}$, for every $a \subseteq \kappa$.

Lemma 3.4. *If the ideal $\mathcal{I}$ is prime, then*

$$\mathcal{D}_{\mathcal{I}} = l^0_{\mathcal{I}} = \{A \subseteq {}^\omega\kappa : (\forall p \in \mathbb{L}_{\mathcal{I}})\, A \cap [p] \in \mathcal{D}_{\mathcal{I}}\}.$$

Proof. Since $\mathcal{D}_{\mathcal{I}} \subseteq \{A \subseteq {}^\omega\kappa : (\forall p \in \mathbb{L}_{\mathcal{I}})\, A \cap [p] \in \mathcal{D}_{\mathcal{I}}\} \subseteq l^0_{\mathcal{I}}$, it remains to check the inclusion $l^0_{\mathcal{I}} \subseteq \mathcal{D}_{\mathcal{I}}$. However, we first show that if $A \subseteq {}^\omega\kappa$ is a strongly $\mathcal{I}$-dominating set, then there is $s \in {}^{<\omega}\kappa$ such that

$$(\forall f : {}^{<\omega}\kappa \to \mathcal{I})(\exists x \in A \cap [s])(\forall n \geq |s|)\, x(n) \notin f(x{\restriction}n). \qquad (\triangleleft)$$

So suppose by contraposition that there is no such $s \in {}^{<\omega}\kappa$. Then for each $s \in {}^{<\omega}\kappa$ fix a function $f_s : {}^{<\omega}\kappa \to \mathcal{I}$ so that $(\forall x \in A \cap [s])(\exists n \geq |s|)\, x(n) \in f_s(x{\restriction}n)$ and define $f : {}^{<\omega}\kappa \to \mathcal{I}$ such that $f(t) = \bigcup\{f_s(t) : s \in {}^{<\omega}\kappa \text{ and } s \subseteq t\}$. Consequently for every $x \in A$ and every $k \in \omega$ there is $n \geq k$ so that $x(n) \in f_{x\restriction k}(x{\restriction}n) \subseteq f(x{\restriction}n)$ and therefore $A \in \mathcal{D}_{\mathcal{I}}$.

To show $l^0_{\mathcal{I}} \subseteq \mathcal{D}_{\mathcal{I}}$ let $A \notin \mathcal{D}_{\mathcal{I}}$ be arbitrary. Then there is $s \in {}^{<\omega}\kappa$ such that $(\triangleleft)$ holds. Next suppose that a tree $q \in \mathbb{L}_{\mathcal{I}}$ with stem s is given. Define $f : {}^{<\omega}\kappa \to \mathcal{I}$ such that $f(t) = \kappa \setminus \mathrm{br}_q(t)$, if $t \in q$ and $t \supseteq s$, otherwise let $f(t) = \emptyset$. Then for every $x \in [s] \setminus [q]$ there is the smallest $n \geq |s|$ so that $x{\restriction}(n+1) \notin q$, i.e. $x(n) \in f(x{\restriction}n)$. Hence $A \cap [q] \neq \emptyset$, due to $(\triangleleft)$.

We have just shown that if $p \in \mathbb{L}_{\mathcal{I}}$ is the tree with $[p] = [s]$ and $q \in \mathbb{L}_{\mathcal{I}}$ is a subtree of p with stem s, then $A \cap [q] \neq \emptyset$. Hence by previous lemma $A \notin l^0_{\mathcal{I}}$. □

In case that the ideal $\mathcal{I}$ is not prime, the situation is a bit more complex. For the price of additional assumptions on cardinal κ and the ideal $sl^0_{\mathcal{I}}$ as well, we get a result involving even the ideals $\mathcal{U}_{\mathcal{I}}$ and $sl^0_{\mathcal{I}}$.

Lemma 3.5. *If $2^\kappa = \mathfrak{c}$, the ideal $\mathcal{I}$ is not prime and* $\mathrm{non}(sl^0_{\mathcal{I}}) = \mathfrak{c}$*, then there is a set $A \subseteq {}^\omega\kappa$ such that $A \in sl^0_{\mathcal{I}}$ and $A \notin \mathcal{D}_{\mathcal{I}}$. Hence $\mathcal{U}_{\mathcal{I}} \subsetneq sl^0_{\mathcal{I}}$ and $\mathcal{D}_{\mathcal{I}} \subsetneq l^0_{\mathcal{I}}$.*

Proof. Since the ideal $\mathcal{I}$ is not prime, we fix disjoint sets $a, b \in \mathcal{I}^+$ with $a \cup b = \kappa$ and we denote

$$\mathbb{S}_{\mathcal{I}}^{a,b} = \{p \in \mathbb{S}_{\mathcal{I}} : (\forall t \in p)(\mathrm{br}_p(t) \in \mathcal{I}^+ \to \mathrm{br}_p(t) \in \{a, b\})\}.$$

