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Strategies used for the solution of a nonroutine word problem: a comparison of secondary school pupils and pre-service mathematics teachers

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Abstract. Word problems are perceived as an essential part of mathematics education, since they require both the conceptual as well as procedural understanding of mathematics. Our study, centered around a nonroutine word problem, involved 25 secondary school pupils aged 15-16 and 19 preservice mathematics teachers. The results showed that while pupils used the whole range of available strategies, preservice teachers utilized almost exclusively algebraic approach. Though, this approach did not lead to greater solution success. Then we compared the preservice teachers' own problem-solving behavior with the way in which they evaluated pupils' solutions. While the pre-service teachers clearly preferred the use of algebra, they did not particularly appreciate this approach in pupils; they put a much greater emphasis on the correctness of the obtained numerical results.

Keywords: Nonroutine word problem, Solution strategy, Algebra, Pre-service teachers

1 Introduction

A common belief of students and teachers that word problems are more difficult than 'pure mathematical problems is supported by several experimental studies [3], surveys of teachers and mathematics educators [11], analyses of textbooks [13],as well as reflected in the learning science literature [9].

The main reason for this difficulty may be explained by the fact that solving word problems involves more than simple arithmetic or algebra and mechanical use of a known algorithm. Errors in the comprehension of the text, inability to create a suitable mathematical model and errors in the solution phase all add together and this results in smaller success rates than those achieved for simple matched equations.

Still, in spite of a great number of research studies devoted to word problems solution process, a satisfactory theory with clear recommendations for the educational process is still not at hand. It remains unclear what factors affecting the solution path were actually adopted by a given solver. Several researchers called for further research into the intersecting cognitive and affective domains and for the application of a wider variety of research approaches, including clinical interviews and observations. This study presents another contribution to a better understanding of the underlying processes.

The aim of this paper is to study different strategies used in the solution of a nonroutine word-problem by two groups of participants: secondary school pupils aged 15-16 years and preservice mathematics teachers (students). Our findings confirm previous observations that the kind of model produced depends on the mathematical preparation of the solver. Correct logical reasoning and calculation may not support an algebraic solution, use of algebra is not always associated with higher solution success and we try to identify possible explanatory factors. We further explore the attitudes of future teachers to the assessment of pupils work. Our findings yield some implications not only for mathematical education but also for education of mathematics teachers.

2 Theoretical and empirical backgroud

To better understand the difficulty of word problems, it is important to understand the process of its solution. For solving a non-standard problem, Plya [15] and Schoenfeld [16] identified the following four phases: (a) understanding the problem, (b) making a plan, (c) carrying out the plan, and (d) looking back. Yimer and Ellerton [23] , based on empirical data from a study of preservice teachers engaged in non-routine mathematics problem solving, propose a five-phase model to describe the range of cognitive and metacognitive approaches used: engagement, transformation-formulation, implementation, evaluation-internalization. De Corte et al. [5] distinguish even 6 phases: (1) the construction of an internal model of the problem situation; (2) the transformation of this situation model into a mathematical model of the elements and relations; (3) working through the mathematical model to derive mathematical result(s); (4) interpreting the outcome of the computational work; (5) evaluating if the interpreted mathematical outcome is computationally correct and reasonable; and (6) communicating the obtained solution. However, for our purposes, the division into into a comprehension phase and a solution phase provided by Koedinger and Nathan [9] will be sufficient. In the comprehension phase, problem solvers process the text of the story problem and create corresponding internal representations of the quantitative and situation-based relationships expressed in that text. In the solution phase problem solvers use or transform the quantitative relationships that are represented both internally and externally to arrive at a solution.

The comprehension and solution phases typically are interleaved rather than performed sequentially. Problem solvers iteratively comprehend first a small piece of the problem statement (e.g., a clause or sentence) and then produce a piece of corresponding external representation (e.g., an arithmetic operation or algebraic expression), often as an external memory aid. A number of researchers have provided convincing evidence that errors in the comprehension phase add to those arrising in the solution phase, leading to higher rates of failures [18].

The strategy used by a solver when solving a word problem reveals a lot about the solvers understanding of the relationships expressed in the situations and how he/she uses this information, especially if presented a non-routine problem, i.e., such that does not immediately invoke a use of a learned algorithm. The choice of a solution strategy is influenced not only by his/her knowledge base, but also by his/her perception of the problem being arithmetic and algebraic. In accordance with Stacey and MacGregor [18], Van Dooren et al. [20, 21], a problem is considered arithmetic if its solution can be obtained by consecutively performing operations on known numbers, while their meaning remains easily connected with the original problem context. The solution becomes a collection of calculations, each one with a numerical answer that can be related to some feature of the situation. Here, a solver may choose a simple *gues-and-check strategy* (allowing even finer distinction according whether a guessing was completely random, or sequential or in some way systematic) [4] or *working backward* strategy (unwind strategy in terminology of Koedinger and Nathan [9]).

By contrast, algebraic way of thinking and problem solving requires the ability to recognise an underlying structure in the problem situation, to represent the relationships in this structure symbolically, and to manipulate symbolic expressions containing unknown values, while the concrete meaning of these manipulations in relation to the problem is often not so transparent, and ultimately find the answer of the problem.

Usually, there may be more than one way of solving the problem and during the solution of one problem pupils often switch between strategies used: a nice overview of various routes encountered in solution protocols is presented by Stacey and MacGregor [18]. If often happens that a pupil starts with an algebraic approach and then resorts to working backwards, or after guess-check-revise uses working backwards etc. Several studies have also revealed that the choice of strategy is strongly influenced by the changes is wording of the problem or by its context [22].

Athough some teachers promote non-algebraic methods because they believe they are easier for students [18], learning algebraic methods is important from several aspects. However, the transition from the arithmetic to the algebraic approach presents a serious barrier for many pupils [17, 18, 3]. In teachers education it is important to prepare for the difficulty experienced by their future pupils, but the teachers themselves must be prepared to be able to use this approach successfully!

3 Description of the experiment

The purpose of the our research was to compare the types of strategies used by pupils and preservice teachers when solving a non-routine word problem. More specifically, we wanted to find answers for the following research questions:

(a) What strategies will pupils and students use?