Due to assumption $2^\kappa = \mathfrak{c}$ we have $|\mathbb{S}_{\mathcal{I}}^{a,b}| = \mathfrak{c}$. Let $\{p_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the set $\mathbb{S}_{\mathcal{I}}^{a,b}$. For every $\alpha < \mathfrak{c}$ we denote $A_\alpha = {}^\omega\kappa \setminus \bigcup\{[p_\beta] : \beta < \alpha\}$ and we show that the set A_α is strongly $\mathcal{I}$-dominating. So let $f : {}^{<\omega}\kappa \to \mathcal{I}$ be given. The set

$$C = \{x \in {}^\omega\kappa : (\forall n)(x(n) = \min(a \setminus f(x{\restriction}n)) \text{ or } x(n) = \min(b \setminus f(x{\restriction}n)))\}$$

is a Cantor perfect set such that $(\forall x \in C)(\forall n)\, x(n) \notin f(x{\restriction}n)$ and $|C \cap [p_\alpha]| \leq 1$, for every $\alpha < \mathfrak{c}$. Consequently for $\alpha < \mathfrak{c}$ we have $C \cap A_\alpha \neq \emptyset$ and hence $A_\alpha \notin \mathcal{D}_{\mathcal{I}}$.

Again, due to assumption $2^\kappa = \mathfrak{c}$, let $\{f_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the set of functions $f : {}^{<\omega}\kappa \to \mathcal{I}$. For each $\alpha < \mathfrak{c}$ pick $x_\alpha \in A_\alpha$ with $(\forall n)\, x_\alpha(n) \notin f_\alpha(x_\alpha{\restriction}n)$ and let $A = \{x_\alpha : \alpha < \mathfrak{c}\}$. Clearly the set A is strongly $\mathcal{I}$-dominating.

We now prove that $A \in sl^0_{\mathcal{I}}$. Let $p \in \mathbb{S}_{\mathcal{I}}$ be given. Find $\alpha < \mathfrak{c}$ so that $q = p \cap p_\alpha$ is an $\mathcal{I}$-superperfect tree with $Z_q \subseteq Z_p$. Since $[q] \cap A \subseteq [p_\alpha] \cap A \subseteq \{x_\beta : \beta \leq \alpha\}$ and $\mathrm{non}(sl^0_{\mathcal{I}}) = \mathfrak{c}$, there is a tree $r \in \mathbb{S}_{\mathcal{I}}$ with $Z_r \subseteq Z_q$ and $[r] \cap ([q] \cap A) = \emptyset$. Therefore $A \in sl^0_{\mathcal{I}}$. □

In the first place observe that CH together with Lemma 3.4 and Lemma 3.5 imply that the equality $\mathcal{D}_{\mathcal{I}} = l^0_{\mathcal{I}}$ holds, if and only if the ideal $\mathcal{I}$ is prime.

Without CH we need extra notation to spare the assumption $\mathrm{non}(sl^0_{\mathcal{I}}) = \mathfrak{c}$. At first observe that $\mathrm{non}(l^0) = \mathfrak{c}$, since every Laver perfect set can be decomposed into $\mathfrak{c}$ many Laver perfect sets. In order to isolate ideals which this argument goes through for, we denote as $\mathcal{I}{\restriction}a$ the *restriction* of the ideal $\mathcal{I}$ to a set $a \subseteq \kappa$, i.e. we let $\mathcal{I}{\restriction}a = \{I \cap a : I \in \mathcal{I}\}$.

Now suppose that for every $a \in \mathcal{I}^+$ the ideal $\mathcal{I}{\restriction}a$ is not prime and let $p \in \mathbb{S}_{\mathcal{I}}$ be given. For each $t \in p$ with $\mathrm{br}_p(t) \in \mathcal{I}^+$ find disjoint sets $a_0(t), a_1(t) \in \mathcal{I}^+$ such that $\mathrm{br}_p(t) = a_0(t) \cup a_1(t)$ and define $\rho(t) = |\{n < |t| : \mathrm{br}_p(t{\restriction}n) \in \mathcal{I}^+\}|$. Next for each $x \in {}^\omega 2$ define the tree $q_x \in \mathbb{S}_{\mathcal{I}}$ by induction so that $\mathrm{br}_{q_x}(t) = a_{x(\rho(t))}(t)$, if $t \in q_x$ and $\mathrm{br}_p(t) \in \mathcal{I}^+$, otherwise let $\mathrm{br}_{q_x}(t) = \mathrm{br}_p(t)$. Clearly $[p] = \bigcup_{x \in {}^\omega 2}[q_x]$, $Z_{q_x} \subseteq Z_p$ and $[q_x] \cap [q_y] = \emptyset$, whenever $x \neq y$. Consequently $\mathrm{non}(sl^0_{\mathcal{I}}) = \mathfrak{c}$.

It is not difficult to see that even a weaker property of the ideal $\mathcal{I}$ is sufficient for Lemma 3.5 to go through. It is enough if the ideal $\mathcal{I}$ is "everywhere not prime" just locally, to be precise if there is a set $a \in \mathcal{I}^+$ such that for every set $b \in (\mathcal{I}{\restriction}a)^+$ the ideal $\mathcal{I}{\restriction}b$ is not prime.

Hence what remains is the situation when the ideal $\mathcal{I}$ is *locally prime*, i.e. when for every set $a \in \mathcal{I}^+$ there is a set $b \in (\mathcal{I}{\restriction}a)^+$ so that the ideal $\mathcal{I}{\restriction}b$ is prime. However, we have just proven the following theorem.

Theorem 3.6. *If the ideal $\mathcal{I}$ is not locally prime, then $\mathcal{U}_{\mathcal{I}} \subsetneq sl^0_{\mathcal{I}}$ and $\mathcal{D}_{\mathcal{I}} \subsetneq l^0_{\mathcal{I}}$. What is more, in ZFC+CH we have $\mathcal{D}_{\mathcal{I}} \subsetneq l^0_{\mathcal{I}}$ if and only if the ideal $\mathcal{I}$ is not prime.*

□

4 Subsets of ${}^{\omega}\kappa \times {}^{\omega}\omega{\uparrow}$ with $\mathrm{DU}_{\mathcal{I}}$-property

We denote as ${}^{\omega}\omega{\uparrow}$ the subset of ${}^{\omega}\omega$ of *strictly* increasing sequences. Now assume that $f : {}^{<\omega}\kappa \to \mathcal{I}$ and $x \in {}^{\omega}\kappa$ are given and $(\exists^{\infty} n)\, x(n) \notin f(x{\restriction}n)$ holds. In such a case there is a sequence $z \in {}^{\omega}\omega{\uparrow}$ so that $(\forall n)\, x(z(n)) \notin f(x{\restriction}z(n))$. Moreover, if the sequence z has the property $(\forall n)\, z(n+1) = z(n)+1$, then it is clear that also $(\forall^{\infty} n)\, x(n) \notin f(x{\restriction}n)$ holds. It follows that definitions of strongly $\mathcal{I}$-dominating and strongly $\mathcal{I}$-unbounded set differ only in the requirements on the sequence z.

Definition 4.1. We say that a set $B \subseteq {}^{\omega}\kappa \times {}^{\omega}\omega{\uparrow}$ has $\mathrm{DU}_{\mathcal{I}}$-property, and sometimes write $\mathrm{DU}_{\mathcal{I}}(B)$ instead, whenever the following condition is satisfied

$$(\forall f : {}^{<\omega}\kappa \to \mathcal{I})(\exists (x,z) \in B)(\forall^{\infty} n)\, x(z(n)) \notin f(x{\restriction}z(n)).$$

Let π_0 and π_1 be the projections into first and second coordinate, respectively. For an arbitrary set $A \subseteq X \times Y$ and $x \in X$, $y \in Y$ denote *cuts*

$$A^{(x,\cdot)} = \{y \in \pi_1[A] : (x,y) \in A\} \quad \text{and} \quad A^{(\cdot,y)} = \{x \in \pi_0[A] : (x,y) \in A\}.$$

Also define *shift* $\mathrm{sh} : {}^{\omega}\omega \to {}^{\omega}\omega$ such that $\mathrm{sh}(y) = \langle y(n+1) : n \in \omega\rangle$. We say that a set $X \subseteq {}^{\omega}\omega$ is *shift-closed*, if it is closed under the shift operation sh. We denote as $\mathrm{cl}_{\mathrm{sh}}(X)$ the *closure* of the set X under the shift operation sh.

Example 4.2. It is not difficult to see that a set $A \subseteq {}^{\omega}\kappa$ is strongly $\mathcal{I}$-dominating, if and only if the set $A \times \mathrm{cl}_{\mathrm{sh}}(\{\mathrm{id}_{\omega}\})$ has $\mathrm{DU}_{\mathcal{I}}$-property. Similarly a set $A \subseteq {}^{\omega}\kappa$ is strongly $\mathcal{I}$-unbounded, if and only if the set $A \times {}^{\omega}\omega{\uparrow}$ has $\mathrm{DU}_{\mathcal{I}}$-property. Observe that in both cases cuts $A^{(x,\cdot)}$ are shift-closed.

If $p \subseteq {}^{<\omega}\kappa$ is an $\mathcal{I}$-superperfect tree p, then the set Z_p has $\mathrm{DU}_{\mathcal{I}}$-property. Moreover, the tree p is an $\mathcal{I}$-Laver tree, if and only if $Z_p \subseteq {}^{\omega}\kappa \times \mathrm{cl}_{\mathrm{sh}}(\{\mathrm{id}_{\omega}\})$.

Furthermore, we denote as $\mathcal{DU}_{\mathcal{I}}$ the family of those subsets of ${}^{\omega}\kappa \times {}^{\omega}\omega{\uparrow}$ that do not have $\mathrm{DU}_{\mathcal{I}}$-property, i.e. $\mathcal{DU}_{\mathcal{I}} = \{B \subseteq {}^{\omega}\kappa \times {}^{\omega}\omega{\uparrow} : \neg\mathrm{DU}_{\mathcal{I}}(B)\}$. It is easy to see that the following lemma holds.

Lemma 4.3. *The family $\mathcal{DU}_{\mathcal{I}}$ of sets without $\mathrm{DU}_{\mathcal{I}}$-property form a σ-ideal with base consisting of $\mathbf{\Pi}^0_2$ sets.*

Proof. Let $\langle B_k : k \in \omega\rangle$ be a sequence of sets that does not have $\mathrm{DU}_{\mathcal{I}}$-property. For each $k \in \omega$ fix a function $f_k : {}^{<\omega}\kappa \to \mathcal{I}$ such that

$$(\forall (x,z) \in B_k)(\exists^{\infty} n)\, x(z(n)) \in f_k(x{\restriction}z(n))$$

Define $f : {}^{<\omega}\kappa \to \mathcal{I}$ so that $f(s) = \bigcup\{f_k(s) : k \le |s|\}$. Then for every $k \in \omega$ and every $(x,z) \in B_k$ there are infinitely many $n \in \omega$ with $x(z(n)) \in f(x{\restriction}z(n))$. Thus the set $\bigcup_{k\in\omega} B_k$ does not have $\mathrm{DU}_{\mathcal{I}}$-property.