(b) How will the knowledge and training of the participants influence their choice of strategy?

(c) How will preservice teachers evaluate the use of strategies by pupils?

3.1 Participants

Our study involved two groups of participants who were administered a nonroutine word problem called POTS. Group I consisted of 25 pupils 15-16 years old and Group II of 19 pre-service mathematics teachers. For brevity, participants from the first group will sometimes be referred to as pupils, participants from the second group as students. The characteristics of the two groups and the differences between them are described in more detail below. The names of the participants used in this text were changed to ensure anonymity.

In the second phase of the experiment we interviewed a few selected participants from the the second group (pre-service teachers) to find out their attitudes to various solutions produced by pupils.

Participants in Group I were pupils of the first year of a secondary school (Lyceum) in Krakow, Poland. The pupils were 15-16 years old, and their class had an extended mathematical instruction that was, at least in some part, given by a university professor with specialization in didactics of mathematics. The experiment was performed with one complete class within one regular lesson lead by one of the authors of the present paper. During the experimental lesson, the teacher did not communicate with students, he just handed the sheets with the problem texts and collected the solutions. Pupils were solving the problem individually, raising their hands when finished. After approximately half of the class was ready, all the others were asked to finish their work. Notice that the time needed for the solution of the pots problems was 7 minutes.

Group II consisted of 19 pre-service teachers in their fourth year of study at a university in Slovakia. At the time of the experiment, they all had completed a three-year bachelor degree program in a combination of two subjects (Mathematics plus Physics, Chemistry, Biology, Informatics or Geography). Their mathematics preparation at the university consists of several courses in ‘higher’ mathematics (Calculus, Linear and general algebra, Geometry, Logic and set theory, Discrete mathematics, Probability theory). In addition, they went through a mixture of general courses in psychology, educational and instructional sciences. Between the first experiment and follow-up interviews (the time span was about 18 months) they also took part in several periods of practial placement at schools where they performed the kinds of tasks performed that regular teachers do , including evaluation of pupils work. The test problems were given to them in the final week of the second semester of their study for master degree as a part of their assessment for a mathematics problem-solving course, devoted to school education. The course comprised 13 weekly sessions 2-hours long and the topics treated in this course were number theory, optimization problems, equations and inequalities, word problems, the use of such problems at schools and theoretical background of problem solving. This course was one of the final courses which students needed in order to gain an endorsement to teach mathematics at a secondary school (according to Slovak educational system, these schools are for children aged 10-18 years).

As our primary aim was to explore the attitudes of future teachers for lower elementary and secundary education, the first group consisting of pupils with extended mathematical education was an intended choice. Such teachers should be able to work with gifted pupils and we expected the mathematical competencies of the two groups to be comparable. We believe that in this way the particular way of mathematical training would be the decisive factor to be used in explaining differencies (if any) in strategy choices and results.

4 Description of the problem and its solution

Participants were given a paper-and-pencil test. In this paper we concentrate on the word problem that we shall call the POTS problem. The text was presented to the participants in their native languages (Polish and Slovak). The English translation that follows is as literal as possible. However, here we have for easier reference explicitly separated the two formulated questions that were in the original text simply stated one after another. (We hypothethise that this might have been a factor contributing to the lower success rate for the answers to the second questions.)

> **The POTS problem.** *There are two pots with certain amount of water. If we pour from the first pot into the second one exactly as much water as* **it** *already contains and then from the second pot into the first one as much water as* **it** *already contains, each pot will contain exactly 120 litres of water.*
> *(i) How many litres of water did each pot contain originally?*
> *(ii) What is the minimal possible capacity of each pot?*

The POTS problem is an exemplification of a nonroutine matematical word problem, that in the first place requires a sufficient understanding of the problem formulation. We expected the participants to have difficulties in the comprehension phase with the word **it** used in the problem formulation twice (emphasized in bold). Does this pronoun refer to the pot **from** which water is poured, or to the pot **into** which water is poured? Really, both interpretations would be possible, if the final contents of water were not given. A little thought reveals that the former interpretation means that after the first pouring the first pot becomes empty and after the second pouring all the water is transferred back to the first pot. Obviously this interpretation does not allow for the final redistribution of water as stated in the text. In the vast majority of the solution protocols produced by the participants of our study it seemed that the solvers immediately perceived the second meaning. A nice example of how the solution phase influences back the comprehension phase is provided by the solution protocol of Patrik, Figure 1. The illustration in the left half of the sheet implies that the pupil started with the first interpretation. When he saw that after the two pourings the second pot remains empty, he crossed out the pictures and went on with the second meaning.

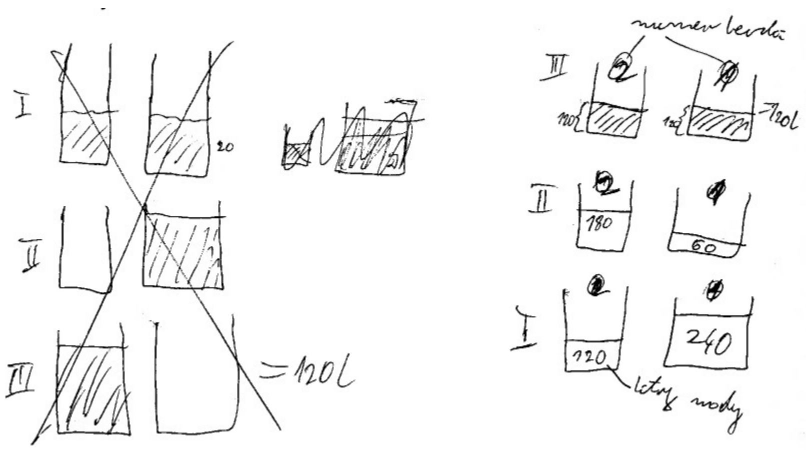

Figure 1: Solution of Patrik

Another difficult point in the text was how to understand the phrase **minimal possible capacity** in question (ii). An easy answer, consistent with the keyword *minimal* is: minimal capacity of each pot is 0. However, what is meant is *he minimal possible capacity for the described process to be viable.* In the terminology of Koedinger and Nathan [9], this is a typical example of an inconsistent problem formulation, as to find the correct answer, the word *minimal* in fact requires taking the *maximal* amounts in each pot during the pouring process.