The last assertion follows from the fact that for every $f : {}^{<\omega}\kappa \to \mathcal{I}$ the set

$$\{(x,z) \in {}^{\omega}\kappa \times {}^{\omega}\omega{\uparrow} : (\exists^{\infty} n)\, x(z(n)) \in f(x{\restriction}z(n))\}.$$

is $\mathbf{\Pi}^0_2$ and does not have $\mathrm{DU}_{\mathcal{I}}$-property. □

Observation 4.4. Notice that if for every $x \in \pi_0[B]$ the cut $B^{(x,\cdot)}$ is shift-closed and the set B has $\mathrm{DU}_{\mathcal{I}}$-property, then it satisfies the following stricter condition

$$(\forall f : {}^{<\omega}\kappa \to \mathcal{I})(\exists (x,z) \in B)(\forall n)\, x(z(n)) \notin f(x \restriction z(n)).$$

Since in both special cases we are primarily interested in cuts are shift-closed, it is possible to define $\mathrm{DU}_{\mathcal{I}}$-property with the above formula. However, if we did so, the family $\mathcal{DU}_{\mathcal{I}}$ would not be an ideal.

With a set $B \subseteq {}^{\omega}\kappa \times {}^{\omega}\omega\!\uparrow$ we may also associate a two player game $\mathrm{G}_{\mathcal{I}}(B)$ defined as follows

I	s_0		s_1		s_2		$\ldots$	
II		I_0		I_1		I_2		$\ldots$

where each s_n is a finite sequence from ${}^{<\omega}\kappa$ and each I_n is a set from the ideal $\mathcal{I}$. Player I wins, iff

1. $s_{n+1} \neq \emptyset$ and $s_{n+1}(0) \notin I_n$, for every $n \in \omega$,

2. If $x = s_0{}^\frown s_1{}^\frown \ldots$ and $z(n) = \sum_{i \leq n} |s_i|$, for $n \in \omega$, then $(x,z) \in B$.

It is not difficult to see that the game $\mathrm{G}_{\mathcal{I}}(B)$ defined above is just a variant of the two player game $G_X(A)$, for suitable sets X and $A \subseteq {}^{\omega}X$. Moreover, the set A may be chosen Borel, whenever the set B is Borel. It follows by the Borel determinacy that the game $\mathrm{G}_{\mathcal{I}}(B)$ is determined, whenever the set B is Borel.

The next lemma shows the relationship between $\mathrm{DU}_{\mathcal{I}}$-property and the above two player game. What is more, it implies the perfect set theorem for Borel sets with $\mathrm{DU}_{\mathcal{I}}$-property.

Lemma 4.5. *Let $B \subseteq {}^{\omega}\kappa \times {}^{\omega}\omega\!\uparrow$ be a given set such that for every $x \in \pi_0[B]$ the cut $B^{(x,\cdot)}$ is shift-closed. Then*

1. *Player I has a winning strategy in the game $\mathrm{G}_{\mathcal{I}}(B)$, if and only if there is an $\mathcal{I}$-superperfect tree $p \subseteq {}^{<\omega}\kappa$ with stem s such that $Z_p \subseteq B$.*

2. *Player II does not have a winning strategy in the game $\mathrm{G}_{\mathcal{I}}(B)$, if and only if the set B has $\mathrm{DU}_{\mathcal{I}}$-property.*

Proof. (2) We prove just the implication from left to right. So let $\tau : {}^{<\omega}({}^{<\omega}\kappa) \to \mathcal{I}$ be a winning strategy for player II. Define $f : {}^{<\omega}\kappa \to \mathcal{I}$ such that

$$f(t) = \bigcup \{\tau(s_0, \ldots, s_n) : n \in \omega \text{ and } (\forall i \leq n)\, s_i \in {}^{<\omega}\kappa \text{ and } t = s_0{}^\frown \ldots {}^\frown s_n\}.$$

Now let $(x,z) \in B$ be arbitrary. Fix $\langle s_n : n \in \omega\rangle \in {}^{<\omega}({}^{<\omega}\kappa)$ so that $x = s_0{}^\frown s_1{}^\frown \ldots$ and $z(n) = \sum_{i \leq n} |s_i|$. Since τ is a winning strategy for player II, there is $n \in \omega$ such that $s_{n+1}(0) \in \tau(s_0, \ldots, s_n)$. Therefore $x(z(n)) \in f(x \restriction z(n))$ and the set B does not have $\mathrm{DU}_{\mathcal{I}}$-property, due to observation 4.4.