It turned out that the percentage of correct solutions for question (ii) was much smaller than that for question (i). This failure could be explained not only by the inconsistency of the word *minimal*, but also by the imperfect mental picture the participants created about the described process. Some participants implicitly asumed that the capacities of the two pots have to be the same, although the text has not formulated such an assumption. As said above, another possible explanation might simply be a distraction of some participants, as question (ii) was not explicitly typographically separated in the text.

In the solutions protocols we encountered three different strategies, that we now describe in the order of progressive abstraction. While the first two of them can be described as arithmetic (called informal by Koedinger and Nathan 2004 [9]), the third one is clearly algebraic. Notice also, that participants were not encouraged to use any particular approach, specifically, they were not told that an algebraic solution was required.

4.1 Strategy A. Guess-check-revise

In this strategy the solvers guess at the unknown value and then follow the arithmetic operations as described in the text. They compare the outcome with the desired result from the problem statement and when the outcome differs, they try again. A detailed study of various variants of this strategy and possible outcomes of a solution process that employs this stragety can be found in Capraro et al 2012 [4].

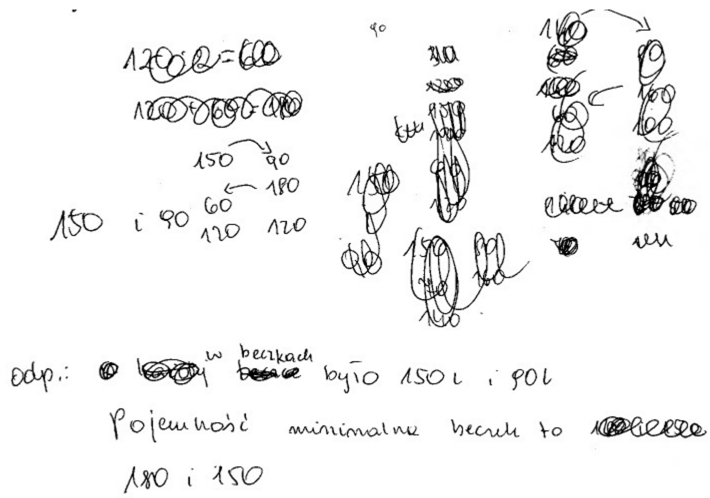

Figure 2: Solution of Patricia

A model example of this strategy applied to the POTS problem is provided by the solution of Patricia, given in Figure 2. A solution protocol pointing to the use of the guess-check-revise strategy is typically characterized by great portions of crossed out text and pictures, indicating abandoned guesses. Patricia's protocol does not provide any other explanation of her flow of thought. Still, with sufficient patience, even when the solver has not created a more complete mental picture of the situation described in text, he/she may eventually arrive at the correct solution, as was also the case of Patricia.

Another sample of guess-check-revise strategy is illustrated in the solution protocol of Dorota given in Figure 3. Dorota first summarized the quantitative data given in the text (the first line says '120 litres in the end in each pot) and made the observation about the total amount of water (the second line). Then she probably produced a simple illustration

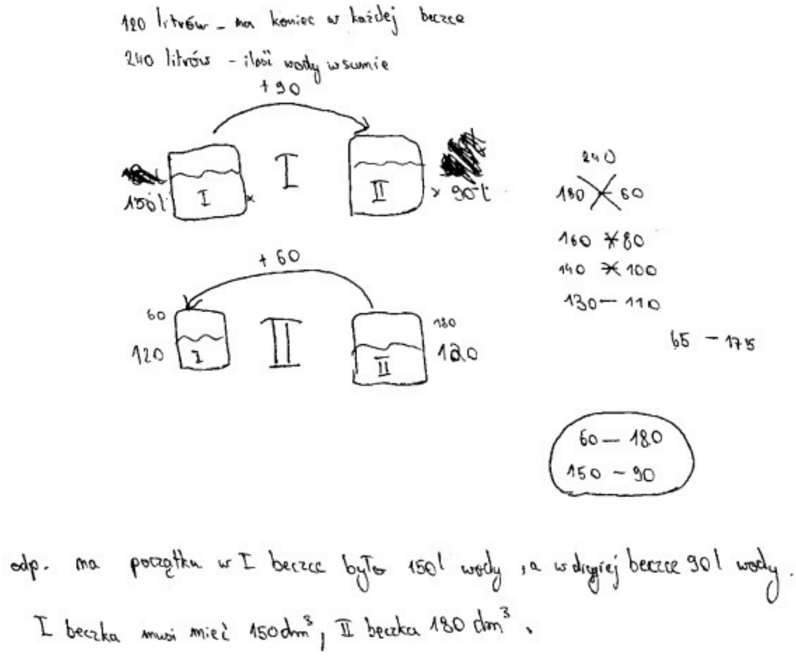

Figure 3: Solution of Dorota

of the pots, indicating the flow of water by arrows. She launched her computations by trying the starting amounts 180 and 60 litres, then she decreased the amount of water in the first pot by 20 litres (twice) and after that she slowed down by decreasing the amount in the first pot by 10 litres only. After these initial trials, it seems that she got a feeling for what is going on and built a more abstract model of the problem. The circled numbers in the right lower part of the sheet together with the numbers acompanying the arrows in her illustration let us suspect that in the end she switched to the second strategy working backwards. Still, the several trials with different starting amounts let us classify this solution in group A.

Dorota's mental picture of the situation was more complete than Patrica's and she did not make completely random guesses. She realized that the total amount of water in both pots is 240 litres and this total amount does not change during the whole process. By trying various values of the starting volumes she simulated the described pouring (obviously by heart) to see whether she arrives at the given final values. The graphicall illustration also shows that she carefully noticed the amounts of water after each pouring (see the small numbers written close to the pictures of pots) and so wrote the correct answer for question (ii): minimal possible capacities are 150 and 180 litres respectively.

4.2 Strategy B. Working backwards

To find the unknown starting values, a solver who uses this strategy (called also the *unwind* strategy in Koedinger and Nathan 2004 [9]) reverses the process described in the problem.

This strategy applied to the POTS problem consists of the following observations and computational steps. In the end, both pots contain 120 litres of water. The second pouring doubled the amount of water in the first pot, so 60 litres were tranferred from the second pot into the first one. Hence before the second pouring the first pot contained 60 litres and the second one 180. The first pouring doubled the amount of water in the second pot. Hence before the first pouring the second pot contained 90 litres and the first one 150. By looking at the successive amounts of water in the pots, the minimal possible capacities 150 and 180 litres, respectively, are obtained.

A nice explanation of the students mental picture of the problem can be derived from the illustration that accompanied the solution of Sara, see Figure 4. In line with the characterisation provided in [2], this illustration can be described as *pictorial.* It is a drawing that could be easily related to reality (observe not only the realistic shape of the pots and the arrows indicationg the flow of water, but also its changing levels in the pots reflecting the problem formulation), yet it requires some kind of arithmetic interpretation.

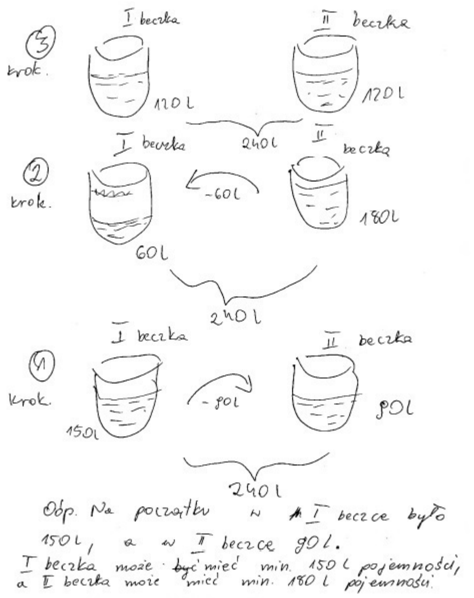

Figure 4: Solution of Sara

We classified a solution protocol as one that uses strategy B if it contained any indication of the reversal of the process, expressed either verbally or by an illustration similar to the one produced by Sara.

4.3 Strategy C. Algebra - setting up a system of equations

The core of the algebraic method is in capturing all known and unknown values in one overall static representation. Let us denote the starting amount of water in the first pot by x and the starting amount in the second pot by y. The water volumes in the pots change in accordance with Table 1.

	Pot no. 1	Pot no. 2
Beginning	x	y
After the first pouring	$x-y$	$2y$
After the second pouring	$2(x-y)$	$2y-(x-y)$

Table 1: Setting the equation system

As the amounts of water after the second pouring in both pots are equal to 120 litres, the corresponding system of linear equations reads as:

$$2(x-y)=120 \quad \text{and} \quad 3y-x=120. \tag{1}$$

The unique solution of this system is $x=150$ and $y=90$. This gives the answer for question (i) of the problem: at the beginning the first pot contained 150 litres of water and the second one 90 litres of water.

Notice that to obtain the correct answer for (ii), it does not suffice to take just the obtained solution of the equation system. One has to substitute the computed values of x and y into the entries in Table 1 and check the columns of this table: the first pot contains successively 150, 60 and 120 litres of water; the second one 90, 180 and 120. Hence the minimum necessary capacities complying with the described process are equal to the respective column maxima: 150 litres for the first pot and 180 litres for the second one.

The prerequisite for a successful use of strategy C is the ability of the solver to translate the situation described in words into a mathematical language and represent it in a form of an algebraic equation (or, as in our case, of a system of equations). Notice that the form of the obtained equation system (1) does not in the least resemble the reality described in the text.

In our study, all the participants who opted for strategy C were immediately able to make a suitable choice of the variables (unknowns): the starting water volumes. We did not encounter difficulties described in [18], like denoting by the same letter different quantities, or any unknown quantity or a combination of quantities. Those who failed to obtain a correct solution, did so because they were either not able to work with symbolic expressions to obtain the new amounts of water, or to express the relations described in the text, or made a mistake in algebraic manipulations when solving the equation system.

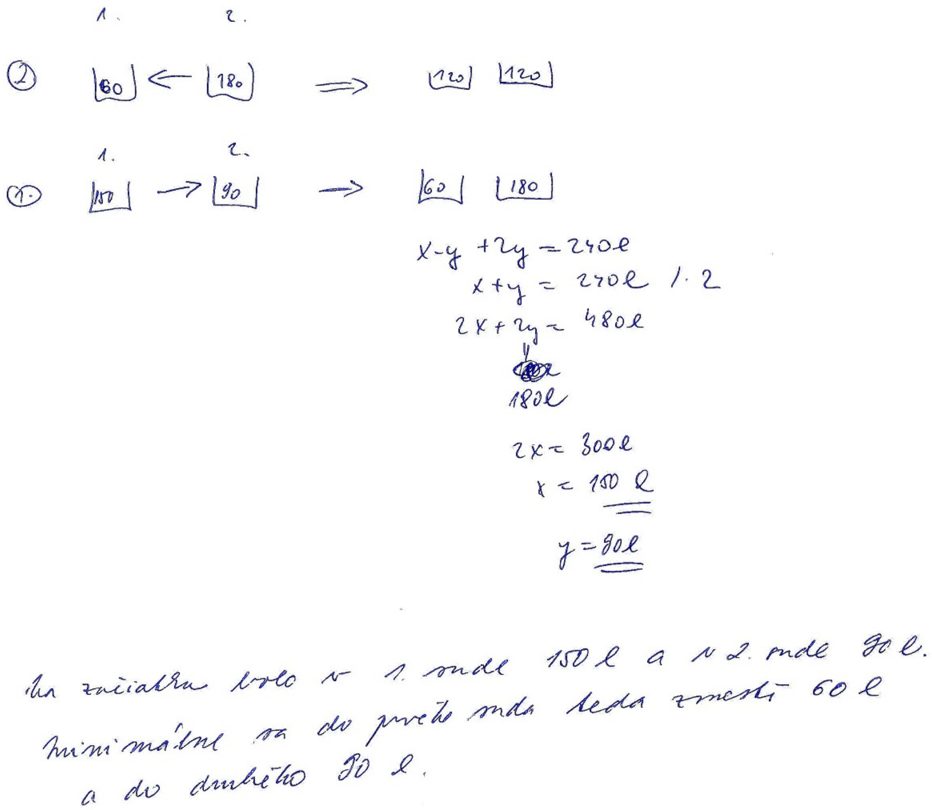

Figure 5: Solution of Kamila

An example of such a failure is the solution of Kamila, given in Figure 5. Her first equation $x - y + 2y = 240$ tells that the sum of the volumes of water in the two pots after the first pouring is 240 litres. The second equation is obtained from the first one just by a simple algebraic manipulation of its left-hand side. The third equation is derived by a useless multiplication of the second one by 2. The final value of the unknown y, indicated by a downward pointing arrow, could by no means be deduced from the equation itself. The illustration provided by Kamila in the upper part of the sheet indicates that in fact she found the correct solution by working backwards, most probably before she resolved to writing equations. Perhaps she considered equations 'compulsory' for word problems, and though she is not able to properly use them, she at least pretends this ability and hopes for a better mark.