Furthermore, notice that the part of the lemma just proven is the only part where shift-closed cuts are needed. ∎

5 Parametrized $\mathrm{DU}_{\mathcal{I}}$-property

There is a standard technique used when dealing with analytic determinacy, known as *Solovay's unfolding trick*. The application of the unfolding trick in our setting is based on the following modification of the game $\mathrm{G}_{\mathcal{I}}(B)$. First of all fix λ, some cardinal. Instead of playing only finite sequences $s \in {}^{<\omega}\kappa$, player I plays couples $(s, \xi) \in {}^{<\omega}\kappa \times (\lambda+1)$, where $\lambda+1 = \lambda \cup \{\lambda\}$ is the ordinal successor of λ. Ordinal $\xi < \lambda$ is called a *witness*. The case when player I plays $\xi = \lambda$ is interpreted such that player I waits with playing witnesses. Player II plays sets $I \in \mathcal{I}$, as before.

For $y \in {}^{\leq\omega}(\lambda+1)$ denote as $y^* = \langle y(n) : n \in \omega \text{ and } y(n) < \lambda\rangle$ the sequence we obtain by omitting all λ terms in y, provided that the order of remaining terms is preserved. For example $\langle 0, \lambda, 1, \lambda, \dots\rangle^* = \mathrm{id}_\omega$. Observe that the mapping $y \mapsto y^*$ is continuous.

With a set $C \subseteq {}^\omega\kappa \times {}^\omega\lambda \times {}^\omega\omega\!\uparrow$ we may associate a two player game $\widetilde{\mathrm{G}}_{\mathcal{I}}(C)$ defined as follows

I	(s_0, ξ_0)		(s_1, ξ_1)		(s_2, ξ_2)		$\dots$
II		I_0		I_1		I_2	$\dots$

where each s_n is a finite sequence from ${}^{<\omega}\kappa$, each ξ_n is an ordinal smaller or equal λ and each I_n is a set from the ideal $\mathcal{I}$. Player I wins, iff

1. $s_{n+1} \neq \emptyset$ and $s_{n+1}(0) \notin I_n$, for every $n \in \omega$,
2. $\xi_n < \lambda$, for infinitely many $n \in \omega$,
3. If $x = s_0 ^\frown s_1 ^\frown \dots$, $y = \langle \xi_n : n \in \omega\rangle$ and $z(n) = \sum_{i \leq n} |s_i|$, for $n \in \omega$, then $(x, y^*, z) \in C$.

It is easy to see that the Borel game $\widetilde{\mathrm{G}}_{\mathcal{I}}(C)$ is a variant of the Borel game $G_X(D)$, for suitable sets X and $D \subseteq {}^\omega X$. Consequently we may use the Borel determinacy as before.

We now introduce parametrized version of $\mathrm{DU}_{\mathcal{I}}$-property in order to simplify expression.

Definition 5.1. We say that a set $C \subseteq {}^\omega\kappa \times {}^\omega\lambda \times {}^\omega\omega\!\uparrow$ has $\widetilde{\mathrm{DU}}_{\mathcal{I}}$-property, in case that the following condition holds

$$(\forall f : {}^{<\omega}\kappa \times {}^{<\omega}\lambda \to \mathcal{I})(\exists (x, y, z) \in C)(\forall k)(\forall^\infty n)\, x(z(n)) \notin f(x{\restriction}z(n), y{\restriction}k).$$

We naturally extend notions of projections and cuts to products of more than two spaces, e.g. it is easy to see that, if a set C has $\widetilde{\mathrm{DU}}_{\mathcal{I}}$-property, then the projection

$$\pi_{02}[C] = \{(x, z) \in {}^\omega\kappa \times {}^\omega\omega\!\uparrow : (\exists y \in {}^\omega\lambda)\, (x, y, z) \in C\}$$

has $\mathrm{DU}_{\mathcal{I}}$-property. Moreover, the inverse implication holds too, provided that λ is sufficiently small, i.e. $\lambda < \mathrm{add}(\mathcal{DU}_{\mathcal{I}})$. One can easily see that the following a bit more general result implies this assertion.

Lemma 5.2. *Assume that a set $C \subseteq {}^\omega\kappa \times {}^\omega\lambda \times {}^\omega\omega\!\uparrow$ does not have $\widetilde{\mathrm{DU}}_\mathcal{I}$-property. Then there is a sequence $\langle f_\alpha : \alpha < \lambda\rangle$ with $f_\alpha : {}^{<\omega}\kappa \to \mathcal{I}$, for each $\alpha < \lambda$, such that*

$$(\forall (x,z) \in \pi_{02}[C])(\exists \alpha < \lambda)(\exists^\infty n)\, x(z(n)) \in f_\alpha(x{\restriction}z(n)).$$

Proof. By the assumption there is a function $f : {}^\omega\kappa \times {}^\omega\lambda \to \mathcal{I}$ such that

$$(\forall (x,y,z) \in C)(\exists k)(\exists^\infty n)\, x(z(n)) \in f(x{\restriction}z(n), y{\restriction}k).$$

For each $u \in {}^{<\omega}\lambda$ define $f_u : {}^{<\omega}\kappa \to \mathcal{I}$ so that $f_u(s) = f(s,u)$, and we are done. □

Corollary 5.3. *Let $\lambda < \mathrm{add}(\mathcal{DU}_\mathcal{I})$ and let $C \subseteq {}^\omega\kappa \times {}^\omega\lambda \times {}^\omega\omega\!\uparrow$ be a given set. Then the set C has $\widetilde{\mathrm{DU}}_\mathcal{I}$-property, iff the projection $\pi_{02}[C]$ has $\mathrm{DU}_\mathcal{I}$-property.* □

The following lemma gives another characterization of sets with $\widetilde{\mathrm{DU}}_\mathcal{I}$-property. Here $[\omega]^\omega$ denotes the family of infinite subsets of ω. The presence of the infinite set a is exactly due to possibility for player I to wait with playing witnesses.

Lemma 5.4. *A set $C \subseteq {}^\omega\kappa \times {}^\omega\lambda \times {}^\omega\omega\!\uparrow$ has $\widetilde{\mathrm{DU}}_\mathcal{I}$-property, if and only if*

$$(\forall f : {}^\omega\kappa \times {}^\omega\lambda \to \mathcal{I})(\exists (x,y,z) \in C)(\exists a \in [\omega]^\omega)(\forall^\infty n)\, x(z(n)) \notin f(x{\restriction}z(n), y{\restriction}|a \cap n|).$$

Proof. Assume that the set C has $\widetilde{\mathrm{DU}}_\mathcal{I}$-property and let $f : {}^\omega\kappa \times {}^\omega\lambda \to \mathcal{I}$ be given. Then there is $(x,y,z) \in C$ such that $(\forall k)(\forall^\infty n)\, x(z(n)) \notin f(x{\restriction}z(n), y{\restriction}k)$. Set $n_0 = 0$ and for each $k > 0$ define

$$n_k = \min\{m \in \omega : (\forall n \geq m)\, x(z(n)) \notin f(x{\restriction}z(n), y{\restriction}k) \text{ and } m > n_{k-1}\}.$$

Denote $a = \{n_k : k \in \omega \setminus 1\}$. For $n > n_1$ pick the greatest k such that $n_k < n$. Then $|a \cap n| = k$ and hence $x(z(n)) \notin f(x{\restriction}z(n), y{\restriction}k) = f(x{\restriction}z(n), y{\restriction}|a \cap n|)$.

To prove the converse consider only functions $f : {}^\omega\kappa \times {}^\omega\lambda \to \mathcal{I}$ nondecreasing in both variables. □

It is useful to define trees also on set ${}^{<\omega}\kappa \times {}^{<\omega}\lambda$. A set $q \subseteq {}^{<\omega}\kappa \times {}^{<\omega}\lambda$ is a *tree*, if $|s| = |t|$ and $(s{\restriction}k, t{\restriction}k) \in q$, whenever $(s,t) \in q$ and $k < |s|$. Besides, let

$$[q] = \{(x,y) \in {}^\omega\kappa \times {}^\omega\lambda : (\forall n)\, (x{\restriction}n, y{\restriction}n) \in q\}.$$

Observe that there is a one to one correspondence between the set of trees just defined and the set of trees defined on the set ${}^{<\omega}(\kappa \times \lambda)$.

Lemma 5.5. *Let $C \subseteq {}^\omega\kappa \times {}^\omega\lambda \times {}^\omega\omega\!\uparrow$ be a given set, so that for every $(x,y) \in \pi_{01}[C]$ the cut $C^{(x,y,\cdot)}$ is shift-closed. Then*

1. *Player I has a winning strategy in the game $\widetilde{\mathrm{G}}_\mathcal{I}(C)$, if and only if there is a tree $q \subseteq {}^{<\omega}\kappa \times {}^{<\omega}(\lambda + 1)$ such that $p = \pi_0[q]$ is an $\mathcal{I}$-superperfect tree and $\{(x, y^*, z^p_x) : (x,y) \in [q]\} \subseteq C$.*

2. *Player II does not have a winning strategy in the game $\widetilde{\mathrm{G}}_\mathcal{I}(C)$, if and only if the set C has $\widetilde{\mathrm{DU}}_\mathcal{I}$-property.*

Proof. (2) At first assume that the set C does not have $\widetilde{\mathrm{DU}}_{\mathcal{I}}$-property and fix a function $f : {}^{<\omega}\kappa \times {}^{<\omega}\lambda \to \mathcal{I}$ such that

$$(\forall (x,y,z) \in C)(\exists k)(\exists^\infty n)\, x(z(n)) \in f(x{\restriction}z(n), y{\restriction}k).$$

We may assume that the function f is nondecreasing in the second variable. Define a strategy for player II $\tau : {}^{<\omega}({}^{<\omega}\kappa \times (\lambda+1)) \to \mathcal{I}$ such that

$$\tau((s_0,\xi_0),\dots,(s_n,\xi_n)) = f(s_0^\frown\dots{}^\frown s_n, \langle \xi_0,\dots,\xi_n\rangle^*).$$

Now let $\langle (s_n,\xi_n) : n \in \omega\rangle$ be a sequence of moves for player I in the game $\widetilde{\mathrm{G}}_{\mathcal{I}}(C)$ and denote $x = s_0^\frown s_1^\frown \dots$, $y = \langle \xi_n : n \in \omega\rangle$ and $z(n) = \sum_{i\le n} |s_i|$, for $n \in \omega$. Besides, we may suppose that $s_{n+1} \neq \emptyset$ for every $n \in \omega$ and both rules (2) and (3) are fulfilled. Since the set C does not have $\widetilde{\mathrm{DU}}_{\mathcal{I}}$-property, there are $k, n \in \omega$ such that $x(z(n)) \in f(x{\restriction}z(n), y^*{\restriction}k)$ and $|\langle \xi_0,\dots,\xi_n\rangle^*| \geq k$. Consequently

$$s_{n+1}(0) = x(z(n)) \in f(x{\restriction}z(n), y^*{\restriction}k) \subseteq \tau((s_0,\xi_0),\dots,(s_n,\xi_n))$$

and τ is a winning strategy for player II.