By contrast, the equation system produced by Joanna (Figure 6), is correct. Notice the highly concise illustration provided. Using the words of Booth and Thomas [2], this illustration is already stylized, it can be characterized much more as a theoretical object than a realistic representation of the problem formulation. Such an abstract diagram

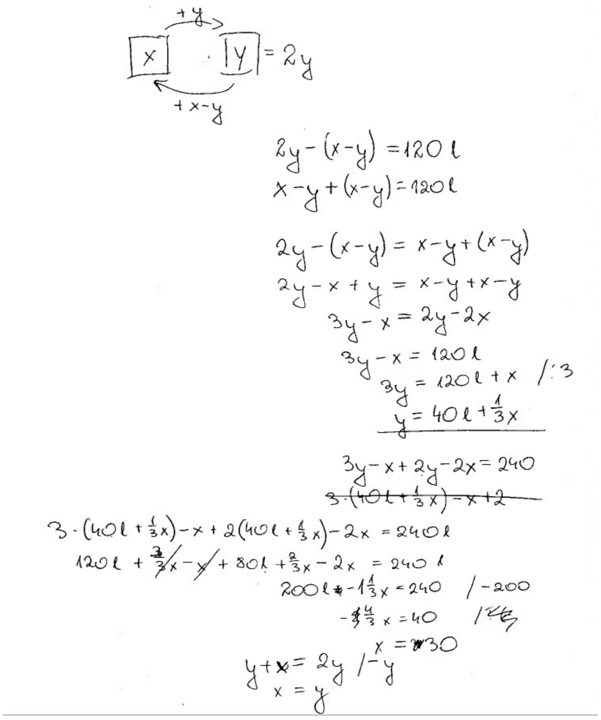

Figure 6: Solution of Joanna

requires mathematical knowledge and interpretation.

Joanna 'looks through the diagram' [2] in order to abstract from it key data necessary for the formulation of the equations. She shows the ability to translate the word problem into a diagramatic representation as well as interpret its relevance for the problem situation.

Unfortunately, Joanna failed in the solution phase. Instead of working with the two equations as a system, she put into equality their left-hand sides. By making this step, she lost the only numerical data given in the problem formulation and obtained an underdetermined equation, from which only a relation between x and y can be obtained. Joanna indeed did this (notice the incorrext equality $x = y$ at the bottom of the sheet), but she was not able (perhaps because of the lack of time) to return to the beginning and find another way to obtain the final answer. Joanna's protocol is a textbook example of a successful comprehension phase and a failure in the solution phase.

5 Data analysis

We performed a detailed strategy and error analysis of written solution protocols. Every solution protocol was scored in two ways. First, we determined the strategy that was used to obtain the solution to the problem. Protocols were classified as strategy C as soon as any variable or any kind of equation appeared. For example, protocol of Kamila (Group II, see Figure 5), who solved the problem correctly in both questions, was assigned to strategy C although she got the solution in fact by working backwards and the equation system in her solution was incorrect. A solution protocol was classified as one using strategy B if its author indicated in any way (in words, or by the order of illustrations) that he/she processed the steps described in the problem text in the reversed order. Not all protocols were so clear as the solution of Sara, so our distinction between strategies was also supported by our long term experience with the analysis of written solutions of students and observing them at work. Other protocols were classified as A. These solutions typically contained several numerical values and often many crossings (see examples in Figures 2 and 3).

Then we coded a solution 0, (i) or (i) +(ii) according to whether no question, only question (i) or both questions (i) and (ii) were answered correctly, respectively. An answer was considered correct if both numbers (for the first as well as for the second pot) were correct. A solution protocol was coded by 0 if no correct answer or only one correct number was given.

The numbers of participants using a particular strategy (successfully or not) and the percentages of them (within a group or within a strategy used) are given in Table 2 and the graphical representation of the results can be seen in Figure 7.

	Strategy A			Strategy B			Strategy C			Total
Success	0	(i)	$(i)+(ii)$	0	(i)	$(i)+(ii)$	0	(i)	$(i)+(ii)$	
Group I	0 0%	1 4%	3 12%	1 4%	0 0%	8 32%	2 8%	4 16%	6 24%	25
Group II	0 0%	1 5.3%	0 0%	0 0%	1 5.3%	1 0%	4 21%	8 42%	5 26.4%	19
Total for strategy	0 0%	2 40%	3 60%	1 10%	1 10%	8 80%	6 20.7%	12 41.4%	5 37.9%	44

Table 2: Results

Failure to find a correct solution for question (ii), apart from an error in understanding the request, was often caused by lack of concentration of pupils and/or students: they simply did not notice that something more should be answered. This explanation is also supported by the course of the interviews conducted in the second phase of the experiment. Other incorrect solutions can be attributed either to numerical errors, or to failures in algebraic manipulation skills.