To prove the converse suppose that $\tau : {}^{<\omega}({}^{<\omega}\kappa \times (\lambda+1)) \to \mathcal{I}$ is a winning strategy for player II. Define $f : {}^{<\omega}\kappa \times {}^{<\omega}\lambda \to \mathcal{I}$ such that

$$f(s,t) = \bigcup_{n\in\omega} \{\tau((s_0,\xi_0),\dots,(s_n,\xi_n)) : s = s_0^\frown\dots{}^\frown s_n \text{ and } \langle \xi_0,\dots,\xi_n\rangle^* = t\}.$$

We use Lemma 5.4 to show that the set C does not have $\widetilde{\mathrm{DU}}_{\mathcal{I}}$-property. So let $(x,y,z) \in C$ and $a = \{a_n : n \in \omega\} \in [\omega]^\omega$ be arbitrary. Set $\xi_{a_n} = y(n)$ and $\xi_k = \lambda$, if $k \notin a$. Clearly $\langle \xi_0, \xi_1, \dots\rangle^* = y$. Next find $\langle s_n : n \in \omega\rangle \in {}^{<\omega}({}^{<\omega}\kappa)$ such that $x = s_0^\frown s_1^\frown \dots$ and $z(n) = \sum_{i\le n} |s_i|$. Since τ is a winning strategy for player II, there is n so that $s_{n+1}(0) \in \tau((s_0,\xi_0),\dots,(s_n,\xi_n))$. Moreover, $\langle \xi_0,\dots,\xi_n\rangle^* = y{\restriction}|a \cap n|$, therefore

$$x(z(n)) = s_{n+1}(0) \in \tau((s_0,\xi_0),\dots,(s_n,\xi_n)) = f(x{\restriction}z(n), y{\restriction}|a\cap n|).$$

We have thus proved that for a fixed $a \in [\omega]^\omega$ and arbitrary $(x,y,z) \in C$ there is n such that $x(z(n)) \in f(x{\restriction}z(n), y{\restriction}|a \cap n|)$. Since the cut $C^{(x,y,\cdot)}$ is closed under the shift operation sh, there are infinitely many such $n \in \omega$. □

Due to Borel determinacy we immediately obtain the following corollary.

Corollary 5.6. *If $C \subseteq {}^\omega\kappa \times {}^\omega\lambda \times {}^\omega\omega{\uparrow}$ is a Borel set so that for every $(x,y) \in \pi_{01}[C]$ the cut $C^{(x,y,\cdot)}$ is shift-closed, then the following conditions are equivalent:*

1. *The set C has $\widetilde{\mathrm{DU}}_{\mathcal{I}}$-property.*
2. *Player I has a winning strategy in the game $\widetilde{\mathrm{G}}_{\mathcal{I}}(C)$.*
3. *There is a tree $q \subseteq {}^{<\omega}\kappa \times {}^{<\omega}(\lambda+1)$ such that $p = \pi_0[q]$ is an $\mathcal{I}$-superperfect tree and $\{(x, y^*, z_x^p) : (x,y) \in [q]\} \subseteq C$.*

□

We may now prove the perfect set theorem for sets with $\mathrm{DU}_{\mathcal{I}}$-property.

Theorem 5.7. *Let $\lambda < \mathrm{add}(\mathcal{DU}_{\mathcal{I}})$ and let $B \subseteq {}^{\omega}\kappa \times {}^{\omega}\omega{\uparrow}$ be a λ-Suslin set such that for each $x \in \pi_0[B]$ the cut $B^{(x,\cdot)}$ is shift-closed. Then the following conditions are equivalent:*

1. *The set B has $\mathrm{DU}_{\mathcal{I}}$-property.*
2. *Player I has a winning strategy in the game $\mathrm{G}_{\mathcal{I}}(B)$.*
3. *There is an $\mathcal{I}$-superperfect tree $p \in \mathbb{S}_{\mathcal{I}}$ with $Z_p = \{(x, z^p_x) : x \in [p]\} \subseteq B$.*