The results presented in Table 2 lead to the following observations. First, the vast majority of students, 89.4% used strategy C (algebraic), while the proportion of pupils using this strategy was only 48%. However, this strategy did not ensure a correct solution, notice that clearly the most successful was the strategy B, working-backwards. Further,

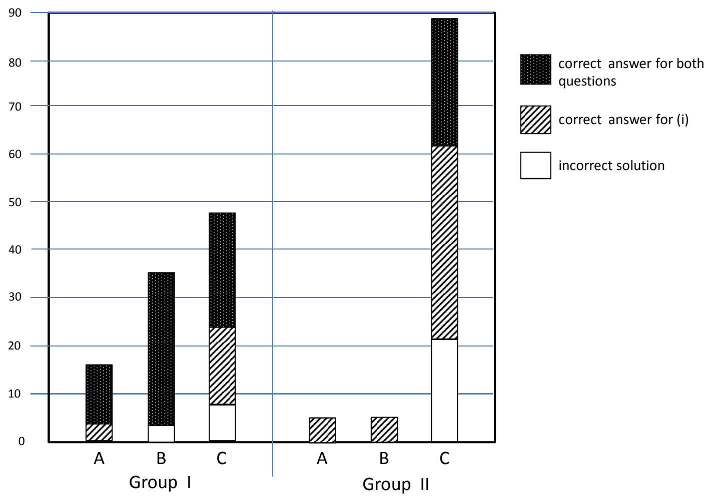

Figure 7: Graphical representation of results

only 29.4% of students who used strategy C obtained all four numbers correctly, compared to 50% of correct solutions among pupils solutions involving strategy C. The smaller success rate of students remains also if one compares partial success, i.e. the correct answer for at least (i) among users of strategy C: 76,5% students against 83,3% pupils. Irrespectively of the strategy used, 68% of pupils obtained the completely correct solution, while only 26,3% of students.

As far as the choice of strategy by our future teachers is concerned, we came to the same conclusion as expressed in [21]. It seems evident that the problem solving behaviour of the student teachers is to a large extent shaped by their particular experiences during their teacher education. Recall that the majority of the mathematical courses passed by our participants were the courses of 'higher' mathematics.

What is the possible explanation of the significantly lower solution success of students? Students showed a much lower fexibility in the use of strategies. The results confirm that the algebraic method for solving word-problems is really more demanding than simpler informal strategies. Besides being able to create a model that is a purely formal representation of the situation presented in the text, the successful manipulation of symbols requires extensive symbolic comprehension skills. If they are not present, the errors in the solution phase add to the errors in the comprehension phase which results in the overall smaller success rates. It seems that students weak symbolic comprehension skills contrast with their existing skills for comprehending and manipulating quantitative constraints written in natural language.

6 Follow-up interviews

Approximately 18 months after the students participated in the first phase of the experiment, we invited three of them (Kamila - her solution protocol from the first phase of the experiment is given in Figure 5, Daniela and Julian) for a follow-up interview. We chose these three participants to have a representative of each success group, i.e. incorrect solution, partially correct and completely correct solution. We allowed for such a long time span between the two experiment phases, because in the meantime the chosen students completed university courses devoted specifically to the teaching competencies, in particular, they took part also in practical placements. Our intentions were to get a more detailed insight into their way of thinking, into their assessment of the use of different strategies by other solvers and the relationship between the pre-service teacher's own spontaneous response to a problem and his/her evaluation of pupils' strategies when approaching the same problem. We wanted to know whether our participants would be able to recognize different strategies of pupils and how would they appreciate them.

We chose handwritten pupils' solutions (Patricia, Dorota, Sara, Patrik, Pavlina and Joanna), representing each strategy. Unlike in the study of Van Dooren et al [20], our sample contained correct as well as incorrect solutions, as our concern was not only whether the students would recognize pupils' strategies, but we also wanted to see their attitudes to problem evaluation as such.

The task of our interviewees was to assess the pupils solutions and evaluate each solution in the sample on the 8-points scale. Our interviewees could first solve the problem for themselves if they decided so. Indeed, each one first tried to produce her/his own solution (in the meantime they had all completely forgotten the problem) and only afterwards started to examine the pupils' solutions. The interviewers' interventions were mainly in a form of questions, encouraging the students to think aloud while examining the pupilssolutions, asking what they were seeing in the protocols and what were their reasons for the decisions concerning the allocation of points. In particular, we tried not to influence them in their evaluations of the solution protocols. The interviews were recorded and then transcribed and also field notes were taken.

The second specialization of each interviewee (G is geography, F is physics, I is informatics), the strategy and correctness of his/her solution in the first phase of the experiment and the numbers of points he/she gave to the pupils' solutions are given in Table 3. Each interview took approximately 45 minutes. Now we describe their course in more detail.

	Own solution		Strategy A		Strategy B		Strategy C	
	strategy	result	Dorota correct	Patricia correct	Sara correct	Patrik incorrect	Pavlina correct	Joanna incorrect
Kamila (G)	C	0	7	3	7	3	7	4
Daniela (F)	C	(i)	7	7	8	3	8	4
Julian (I)	C	$(i)+(ii)$	7	4	8	2	7	2

Table 3: Results

6.1 Kamila

During the interview, she first set up a correct equation system, her solution procedure was also correct, leading to a solution for both unknowns. However, during algebraic manipulations she made a numerical mistake and the values she obtained were incorrect. Then she completely ignored question (ii) and immediately went to see the first pupil's solutions. She did not try to allocate the points in a systematic manner. She deduced one point from correct solutions (Dorota, Sara, Patrik) because she considered the pupils' answers to question (ii) illogical. According to her, the minimal possible capacity of each pot is 0. She did not try to find an explanation for pupils' answers, neither corrected her own incorrect numerical values according to pupils' solution protocols.

6.2 Daniela.

She denoted by x and y the starting amounts of water in the two pots. Then she derived the expressions for amounts of water after the first pouring, but failed to manipulate the algebraic symbols to find the expressions for the result of the second pouring. So she switched to working-backwards strategy and quickly found the (incorrect) solution 90 and 250 litres. She also ignored question (ii).

The first solution she looked at was the solution of Dorota, then she checked the protocol of Pavlina. When she saw the different solution values, she first wanted to correct them according to her own solution. Then she went to the solution protocol of Sara. Clearly, she was able to understand it best, so Daniela realized the incorrectness of her own solution. Based on Sara's solution, Daniela corrected her numerically incorrect answer. When assessing the pupils' solutions, she did not recognize the guess-check-revise strategy of Dorota and Patricia, she meant that they had derived their solutions by working backwards. Her own comment to the solution of Patricia :

> *This one has left only the final numbers … but, when I look at it, it is the same method as the previous one (authors" comment: Sara), just he imagined it through numbers, just he crossed it a lot. As if he again divided 120 into two parts, hence here he has 60, he added, here he has 180 … Here, under this scrawling, I can't see how it is divided, but he did it analogously, too, hence he got the correct answer. He just crossed out the solution.*

Was she influenced by her own approach? It is interesting to note that unlike the other interviewees she rewarded both guess-check-revise solutions equally by 7 points. Now we illustrate how Daniela explained the different number of points given to Patrik and Joanna:

> *Hmm... when I compare these solutions, he only saw it as if practically when he was pouring... he could imagine the first pouring, but he (Patrik) failed in the second one .... she (Joanna) could define it exactly, she got some insight into the problem and even could represent it mathematically... so she was better because she could write it matematically.*

We wanted to know, how would she evaluate the solutions, if the aim of the test were to evaluate the pupils' ability to formulate and solve equations. Her response was:

> *Well, I would for sure deduce some points ... hmm ... if they were told to solve it by equations ... but I would not deduce all points, since in fact some things are solved correctly, even if not by equations. But for sure there would be some penalty for not using equations when they were told to do so.*

6.3 Julian.

His solution in the first phase of the experiment was one of the best organized and correct solutions we obtained, morever, he was the only one who denoted the unknowns not by standard letters for used in school mathematics for unknowns x and y, but by p_1 and p_2 (for pot 1 and pot 2). However, during the inteview, he had difficulties in setting up the system of equations. He correctly found the expressions for the amounts of water after the first as well as the second pouring, but he was not able to make a sensible system of equations. After some help from the interviewers (Which quantities do your expressions represent?) he quickly wrote the correct system, solved it and derived the solution for (ii) too.

Then he started to look at the pupils' solutions. He immediately recognized that they are difficult to compare because of different methods used and so he settled down to an attempt to classify the steps in solutions and formulate an allocation rule for the points. His final decision was: 1 point for each correct answer (number) out of four desired, 2 points for an abstract representation of the problem formulation either in a form of simplified text or a diagram (his comment: *It is important to be able to abstract the important data from the problem text*, he also especially appreciated the diagram produced by Joanna), 2 points if the pupil was able to perform the solution method chosen by him/her correctly. While allocating points, Julian hesitated a lot, saying that the aims of this test were not clear: if the primary aim were to teach pupils equations, the teacher should take this into account. He was also hesitant with respect to solutions of Dorota and Patricia. He could not see any other difference between them than the aestetic appearance. He expressed his hesitation by the question: *Is it correct to give points if the pupil commented his/her solution to please the teacher?*

6.4 Comparison of students' approaches to pupils' solutions

Looking at the numbers of allocated points for different solutions in Table 3, two things are most striking. First, the high variability of evaluations of the same solution (Patricia), and also for two different solutions that were both completely correct (as far as the obtained numerical values are concerned), both using just guess-check-revise strategy, but different in their aesthetic appearance. However, as was evident from the course of interviews, our students did not mind the guessing strategy, they have not made a conclusion that pupils who use it show significant shortcommings in their mathematical skills.

Secondly, our inteviewees showed a clear inclination to appreciate the correct numerical result more than the modelling skills and ability to create formal models and work in them. This is in particular evident in the evaluation produced by Julian, especially in the point allocation rules he created. Notice that according to him, Patricia, who only guessed the correct solution, received twice as many points as Joanna, who created a correct formal model but failed in solving the equation system.

7 Discussion

Our data provide a further support for the theory (see [12, 10, 14]) distinguishing three stages of interpretation and modelling when solving a word problem: the propositional text base, a cognitive model of the events described (the situation model) and a formal model of mathematical relationships in the problem (the problem model).

The *propositional text base* is sufficient for solving the problem by the guess-check-revise method. Participants show comprehension of each piece of given information, without necessarily relating them in a single model. The problem solver guesses an answer, and checks whether it allows each proposition to be true. In our case that meant that the participant was able to reproduce the described pouring process, simulate it, starting with guessed quantities and check if the required final state in both pots was reached with them. Some participants showed ability to process the given information in reducing the number of necessary guesses by realizing that the sum of the starting amounts is 240 litres.

A deeper level of understanding and modelling corresponding to the *situation model* is represented by the ability to work in the opposite direction than the one described by the text of the problem. In our setting, in addition to the ability of reversing the process described in the text, the solvers have to make a logical deduction enabling them to find the numerical value of the amount of water transferred in the second pouring, namely to derive that it was exactly one half the final volume in the first pot.

The *problem model*, necessary for an algebraic solution, represents information stated in the verbal presentation in a form of formal relations. The obtained system of linear equations does no longer represent actions (in our case: pouring water) evident in the situation. The student must construct a hypothetical situation of equality by making inferences.

The three stages are reflected in the three types of strategies: guess-and-check, working backwards and algebraic strategy, i.e., setting and solving the system of equations.

Some researchers, for example, Stacey and MacGregor [18] assert that using guess-and-check can have negative effects on further mathematical learning, arguing that the reliance on this strategy may encourage students to search for a rule that fits particular instances of patterns rather than understanding the general relationship among patterns. They asserted that guess and check obstructed students adopting algebraic methods in formulating and solving equations. The authors of this study, being active researchers in mathematics as well as educators, appreciate very much the need of teaching algebraic approaches to solving problems and teaching the formalism necessary for this approach. On the other hand, having enough experience with creating new mathematical knowledge, i.e., trying to solve problems for which no established algorithm exists so far, they also find heuristic approaches very important. We fully agree with Pólya, who considered 'guessing' immensely important in the process of mathematical exploration, investigation, and conjecturing. In very clear words of himself: *'You have to guess a mathematical theorem before you prove it; you have to guess the idea of the proof before you carry through the details.'* Trying out several possible pathways in a form resembling the guess-check-revise method can help invoke an enlightement leading to a discovery of a completely new method. In case of pupils, such an enlightement may bring about a method that the pupil had learned in the past and forgotten in the meantime, or to a method that for this particular pupil in this particular moment may embody a new discovery.