Proof. We prove just the nontrivial implication (1) → (3). Suppose that the set B has $\mathrm{DU}_{\mathcal{I}}$-property. Since the set B is λ-Suslin there is a closed set $C \subseteq {}^{\omega}\kappa \times {}^{\omega}\lambda \times {}^{\omega}\omega{\uparrow}$ such that $B = \pi_{02}[C]$. At first we show that the set

$$C^{\star} = \{(x, y, \mathrm{sh}(z)) \in {}^{\omega}\kappa \times {}^{\omega}\lambda \times {}^{\omega}\omega{\uparrow} \colon (x, y, z) \in C\}$$

is closed, as well. So let $\langle (x_n, y_n, \mathrm{sh}(z_n)) : n \in \omega\rangle \in {}^{\omega}C^{\star}$ be a sequence converging to $(x, y, \bar{z}) \in {}^{\omega}\kappa \times {}^{\omega}\lambda \times {}^{\omega}\omega{\uparrow}$. At first observe that $z_n(0) < z_n(1) = \bar{z}(1)$, for almost every $n \in \omega$. Hence we may suppose that $z_n(0) = z_{n+1}(0)$ for every $n \in \omega$ and then $z_n = \langle z_n(0)\rangle^\frown \mathrm{sh}(z_n)$ converges to some $z \in {}^{\omega}\omega{\uparrow}$. It follows that $(x, y, z) \in C$, because the set C is closed. Moreover, we have $\mathrm{sh}(z) = \bar{z}$, due to continuity of sh, and consequently the set $C^{\star}$ is closed.

Now let $C_0 = C$ and $C_{n+1} = C_n \cup C^{\star}_n$, for $n \in \omega$. It follows that for every $n \in \omega$ the set C_n is closed and hence the set $C_\omega = \bigcup_{n\in\omega} C_n$ is $\mathbf{\Sigma}^0_2$. Moreover, for each $(x, y) \in \pi_0[C_\omega]$ the cut $C_\omega^{(x,y,\cdot)}$ is shift-closed and by Corollary 5.3 the set C_ω has $\widetilde{\mathrm{DU}}_{\mathcal{I}}$-property, since $\pi_{02}[C_\omega] = B$ and the set B has $\mathrm{DU}_{\mathcal{I}}$-property. Consequently by Corollary 5.6 there is a tree $q \subseteq {}^{<\omega}\kappa \times {}^{<\omega}(\lambda + 1)$ such that the tree $p = \pi_0[q]$ is $\mathcal{I}$-superperfect and $\{(x, y^*, z^p_x) : (x, y) \in [q]\} \subseteq C_\omega$. Clearly $Z_p \subseteq B$. □

To deal with strongly $\mathcal{I}$-unbounded sets and strongly $\mathcal{I}$-dominating sets let us denote $\mathcal{DU}_{\mathcal{I}}|R = \{A \subseteq {}^{\omega}\kappa : A \times R \in \mathcal{DU}_{\mathcal{I}}\}$, for a set $R \subseteq {}^{\omega}\kappa{\uparrow}$. Then it is clear that $\mathcal{U}_{\mathcal{I}} = \mathcal{DU}_{\mathcal{I}}|{}^{\omega}\omega{\uparrow}$ and $\mathcal{D}_{\mathcal{I}} = \mathcal{DU}_{\mathcal{I}}|\,\mathrm{cl}_{\mathrm{sh}}(\{\mathrm{id}_\omega\})$.

It is not difficult to see that a simple modification in above proofs, in particular Corollary 5.3 and Theorem 5.7, leads to the general theorem, which both theorems in the introduction follow from.

Theorem 5.8. *Let $R \subseteq {}^{\omega}\kappa{\uparrow}$ be a nonempty shift-closed set and let $A \subseteq {}^{\omega}\kappa$ be a λ-Suslin set, for some $\lambda < \mathrm{add}(\mathcal{DU}_{\mathcal{I}}|R)$. Then the following conditions are equivalent:*

1. *The set $A \times R$ has $\mathrm{DU}_{\mathcal{I}}$-property.*
2. *Player I has a winning strategy in the game $\mathrm{G}_{\mathcal{I}}(A \times R)$.*
3. *There is an $\mathcal{I}$-superperfect tree $p \in \mathbb{S}_{\mathcal{I}}$ with $Z_p = \{(x, z^p_x) : x \in [p]\} \subseteq A \times R$.*

□

6 Concluding remarks

In the presented paper we have generalized some results, regarding (strongly) unbounded and strongly dominating sets. However, we have completely missed the notion of dominating set studied in [9] and [1]. Despite the proofs in those papers often used special properties of the Baire space and the ideal $[\omega]^{<\omega}$, we may still ask whether a similar generalization is possible in the case of dominating sets.

On the other hand, there are few differences between strongly $\mathcal{I}$-unbounded and strongly $\mathcal{I}$-dominating sets, e.g. it is not clear whether the equality $\mathcal{U}_\mathcal{I} = sl^0_\mathcal{I}$ holds, even for prime ideals $\mathcal{I}$. This is somewhat related to the following question.

In the proof of Lemma 3.4 we have shown that for every strongly $\mathcal{I}$-dominating set $A \subseteq {}^\omega\kappa$ there is an $\mathcal{I}$-Laver tree $p \subseteq {}^{<\omega}\kappa$ such that the set $Z_p \cap (A \times \mathrm{cl_{sh}}(\{\mathrm{id}_\omega\}))$ has $\mathrm{DU}_\mathcal{I}$-property. Hence we may ask if for every strongly $\mathcal{I}$-unbounded set $B \subseteq {}^\omega\kappa$ there is an $\mathcal{I}$-superperfect tree q such that the set $Z_q \cap (B \times {}^\omega\omega\uparrow)$ has $\mathrm{DU}_\mathcal{I}$-property.

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