Unlike Johanning [8] we believe that it make sense to teach (not only) future teachers explicitly various strategies for solving problems, like guess-and-check and working backwards, and relating them to algebraic thinking. This might help to explain the methodological aspects of the 'didactic cut" between arithmetical and algebraic solutions of word problems and to understand the development of pupils' ability to think algebraically while they work with approaches that are sensible to them. In this task, historical observations could also provide a useful help, see e.g. a detailed discussion of this topic in [17].

Since our participants were not explicitly asked to use algebra, each pupil resorted to the solution method that he/she was more comfortable with. On the other hand, the clear prevalence of the use of algebra among the pre-service teachers shows the influence of the intensive training in the abstract mathematics (see also [20, 21]). However, the poor results of preservice teachers invoke serious doubts whether this knowledge and understanding is sufficiently deep and not more formal than internalized. Their almost exclusive use of algebra also shows their low level of flexibility for problems where a less abstract method leads more easily to a correct solution. On the other hand, they did not punish the pupils for other solution strategies, although we cannot be sure about their attitudes to real pupils of their own in future.

We also fully agree with the view expressed in [20, 21]:

> *Possessing both algebraic and arithmetic strategies and being able to switch flexibly between them taking into account the characteristic of the problem, is an important characteristic of skilled mathematical problem-solving behaviour. Future teachers should be able to demonstrate to their pupils the validity and the necessity of the algebraic method as a powerful mathematical tool. One reason is that algebraic symbolism has uses outside of problem solving, including efficiently communicating formulas and facilitating theorem proving. Even within problem solving, equations may well facilitate performance for more complex problem conditions or a single solution for a whole class of problems. On the other hand, the teacher should also be willing to recognize and encourage heuristic approaches and give credit also for using informal strategies instead of pushing their pupils into algebraic method without fully understanding its real contents. Informal strategies in pupils beginning the study of algebra should not be seen as obstacles to learning algebraic methods, rather as real foundations upon which the methods or strategies of algebraic thought are constructed. If we want pupils to become skilled in choosing the most appropriate tool, it is crucial that their teachers possess this disposition themselves and that they act as an appropriate model in their classroom.*

We believe that more empirical studies are necessary to reveal the negative consequences of instruction requiring pupils to use an approach for which they are not yet ready (or teachers teaching methods that they do master themselves!). A student whose apprehension to solve a problem by a method different from the one that is required in mathematics classes gives up any attempt to solve the problem (e.g., because he/she has not enough self-confidence in setting-up and solving algebraic equations) should not be considered a good outcome of the educational process.

The findings in this paper, in spite of the limitations of the study concernig the size of the subjects sample and variety of topics, are similar to the serious concerns expressed by Van Dorren et al. [20] for the situation in Flanders. The achievement of our students, future teachers very close to the start of their profession, were significantly poorer than the results of much younger (although not representative, as far as general abilities of the corresponding population is concerned) pupils. Some of the main causes are easily at hand: the teacher's job has become much less popular recently, mathematics is also considered unattractive and too difficult. So the numbers of students at teachers programmes (at least in the country of the authors) have declined sharply. Moreover, young people with better results in mathematics usually choose economically more prospective jobs, in particular in information technologies. Expected consequences of this situation are easy to predict, not only as subjective feelings of the authors. Several large-scale studies have investigated the significance of teachers' content knowledge and pedagogical content knowledge for high-quality instruction and student progress in secondary-level mathematics, e.g. [6, 1]. Their findings demonstrate that teacher candidates who develop only a limited mathematical content knowledge have also limited pedagogical content knowledge and this produces negative effects on instructional quality and student progress. Moreover, differences in content knowledge that emerge during preservice training persist across the entire teaching career.

Appendix. Solving the generalized POTS problem

A generalization of the POTS problem (in fact, a whole class of the problems with an indentical logical structure) involves presenting the final amounts of water in the the pots not in concrete numbers, but as variables. Strategy gues-and-check cannot be used now, working backwards is possible, but still involves symbolic manipulations. Presenting such a parametric problem to pupils who have not yet 'climbed the step of working with symbols instead of numers, could have grave consequences and therefore should only be reserved for well prepared classes [17]. On the other hand, in our opinion, mathematics teachers, especially secondary school teachers should be able not only understand, but also be able to produce such systems and confidently work with them. Usually, parametric equations or systems of equations are not included in standards for middle schools.

Let us now present the solution of the generalized POTS problem, using elementary knowledge from linear algebra from the viewpoint of linear algebra. If we denote the *given* final amounts of water in the two pots by v_1 and v_2 and the *unknown* starting amounts by x_1 and x_2, the equation system describing the pouring process, as derived from Table 1, but know with symbollically expressed final amounts is

$$2x_1 - 2x_2 = v_1; \qquad -x + 1 + 3x_2 = v_2. \tag{2}$$

Let us stress here again, that in the system of linear equations (2, symbols x_1 and x_2 represent uknowns, while v_1 and v_2 given numerical values.

The most convenient way of solving this parametric system is by using the Cramer's rule (see any textbook in linear algebra, e.g. [19, Chapter4]). Therefore let us compute the determinants

$$D = \begin{vmatrix} 2 & -2 \\ -1 & 3 \end{vmatrix} = 4; \ D_1 = \begin{vmatrix} v_1 & -2 \\ v_2 & 3 \end{vmatrix} = 3v_1 + 2v_2; \ D_2 = \begin{vmatrix} 2 & v_1 \\ -1 & v_2 \end{vmatrix} = v_1 + 2v_2.$$

Since $D \neq 0$, system (2) has a unique solution for any right-hand side. Moreover, as D_1 as well as D_2 are nonnegative for any nonnegative values of v_1 and v_2, we can immediately make a conclusion that for any given final amounts of water, a unique nonnegative solution of the system exists. In other words, the pouring process can be uniquelly realized for any given (nonnegative) final amounts of water v_1 and v_2 given, the starting amounts being

$$x_1 = \frac{3v_1 + 2v_2}{4};\ x_2 = \frac{v_1 + 2v_2}{4}.$$

In the POTS problem presented in this paper we have $v_1 = v_2 = 120$ litres, and one can double check that the derived formulas lead to the already obtained solution $x_1 = 150$litres and $x_2 = 90$ litres.

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